

EXACT ANALYTICAL SOLUTIONS IN MODIFIED GRAVITY MODELS

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In this paper we present a number of exact analytical solutions of the modified gravity equations for one-dimensional vacuum theory with Gauss-Bonnet term as well as the metric $F(R)$ gravity depending on one and two variables.

Action:

$$S = -\frac{c}{2\kappa} \int d^4x \sqrt{-g} R$$

$$\kappa = 1.86 \times 10^{-27} \text{sm} \times g^{-1}; \quad g = \det g_{\mu\nu}$$

Scalar curvature (Ricci scalar):

$$R = g^{\mu\sigma} g^{\nu\tau} R_{\mu\nu\sigma\tau}$$

Curvature tensor:

$$\begin{aligned} R_{\mu\nu\sigma\tau} = & \frac{1}{2} \left(\frac{\partial^2 g_{\mu\tau}}{\partial x_\nu \partial x_\sigma} + \frac{\partial^2 g_{\nu\sigma}}{\partial x_\mu \partial x_\tau} - \frac{\partial^2 g_{\mu\sigma}}{\partial x_\nu \partial x_\tau} - \frac{\partial^2 g_{\nu\tau}}{\partial x_\mu \partial x_\sigma} \right) \\ & + g_{\alpha\beta} \left(\Gamma_{\nu\sigma}^\alpha \Gamma_{\mu\tau}^\beta - \Gamma_{\nu\tau}^\alpha \Gamma_{\mu\sigma}^\beta \right) \end{aligned}$$

Vacuum gravitational field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$ - тензор Риччи

Action:

$$S = \frac{1}{2k^2} \int d^4x \sqrt{-g} (R + F(G))$$

Gauss-Bonnet term:

$$G = R^2 - 4R^{\mu\nu} R_{\mu\nu} + R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$$

Vacuum gravitational field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}F(G) - \left[-2RR_{\mu\nu} + 4R_{\mu\rho}R_{\nu}{}^{\rho} - 2R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau} + 4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\rho\sigma} - 2R\nabla_{\mu}\nabla_{\nu} + 2g_{\mu\nu}R\Box - 4R_{\mu\nu}\Box + 4R_{\nu}{}^{\rho}\nabla_{\rho}\nabla_{\mu} + 4R_{\mu}{}^{\rho}\nabla_{\rho}\nabla_{\nu} - 4g_{\mu\nu}R^{\rho\sigma}\nabla_{\rho}\nabla_{\sigma} + 4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}\nabla_{\rho}\nabla_{\sigma} \right] F_G = 0$$

$$F_G(G) \equiv \frac{dF(G)}{dG}$$

Gravity with Gauss-Bonnet term. Case of dependence on one variable

Friedmann-Lemaître-Robertson-Walker metric:

$$ds^2 = -dt^2 + (a(t))^2 \sum_{i=1,2,3} (dx^i)^2$$

Field equation:

$$24H^3 \dot{G} F_{GG} - G F_G + F + 6H^2 = 0$$

Gauss-Bonnet term:

$$G = 24H^2(\dot{H} + H^2)$$

$$\cdot \equiv \frac{d}{dt}$$

$H = \frac{\dot{a}}{a}$ - Hubble rate

$a(t)$ - scale factor

Scalar curvature $R = 6(\dot{H} + 2H^2)$

General vacuum solution of the model with Gauss-Bonnet term. Case of dependence on one variable

Field equations include the derivatives $\frac{d}{dt}$ and $\frac{d}{dG}$

Ansatz:

$$\sqrt{H^2} \frac{dH^2}{dt} = \Phi(H^2), \quad \Phi(H^2) - \text{an arbitrary function}$$

Let's denote $H^2 \equiv x$

Then the field equation becomes an ODE for the function $F(x)$:

$$\frac{2x\Phi}{\Phi_x + 4x} F_{xx} - \left(\frac{2x\Phi(\Phi_{xx} + 4)}{(\Phi_x + 4x)^2} + \frac{\Phi + 2x^2}{\Phi_x + 4x} \right) F_x + F + 6x = 0$$

