Vacuum polarization effects of pointlike singularities

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Conical spaces

Cosmic string: metric (cylindric coords):

$$ds^2 = dt^2 - dz^2 - d\varrho^2 - \beta^2 \varrho^2 d\varphi^2\,, \qquad 0 < \beta \leqslant 1$$

Geometry: $R = 2(1-\beta)\,\delta_+(\varrho)/\varrho, \qquad \delta\varphi = 2\pi(1-\beta)$

Phase transition energy scale: $\mu \sim \eta^2 = \dfrac{1-\beta^2}{8\pi G}$

For $\eta = \eta_{\rm GUT} \sim 10^{16}$ GeV $\qquad 1-\beta \sim 10^{-5} \qquad a \sim \dfrac{1}{\sqrt{\lambda}\eta} \sim 10^{-29}$cm

Complement: $\beta' \equiv 1 - \beta = \dfrac{\delta\varphi}{2\pi} \qquad \beta' = 4G\mu$

Klein-Gordon: $(\Box + m^2 + \xi R)\,\phi = 0\,,$

$$\left(\partial_t^2 - \Delta + m^2 + \lambda\,\delta^{2,3}(\boldsymbol{x})\right)\phi(t,\boldsymbol{x}) = 0\,.$$

Time factorization: $\phi_\omega^{(\pm)}(t,\boldsymbol{x}) = \mathrm{e}^{\mp i\omega t}\,u_\omega(\boldsymbol{x})\,,$

Schrödinger: $H\,u_\omega(\boldsymbol{x}) = (\omega^2 - m^2)\,u_\omega(\boldsymbol{x})\,.$

Formal Hamiltonian: $H = -\Delta + \lambda\,\delta(\boldsymbol{x})$

Coupling problem & self-adjoint extension

$\lambda \neq 0$: Hu_ω does not belong Hilbert space
$\lambda = 0$: no interaction!

Renormalization of λ or SAE?

Resolution of laplacian: $H = \bigoplus_{l=0}^{\infty} \Big(H_l \bigotimes \underbrace{\mathbf{1}}_{\text{angular}} \Big)$,

where partial Hamiltonians

$$H_l = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2}\,, \qquad l = 0, 1, 2, \dots .$$

H_l are self-adjoint itself for any $l \geqslant 1$,
Self-adjoint extensions of H_0 (s-wave): $-\infty < \alpha \leqslant \infty$

$$H_{0,\alpha} = -\Delta_{0,\alpha} = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr}$$

$$\mathcal{D}(H_{0,\alpha}) = \Big\{ u_\alpha \in L^2((0,\infty); r^2 dr)\,;\ 4\pi\alpha \lim_{r\to+0} r u_\alpha(r) = \lim_{r\to+0}[u_\alpha + r u'_\alpha] \Big\}$$

Eigenvalues/Eigenfunctions to $H_{0,\alpha}$:

$$p > 0\,, \qquad u \sim r^{-1/2}\big[J_{1/2}(pr) + k(\alpha) Y_{1/2}(pr)\big]$$

Hadamard Green's function

$p^2 < 0$: $\phi(x,t) \sim \mathrm{e}^{\pm|p|t} u(x)$

$\alpha < 0$: bound state $u_{0,\alpha}(r) = \sqrt{-2\alpha}\,\mathrm{e}^{4\pi\alpha r}/r$

$\{u_{plm}\}$ — complete set of eigenfunctions of the *free* Laplacian.

Hadamard Green's function:

$$D^{(1)}_\alpha(x,x') = \tfrac{1}{2}\left\langle \phi(x)\,\phi(x') + \phi(x')\,\phi(x)\right\rangle_{\mathrm{vac}}$$

$$D^{(1)}_\alpha = \mathrm{Re}\int\limits_m^\infty d\omega\, e^{-i\omega(t-t')}\left[u_{p\alpha}(x)\,u^*_{p\alpha}(x') + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} u_{plm}(x)\,u^*_{plm}(x')\right].$$

$\alpha = \infty$ = no interaction.

$$D^{(1)}_\infty(x,x') = \mathrm{Re}\int\limits_m^\infty d\omega\, \mathrm{e}^{-i\omega(t-t')}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} u_{\omega lm}(x)\,u^*_{\omega lm}(x')\,.$$

Renormalized Hadamard Green's function: $D^{(1)}_{\mathrm{ren}} = D^{(1)}_\alpha - D^{(1)}_\infty$.