Gauss-Bonnet term: $G = 12(\Phi(x) + 2x^2)$

It is possible to obtain a general solution of this equation for $F(x)$

for an **arbitrary** function $\Phi(x)$

General vacuum solution of the model with a Gauss-Bonnet term. The case of dependence on one variable

General solution is:

$$F(x) = C_0 (2x^2 + \Phi(x)) \left[C_1^{(0)} + \frac{C_2^{(0)}}{C_0^2} \int \frac{\sqrt{x} (\Phi_x + 4x) e^{\int \frac{x dx}{\Phi}} dx}{(\Phi + 2x^2)^2} - \int \frac{3}{C_0} \left(\int \frac{(\Phi + 2x^2) e^{-\int \frac{x dx}{\Phi}} dx}{\sqrt{x} \Phi} \right) \cdot \frac{\sqrt{x} (\Phi_x + 4x) e^{\int \frac{x dx}{\Phi}} dx}{(2x^2 + \Phi)^2} dx \right]$$

By choosing different forms of function $\Phi(x)$ we obtain different functions $F(x)$

From the expression for G we obtain $x(G) \Rightarrow F(G)$

Then from Ansatz we obtain $x(t) \Rightarrow a(t), R(t)$

Simplest form of $\Phi(x)$:

1. $\Phi(x) = C_0 x^2$

$$x(t) \sim \left(-\frac{C_0}{2}t + t_0\right)^{-2}, \quad a(t) \sim \left(-\frac{C_0}{2}t + t_0\right)^{-\frac{2}{C_0}}$$

$$G(t) \sim \left(-\frac{C_0}{2}t + t_0\right)^{-4}, \quad R(t) \sim \left(-\frac{C_0}{2}t + t_0\right)^{-2}$$

1a For $C_0 < 0, t_0 \geq 0$:

scale factor $a(t) \uparrow$ monotonically with $t \uparrow$

H, G and R are continuous functions for all $t > 0$

For $C_0 < 0, t_0 < 0$:

H, G and $R \rightarrow$ but this singularity "in the past" may be purely coordinate one

1b For $C_0 > 0$ the above functions are defined only for $t < t_s$ and have the future singularities at $t = t_s$. Since $3H^2 \rightarrow \infty$ and $|2\dot{H} + 3H^2| \rightarrow \infty$ at $t \rightarrow t_s$ this is the Big Rip (Большой разрыв) singularity and t_s is the so-called Rip time.

This solution describes the universe that ends at the Big Rip singularity in the time t_s .

1c $C_0 = \frac{2}{3}$

$$F(G) = -3\sqrt{2}\sqrt{G} + \frac{cC_1^{(0)}}{8} G + \frac{C_2^{(0)}}{48c} G \ln \frac{G}{32}$$

1d $C_0 \neq \frac{2}{3}$

$$F(G) = \frac{2\sqrt{3(C_0+2)G}}{C_0-2} + \frac{cC_1^{(0)}}{12C_0} G + \frac{2C_0^2 C_2^{(0)}}{\sqrt{3c(2-3C_0)}} \left(\frac{G}{C_0+2}\right)^{\frac{C_0+2}{4C_0}}$$

2. $\Phi(x) = \beta x^{\frac{3}{2}}$

$$x(t) \sim e^{\beta t}, \quad a(t) \sim e^{\gamma e^{\beta t}}$$

And

$$G(x) = 12(2x^2 + \beta x^{\frac{3}{2}})$$

Action:

$$S = \frac{c}{2\kappa} \int d^4x \sqrt{-g} F(R)$$

Vacuum gravitational field equations:

$$F_R(R)R_{\mu\nu} - \frac{1}{2}F(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] F_R(R) = 0$$

$$F_R(R) \equiv \frac{dF(R)}{dR}$$

Friedmann-Lemaître-Robertson-Walker metric:

$$ds^2 = -dt^2 + (a(t))^2 \sum_{i=1,2,3} (dx^i)^2$$

Friedman equation:

$$18H(\ddot{H} + 4H\dot{H}) F_{RR} - 3(\dot{H} + H^2)F_R + \frac{1}{2}F = 0$$

$$\cdot \equiv \frac{d}{dt}$$

$H = \frac{\dot{a}}{a}$ - Hubble rate

$a(t)$ - scale factor

Scalar curvature $R = 6(\dot{H} + 2H^2)$

$F(R)$ gravity. Case of dependence on one variable

Field equations include the derivatives $\frac{d}{dt}$ and $\frac{d}{dR}$

Ansatz:

$$\frac{1}{\sqrt{H^2}} \frac{dH^2}{dt} = \Phi(H^2), \quad \Phi(H^2) - \text{an arbitrary function}$$

Let's denote $H^2 \equiv x$

Then the Friedman equation becomes an ODE for the function $F(x)$:

$$F_{xx} - \left(\frac{\Phi_{xx}}{\Phi_x + 4} + \frac{\Phi + 2x}{2\Phi x} \right) F_x + \frac{\Phi_x + 4}{2\Phi x} F = 0$$

Scalar curvature $R = 3(\Phi(x) + 4x)$

By choosing different forms of function $\Phi(x)$ we obtain different functions $F(x)$

From the expression for R we obtain $x(R) \Rightarrow F(R)$

Then from Ansatz we obtain $x(t) \Rightarrow a(t), R(t)$

Simplest form of $\Phi(x)$:

1. $\Phi(x) = \alpha x^2$

$$F(x) = C_1 \sqrt{x} e^{\frac{2}{x}} + C_2 \left(\frac{2}{3} x^2 - \frac{32}{3} x - \frac{32\sqrt{2\pi}}{3} \sqrt{x} e^{\frac{2}{x}} \operatorname{erf}\left(\sqrt{\frac{2}{x}}\right) \right), \quad x = 4 \pm \sqrt{16 - \frac{2}{3}R}$$

$$a(t) \sim e^{b(\pm t + t_0)^{\frac{2}{3}}}, \quad R(t) \sim (\pm t + t_0)^{-\frac{2}{3}} \left(4 + b(\pm t + t_0)^{-\frac{2}{3}} \right)$$

2. $\Phi(x) = \frac{\alpha}{x}$

$$F(x) = C_1 \sqrt{x} \left(1 + \sum_{n=1}^{\infty} a_n x^n \right) + \frac{C_2}{x} \left(1 + \sum_{n=1}^{\infty} b_n x^n \right), \quad x = \frac{1}{8} \left(\frac{R}{3} \pm \sqrt{\frac{R^2}{9} - 16\alpha} \right)$$

$$a(t) \sim e^{\beta(\pm t + t_0)^{\frac{4}{3}}}, \quad R(t) \sim (\pm t + t_0)^{-\frac{2}{3}} + b(\pm t + t_0)^{\frac{2}{3}}$$

3. $\Phi(x) = C_0x + C_1$, $C_0, C_1 = const$

A. $C_1 = 0$

$$F_1(R) = C_+ R^{k_+} + C_- R^{k_-}, \quad F_2(R) = C_1 R^{\frac{4 \pm \sqrt{6}}{2}} + C_2 R^{\frac{4 \pm \sqrt{6}}{2}} \ln R$$

$$F_3(R) = C_3 R^{\frac{1}{4}(3 + \frac{2}{C_0})} \cos(b \ln R + \varphi_0)$$

$$H(t) \sim (\pm t + t_0)^{-1}, \quad a(t) \sim (\pm t + t_0)^{-\frac{2}{C_0}}, \quad R(t) \sim (\pm t + t_0)^{-2}$$