$$D^{(1)}_{\mathrm{ren}}(x,x') = \mathrm{Re}\int\limits_m^\infty d\omega\, \mathrm{e}^{-i\omega(t-t')}\left[u_{p\alpha}(x)\,u^*_{p\alpha}(x') - u_{p\infty}(x)\,u^*_{p\infty}(x')\right].$$

Finally:

$$D^{(1)}_{\mathrm{ren}}(x,x') = \frac{1}{4\pi^2 rr'}\int\limits_0^\infty dz\, z\,\frac{\cos\left[\sqrt{(4\pi\alpha z)^2+m^2}(t-t')\right]}{\sqrt{z^2+(m/4\pi\alpha)^2}(1+z^2)}\times$$
$$\times\left(\sin\left[4\pi\alpha z(r+r')\right] + z\,\cos\left[4\pi\alpha z(r+r')\right]\right).$$

$\langle\phi^2(x)\rangle_{\rm ren}$

Vacuum field-square: $\langle\phi^2(x)\rangle_{\rm ren} = D^{(1)}_\alpha(x,x) = \dfrac{1}{4\pi^2 r^2}\,\mathcal{J}\Big(8\pi\alpha r, \dfrac{m}{4\pi\alpha}\Big)$

Two-arguments function:

$$\mathcal{J}(\beta,a) \equiv \int\limits_0^\infty dz\,\frac{1}{1+z^2}\,\frac{z}{\sqrt{z^2+a^2}}\Big(\sin\beta z + z\cos\beta z\Big)$$

Lengthy parameters:

the Compton length $l_c = m^{-1}$

the scattering length $d_s = (4\pi\alpha)^{-1}$

$$\langle\phi^2(x)\rangle_{\rm ren} = \frac{1}{4\pi^2 r^2}\,\mathcal{J}\Big(\frac{2r}{d_s}, \frac{d_s}{l_c}\Big).$$

i) for fixed m and d_s, $\langle\phi^2\rangle_{\rm ren}$ monotonically decreases with growth of r, to $\langle\phi^2(r\to\infty)\rangle_{\rm ren} = 0$;

ii) for fixed r and d_s, $\langle\phi^2\rangle_{\rm ren}$ monotonically decreases as m grows;

iii) for fixed r and m, $\langle\phi^2\rangle_{\rm ren}$ monotonically decreases as α grows (d_s falls), since from physical reasons the limit $\alpha\to+\infty$ implies the absence of polarization effect.

$$\mathcal{J}(\beta,a) \equiv \int_0^\infty dz\,\frac{1}{1+z^2}\,\frac{z}{\sqrt{z^2+a^2}}\Big(\sin\beta z + z\cos\beta z\Big)\,.$$

Introduce cosine- and sine-ints:

$$\mathcal{J}_c := \int_0^\infty dz\,\frac{\cos\beta z}{1+z^2}\,\frac{1}{\sqrt{z^2+a^2}}\,, \qquad \mathcal{J}_s := \int_0^\infty dz\,\frac{\sin\beta z}{1+z^2}\,\frac{z}{\sqrt{z^2+a^2}} = -\frac{\partial\mathcal{J}_c(\beta,a)}{\partial\beta}$$

$$\mathcal{J} = K_0(\beta a) + \mathcal{J}_s - \mathcal{J}_c$$

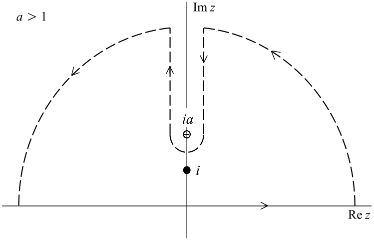
Particular case: $a = 1$:

$$\mathcal{J}(\beta,1) = (1+\beta)K_0(\beta) - \beta K_1(\beta)$$

$a > 1$:

$$\mathcal{J} = K_0(\beta a) - \mathrm{e}^{\beta}\int_\beta^\infty dt\,\mathrm{e}^{-t}\,K_0(at)$$

$a < 1$: works as well



More economy form: $\mathcal{J}(\beta,a) = a\,\mathrm{e}^{\beta}\int_\beta^\infty dt\,\mathrm{e}^{-t}\,K_1(at)\,.$