B. $C_1 \neq 0$

$$F(R) = C {}_2F_1(-k_+; -k_-; -\frac{1}{2}; z) + \tilde{C} z^{\frac{3}{2}} {}_2F_1(-k_+ + \frac{3}{2}; -k_- + \frac{3}{2}; \frac{5}{2}; z)$$

$$z \sim R + \beta = \tanh^2 \left(\frac{\sqrt{C_0 |C_1|}}{2} t - |t_0| \right) < 1$$

$$H \sim \tanh \left(\frac{\sqrt{C_0 |C_1|}}{2} t - |t_0| \right), \quad a(t) \sim \left(\cosh \left(\frac{C_0}{2} \sqrt{\left| \frac{C_1}{C_0} \right|} t + t_0 \right) \right)^{-\frac{2}{C_0}}$$

$$R(t) \sim 3 \left(b + \tanh^2 \left(\frac{\sqrt{C_0 |C_1|}}{2} t - |t_0| \right) \right)$$

$$H = -\sqrt{\left|\frac{C_1}{C_0}\right|} \tanh\left(\frac{\sqrt{C_0|C_1|}}{2} t - |t_0|\right) \Rightarrow \frac{\sqrt{C_0|C_1|}}{2} t - |t_0| < 0$$

$$\frac{\ddot{H}}{H\dot{H}} \ll 1 \Rightarrow 0 < C_0 \ll 1$$

Slow-Roll Parameters:

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \ll 1 \Rightarrow \frac{C_0}{2 \sinh^2\left(\frac{\sqrt{C_0|C_1|}}{2} t_i - |t_0|\right)} \ll 1$$

$$\epsilon_3 = \frac{\dot{F}_R}{2HF_R} \ll 1 \Rightarrow -\frac{C_0}{2 \cosh^2\left(\frac{\sqrt{C_0|C_{00}|}}{2} t - |t_0|\right)} \frac{F_{zzz}}{F_z} \ll 1$$

$$\epsilon_4 = \frac{\ddot{F}_R}{HF_R} \ll 1 \Rightarrow$$

$$-\frac{C_0}{2 \sinh^2\left(\frac{\sqrt{C_0|C_{00}|}}{2} t - |t_0|\right)} + \frac{C_0}{2} - \frac{C_0}{\cosh^2\left(\frac{\sqrt{C_0|C_{00}|}}{2} t - |t_0|\right)} \frac{F_{zzz}}{F_z} \ll 1$$

Metric interval:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -4f(\zeta, \eta) d\zeta d\eta - g_{ab} dx^a dx^b$$

$$g_{ab} = g_{ab}(\zeta, \eta), \quad a, b = 1, 2; \quad \text{signature } (+ - - -);$$

Parameterization g_{ab} :

$$g_{ab} = \begin{pmatrix} \psi & \\ \psi \omega & \psi \omega^2 + \sigma_\alpha^2 \alpha^2 \psi^{-1} \end{pmatrix}, \quad \sigma_\alpha^2 = \pm 1$$

$F(R)$ gravity. Case of dependence on two variable

Vacuum field equations (6 equations):

$$\left(\alpha F_R \left(\ln \frac{\psi}{\alpha}\right)_{,\zeta}\right)_{,\eta} + \left(\alpha F_R \left(\ln \frac{\psi}{\alpha}\right)_{,\eta}\right)_{,\zeta} - \frac{2F_R \psi^2}{\alpha} \omega_{,\zeta} \omega_{,\eta} = 0$$

$$\left(\frac{F_R \psi^2}{\alpha} \omega_{,\zeta}\right)_{,\eta} + \left(\frac{F_R \psi^2}{\alpha} \omega_{,\eta}\right)_{,\zeta} = 0$$

$$(\alpha F_R)_{,\zeta\eta} - \alpha f(F_R R - F) = 0$$

$$\alpha F_R f R + \alpha F_{R,\zeta\eta} - 2F_R \alpha_{,\zeta\eta} - \frac{1}{2}(\alpha_{,\eta} F_{R,\zeta} + \alpha_{,\zeta} F_{R,\eta}) = 0$$

$$(\ln f)_{,\zeta} =$$

$$= \frac{1}{\ln(\alpha F_R)_{,\zeta}} \left(\ln(\alpha F_R)_{,\zeta\zeta} + \frac{\psi^2}{2\alpha^2} (\omega_{,\zeta})^2 + \frac{1}{2} \left(\left(\ln \frac{\psi}{\alpha} \right)_{,\zeta} \right)^2 + \frac{1}{2} \left((\ln \alpha)_{,\zeta} \right)^2 + \left((\ln F_R)_{,\zeta} \right)^2 \right)$$

and a similar equation for $(\ln f)_{,\eta}$

Dependence on two variables. Factorization

Metric interval is diagonal: $g_{\mu\nu} = \text{diag}(f, -f, -|\alpha|, -|\alpha|)$

A. Let's consider

$$\alpha = \alpha_1(\zeta)\alpha_2(\eta)$$

Then

$$F(R) \sim |R|^n, \quad \frac{1 - \sqrt{3}}{2} \leq n \leq \frac{1 + \sqrt{3}}{2}$$

The scalar curvature R and the metric function are:

$$R \sim \alpha_1^{k_1(n)} \alpha_2^{k_2(n)} \quad f \sim (R^{-1})_{,\zeta\eta}$$

Example: in coordinates

$$\alpha_1(\zeta) = \zeta = \frac{1}{2}(t + z), \quad \alpha_2(\eta) = \eta = -\frac{1}{2}(t - z)$$

The interval is:

$$ds^2 = dt^2 - dz^2 - (t^2 - z^2)(dx^2 + dy^2)$$

The scalar curvature is:

$$R \sim -\frac{1}{t^2 - z^2}$$

Dependence on two variables. $F(R) \sim \frac{1}{R}$, general solution

Metric interval is diagonal: $g_{\mu\nu} = \text{diag}(f, -f, -|\alpha|, -|\alpha|)$

B. Let's consider

$$F(R) \sim \frac{1}{R}$$

Ansatz:

$$\alpha(\zeta, \eta) \sim \frac{1}{R^2}, \quad R = R(\zeta, \eta)$$

The solutions for the metric components α , f and for the scalar curvature R are expressed in terms of two arbitrary functions $\varphi_1(\zeta)$ and $\varphi_2(\eta) \neq 0$:

$$\begin{aligned}\alpha &\sim \left(\int \varphi_1(\zeta) d\zeta + \int \varphi_2(\eta) d\eta \right)^{-4} \\ f &\sim \varphi_1(\zeta) \varphi_2(\eta) \left(\int \varphi_1(\zeta) d\zeta + \int \varphi_2(\eta) d\eta \right)^{-4} \\ R &\sim \left(\int \varphi_1(\zeta) d\zeta + \int \varphi_2(\eta) d\eta \right)^2\end{aligned}$$

If $\varphi_1(\zeta) d\zeta \sim d\Phi_1(\zeta)$ and $\varphi_2(\eta) d\eta \sim d\Phi_2(\eta)$ choosing coordinates

$$\Phi_1 \sim (t + z), \quad \Phi_2 \sim -(t - z)$$

we obtain a conformally flat metric. With $\varphi_1(\zeta)$ and $\varphi_2(\eta)$ are constants we obtain a solution in the travelling wave variable

Let us return to:

Metric interval: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -4f(\zeta, \eta) d\zeta d\eta - g_{ab} dx^a dx^b$

Parameterization: g_{ab} :
$$g_{ab} = \begin{pmatrix} \psi & \\ \psi \omega & \psi \omega^2 + \sigma_\alpha^2 \alpha^2 \psi^{-1} \end{pmatrix}, \quad \det g_{ab} = \pm \alpha^2$$