Asymptotics of basic integral

Base of separation: $a\beta$ vs. 1

Typical variants of values of a	Characteristic limiting values of β				
	$\beta \ll a^{-1} \ll 1$	$\beta \ll 1$	$\beta \sim 1$	$\beta \gg 1$	$\beta \gg a^{-1} \gg 1$
$a = 0$	—	$\ln\frac{1}{\beta} - \gamma$	$e^{\beta} E_1(\beta)$	$1/\beta$	—
$a \ll 1$	—	$\left(\ln\frac{1}{\beta} - \gamma\right)(1+\beta) + \beta$	$e^{\beta} E_1(\beta) - \frac{\beta+1}{2} a^2 \ln\frac{2}{a}$	$a K_1(a\beta)$	$a K_0(a\beta)$
$a < 1$	—	$\ln\frac{2}{a\beta} - \gamma - \frac{\operatorname{Arch} a^{-1}}{\sqrt{1-a^2}}$	$a\, e^{\beta} \int_{\beta}^{\infty} dt\, e^{-t} K_1(at)$	$\sqrt{\frac{\pi a}{2\beta}}\, \frac{e^{-a\beta}}{a+1}$	—
$a = 1$	—	$\ln\frac{2}{\beta} - \gamma - 1$	$(1+\beta) K_0(\beta) - \beta K_1(\beta)$	$\sqrt{\frac{\pi}{8\beta}}\, e^{-\beta}$	—
$a > 1$	—	$\ln\frac{2}{a\beta} - \gamma - \frac{\arccos a^{-1}}{\sqrt{a^2-1}}$	$a\, e^{\beta} \int_{\beta}^{\infty} dt\, e^{-t} K_1(at)$	$\sqrt{\frac{\pi a}{2\beta}}\, \frac{e^{-a\beta}}{a+1}$	—
$a \gg 1$	$\ln\frac{2}{a\beta} - \gamma - \frac{\pi}{2a}$	$K_0(a\beta)$	$\sqrt{\frac{\pi}{2a\beta}}\, e^{-a\beta}$	$\sqrt{\frac{\pi}{2a\beta}}\, e^{-a\beta}$	—

Renormalized field-square in doubly logarithmic scale:

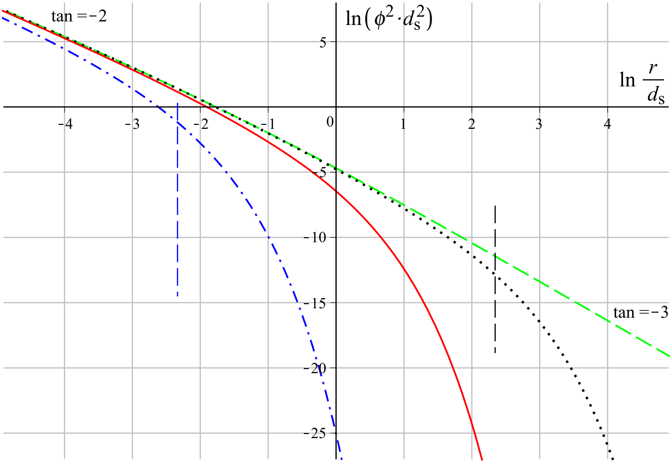


Figure: for massless field (green dashed), for $l_c/d_s = 10$ (black dotted), $l_c/d_s = 1$ (red solid) and $l_c/d_s = 0.1$ (blue dashdotted). The value $r = l_c$ is marked by dash of corresponding color. The value $r = d_s$ corresponds to the ordinate-axis for each curve.

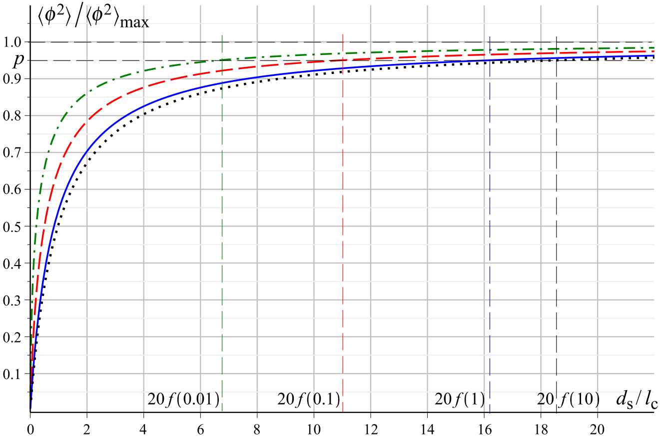


Figure: $\langle\phi^2\rangle$, normalized by $\langle\phi^2\rangle_{\max}$ as a function of d_s (in units $l_c = 1$): for $r/l_c = 0.01$ (green dashdotted), for $r/l_c = 0.1$ (red dashed), $r/l = 1$ (blue solid) and $r/l_c = 10$ (black dotted). The values $d_s^{(p=0.95)}$ are marked by vertical dash lines of corresponding color.