Let us reduce the initial system of field equations to a system of ODEs by moving to the "travelling wave" variable:

$$y(\zeta, \eta) = \zeta + \lambda\eta = \frac{1}{2}((\lambda + 1)z - (\lambda - 1)t)$$

$$y(\zeta, \eta) = \frac{1}{\sqrt{\lambda}} (\zeta + \lambda\eta) = \frac{1}{2\sqrt{\lambda}} ((\lambda + 1)z - i(\lambda - 1)\rho)$$

Dependence on two variables. "Travelling wave"variable

Writing the field equations as a system of ODEs in terms of the "travelling wave"variable we obtain that all functions of the theory are expressed through a **single function** $\chi(y)$ (and its derivatives):

$$\frac{\psi}{\alpha} \sim \cosh \chi(y)$$

$$\omega \sim \omega_0 + \tanh \chi(y), \quad F_R \sim \frac{1}{\alpha \chi_{,y}}$$

The function $(\ln \alpha)_{,y}$ satisfies the Abel equation of the 2 kind:

$$(\ln \alpha)_{,y} \left((\ln \alpha)_{,y} + \frac{\chi_{,yy}}{\chi_{,y}} \right) = \frac{\chi_{,yyy}}{2\chi_{,yy}} ((\ln \alpha)_{,y})^2 + \left(\frac{\chi_{,yy}}{\chi_{,y}} \right)^2 (\ln \alpha)_{,y} +$$

$$\frac{1}{3} \left(\frac{\chi_{,yyyy}}{\chi_{,y}} - \frac{3\chi_{,yyy}\chi_{,yy}}{(\chi_{,y})^2} + \frac{4(\chi_{,yy})^3}{(\chi_{,y})^3} - \frac{(\chi_{,yyy})^2}{\chi_{,y}\chi_{,yy}} + \frac{\chi_{,yyy}(\chi_{,y})^2}{2\chi_{,yy}} - \chi_{,yy}\chi_{,y} \right)$$

which can be solved exactly

Dependence on two variables. Ansatz $\chi_{,y} = \chi_{,y}(R)$

Solution of Abel's equation:

$$(\ln(\alpha\chi_{,y}))_{,y} = \frac{1}{\sqrt{3}} \left[\frac{2\chi_{,yyy}}{\chi_{,y}} - \left(\frac{\chi_{,yy}}{\chi_{,y}} \right)^2 - (\chi_{,y})^2 \right]^{\frac{1}{2}}$$

Metric coefficient f :

$$f \sim \alpha$$

By choosing a different form of the function $\chi(y)$ all variables of the model can be exactly expressed as functions:

$$g_{ab} = g_{ab}(y), \quad f = f(y), \quad R = R(y), \quad F = F(y)$$

Our goal is to obtain $F = F(R)$

Ansatz:

$$\chi_{,y} = \chi_{,y}(R)$$

Dependence on two variables. Resulting equations

Combination of equations leads to:

$$\begin{aligned} \frac{\chi, y}{((\chi, y), R)^2} e^{3 \int \frac{\chi, y}{(\chi, y), R} \left(\frac{F_{RR}}{F_R}\right)^2 dR} \left(\int (\chi, y)^2 (\chi, y), R e^{-3 \int \frac{\chi, y}{(\chi, y), R} \left(\frac{F_{RR}}{F_R}\right)^2 dR} dR + p_0 \right) = \\ = \frac{1}{N^2} \left(\frac{2\varphi_0 f_0}{\lambda} \int \frac{RN dR}{F_R (\chi, y)^2} + N_0 \right) \end{aligned}$$

where
$$N(R) = -\frac{2(\chi, y)_R}{(\chi, y)^2} - \frac{3(\ln F_R)_R}{\chi, y}$$

Choosing a dependency $F_R = F_R(\chi, y)$ (or $\chi, y = \chi, y(F_R)$) we obtain

$$F_R = F_R(R) \text{ and restore } F(R) = \int F_R dR + F_0$$

Coordinate dependency $y(R)$:

$$y + y_0 =$$

$$\pm \int \frac{dR}{\sqrt{\frac{\chi, y}{((\chi, y), R)^2} e^{3 \int \frac{\chi, y}{(\chi, y), R} \left(\frac{F_{RR}}{F_R}\right)^2 dR} \left(\int (\chi, y)^2 (\chi, y), R e^{-3 \int \frac{\chi, y}{(\chi, y), R} \left(\frac{F_{RR}}{F_R}\right)^2 dR} dR + p_0 \right)}}$$

Dependence on two variables. Example

Consider $\chi, y = \chi, y(F_R)$ as:

$$\chi, y \sim (F_R)^k$$

Then

$$F(R) \sim R^{\frac{2k+3}{k+3}}$$

$$y + y_0 \sim R^{\frac{k^2-3}{2(k+3)}} {}_2F_1\left(\frac{1}{2}, \frac{k^2-3}{6k^2-6}; \frac{7k^2-9}{6k^2-6}; C_0 R^{\frac{3k^2-3}{k+3}}\right)$$

$$\chi(R) \sim \sinh^{-1}\left(C_0 R^{\frac{3k^2-3}{2(k+3)}}\right) + \chi_0$$

and all metric functions will depend on R

Dependence on two variables. Example

For some values of k one can express $R = R(y)$ explicitly

Let

$$F(R) \sim R^n, \quad n = \frac{1 \mp \sqrt{3}}{2}$$

Then the metric interval:

$$\begin{aligned} ds^2 &= -4f d\zeta d\eta - \frac{\omega_0}{\Psi_0} w^{\frac{2n-1}{3(n-1)}} \cosh(\Upsilon) (dx^1)^2 - \\ &- 2w^{\frac{2n-1}{3(n-1)}} \cosh(\Upsilon) \left(\tanh(\Upsilon) + \frac{\tilde{\omega}_0 \omega_0}{\Psi_0} \right) dx^1 dx^2 \\ &- \frac{\Psi_0}{\omega_0} w^{\frac{2n-1}{3(n-1)}} \left[\cosh(\Upsilon) \left(\tanh(\Upsilon) + \frac{\tilde{\omega}_0 \omega_0}{\Psi_0} \right)^2 + \sigma_\alpha \cosh^{-1}(\Upsilon) \right] (dx^2)^2 \end{aligned}$$

and

$$\begin{aligned} f &\sim \sinh^{1 \mp \frac{1}{\sqrt{3}}}(\sqrt{C_0 y} + y_0), \quad w \sim R^{\frac{3(n-1)^2}{n-2}} \\ R &\sim \sinh^{-1 \pm \frac{1}{\sqrt{3}}}(\sqrt{C_0 y} + y_0), \quad \Upsilon \sim \sqrt{2} \ln \left| \tanh \frac{\sqrt{C_0 y} + y_0}{2} \right| + \tilde{\varphi}_0 \end{aligned}$$



M. V. Shubina, Exact analytical vacuum solutions of R^n -gravity model depending on two variables, *Annals of Physics*, 451, 169245 (2023) DOI: 10.1016/j.aop.2023.169245



M. V. Shubina, Some exact solutions of Friedmann cosmological equation, *Annals of Physics*, 462, 169613 (2024) DOI: 10.1016/j.aop.2024.169613



M. V. Shubina, General vacuum solution of modified gravity with Gauss-Bonnet term, *Gravitation and Cosmology*, 30, № 4, 455 (2024) DOI: 10.1134/S020228932470035X



M. V. Shubina, Scheme of integration of two-dimensional vacuum $F(R)$ gravity in a travelling wave variable, arXiv:2505.12315

THANK YOU FOR YOUR ATTENTION