Renormalized energy-momentum tensor

$$\langle T^{\nu}_{\nu}\rangle_{\rm ren} = \frac{1}{4\pi^2 r^4}\left[A_{\nu,-1}\mathcal{J}\left(\frac{2r}{d_s},\frac{d_s}{l_c}\right) + A_{\nu,0}K_0\left(\frac{2r}{l_c}\right) + A_{\nu,1}\hat{K}_1\left(\frac{2r}{l_c}\right)\right]$$

(no summation over ν)

Index ν of the diagonal component	Index σ		
	-1	0	1
t	$\left(2\xi-\frac{1}{2}\right)\left(\frac{2r}{d_s}-1\right)+\frac{r^2}{l_c^2}-4\xi\frac{r^2}{d_s^2}$	$-4\xi\frac{r^2}{l_c^2}$	$\frac{1}{2}+\xi\left(\frac{2r}{d_s}-3\right)$
r	$4\xi-\frac{1}{2}+(1-4\xi)\frac{r}{d_s}$	0	$2\xi-\frac{1}{2}$
θ,φ	$\frac{1}{2}-4\xi+(6\xi-1)\frac{r}{d_s}+(1-4\xi)\frac{r^2}{d_s^2}$	$(1-4\xi)\frac{r^2}{l_c^2}$	$\frac{3}{4}-4\xi+\left(2\xi-\frac{1}{2}\right)\frac{r}{d_s}$
Sp	$(1-6\xi)\left(1-\frac{2r}{d_s}+\frac{2r^2}{d_s^2}\right)+\frac{r^2}{l_c^2}$	$2(1-6\xi)\frac{r^2}{l_c^2}$	$(1-6\xi)\left(\frac{3}{2}-\frac{r}{d_s}\right)$

Table: Coefficients $A_{\nu,\sigma}$ for the EMT and for its trace

Renormalized energy-momentum tensor:

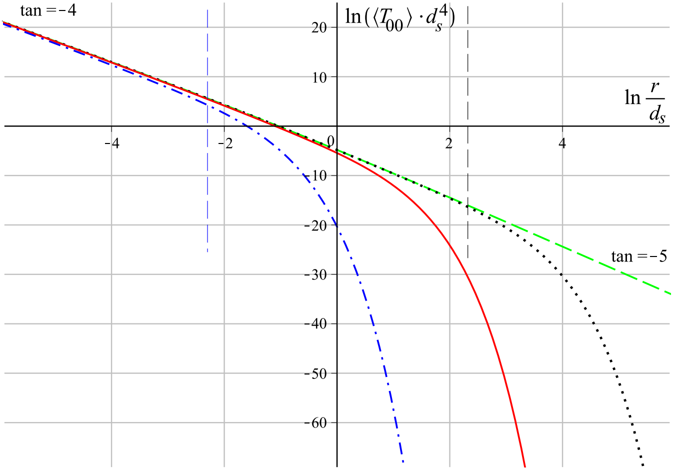


Figure: $\langle T_{00} \rangle$ in doubly logarithmic scale (minimal coupling): for massless field (green dashed), for $l_c/d_s = 10$ (black dotted), $l_c/d_s = 1$ (red solid) and $l_c/d_s = 0.1$ (blue dashdotted)

Conclusions

Renormalization:

$$\frac{1}{\lambda_{\rm ren}} = \frac{1}{\lambda} + \frac{1}{4\pi\epsilon}\,, \qquad \alpha = -\lambda_{\rm ren}^{-1}\,, \qquad \epsilon \to 0^+.$$

- Single-parametric SAE yields the natural answer in terms of finite quantity α (or d_s)
- Vacuum polarization of massive scalar field is computed and has reasonable asymptotic cases
- Presumably, it yields the rule how to work with (3+1)-dimensional pointlike attraction and with (2+1)-dimensional zero-range interaction
- The bare coupling may be directly renormalized what is equivalent here to SAE-concept

Thank you!