Exact solutions for electromagnetic field in the Schwarzschild metric

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Introduction

1. There are several reasons why the problem of field quantization in space with black holes is discussed in the literature. Primordial black holes are candidates for the role of dark matter, and due to Hawking radiation, the black holes with an initial mass of the order of 10^{15} g should evaporate completely in the present epoch. The final stage of evaporation should be accompanied by a gamma-ray burst, the spectrum of which should apparently differ from the thermal one. There are also methodological issues related to the existence or non-existence of the interior regions of black holes, the admissibility of the existence of white holes, etc.

2. A mathematically correct procedure for canonical quantization of a field assumes knowledge of the solutions of the equations of motion of the field under consideration in the black hole metric. From these solutions, a complete system of eigenfunctions is constructed, in which an arbitrary field configuration is expanded. The properties of these solutions, in particular the energy spectrum, are important for describing the interaction of different fields, as well as for obtaining radiation spectra.

3. In this talk we derive the equation of motion for the vector field in the Schwarzschild black hole metric. We will show that the dependence on the angular coordinates θ and φ can be expressed through the well-known spherical vectors, and the dependence on the radial coordinate is described by the confluent Heun functions.

Equations of motion of a vector field in the Schwarzschild metric

Schwarzschild metric (all quantities are normalized to $r_0 \equiv 2M$)

$$ds^2 = f_S dt^2 - \frac{1}{f_S} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \qquad f_S \equiv 1 - \frac{1}{r}$$

Equation of motion for a massless vector field

$$\nabla^\mu F_{\mu\alpha} = \nabla^\mu \nabla_\mu A_\alpha - \nabla^\mu \nabla_\alpha A_\mu = \nabla^\mu \nabla_\mu A_\alpha - \nabla_\alpha \nabla^\mu A_\mu - R^\mu{}_\alpha V_\mu = \nabla^\mu \nabla_\mu A_\alpha - \partial_\alpha \nabla^\mu A_\mu = 0$$

The zero component of this equation is

$$\left[g^{\mu\nu}\partial_\mu\partial_\nu + \frac{3-2r}{r^2}\partial_1 - \frac{1}{r^2}\mathrm{ctg}\,\theta\partial_2 - \frac{1}{2r^3(r-1)}\right]A_0 - \partial_0\left(\frac{1}{r^2}A_1 + \nabla_\mu A^\mu\right) = 0$$

The tensor $F_{\alpha\beta}$ is invariant under gauge transformations of the vector potential A_α

$$A_\alpha(x) \to A'_\alpha(x) = A_\alpha(x) - \partial_\alpha \zeta(x)$$

We can make the component A'_0 equal to zero if we choose a function $\zeta(x)$ of the form

$$\zeta(t,r,\theta,\varphi) = \int_{t_0}^{t} A_0(t',r,\theta,\varphi)\,dt' + \xi(r,\theta,\varphi)$$

where t_0 is an arbitrary constant, and ξ is an arbitrary function that does not depend on time.

Gauge conditions

The zeroth component of the equation now takes the form

$$-\partial_0\left(\frac{1}{r^2}A_1' + \nabla_\mu A'^\mu\right) = 0 \qquad \Rightarrow \qquad \frac{1}{r^2}A_1' + \nabla_\mu A'^\mu = C(r,\theta,\varphi)$$

where $C(r,\theta,\varphi)$ is an arbitrary function of spatial coordinates. By choosing the function ξ appropriately, we can set C to zero . The corresponding function ξ must satisfy the equation

$$\frac{1}{r^2}\partial_r\xi + \Delta\xi = C(r,\theta,\varphi)$$

One can prove that such a function can always be found, and there exists a gauge, in which the following conditions are satisfied

$$A_0(x) = 0 \qquad\qquad \frac{1}{r^2}A_1 + \nabla_\mu A^\mu = 0$$

Equations of motion in the chosen gauge

Taking into account the gauge conditions, let us rewrite the equations in the following form:

$$\frac{r-1}{r}\partial_r\left(r^2 A_1\right)+\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta A_2\right)+\frac{1}{\sin^2\theta}\partial_\varphi A_3=0$$

$$\frac{r^3}{r-1}\partial_t^2 A_1+\frac{1}{\sin\theta}\partial_\theta\left[\sin\theta\left(\partial_\theta A_1-\partial_r A_2\right)\right]+\frac{1}{\sin^2\theta}\partial_\varphi\left(\partial_\varphi A_1-\partial_r A_3\right)=0$$

$$\frac{r^3}{r-1}\partial_t^2 A_2-r^2\partial_r\left[\frac{r-1}{r}\left(\partial_r A_2-\partial_\theta A_1\right)\right]-\frac{1}{\sin^2\theta}\partial_\varphi\left(\partial_\varphi A_2-\partial_\theta A_3\right)=0$$

$$\frac{r^3}{r-1}\partial_t^2 A_3-r^2\partial_r\left[\frac{r-1}{r}\left(\partial_r A_3-\partial_\varphi A_1\right)\right]-\sin\theta\partial_\theta\left[\frac{1}{\sin\theta}\left(\partial_\theta A_3-\partial_\varphi A_2\right)\right]=0$$

Next we expand the vector potential A_q in the spherical vectors $Y_q^{(\lambda)}$ as follows:

$$A_q=e^{-i\omega t}\sum_{\lambda=-1}^{1}F_{(\lambda)}\left(r\right)Y_q^{(\lambda)}\left(\theta,\varphi\right)$$

Spherical vectors

Spherical vectors can be expressed in terms of spherical harmonics as follows

$$Y_q^{(1)}(j,m;\theta,\varphi)=\frac{1}{\sqrt{j(j+1)}}\begin{pmatrix}0 & \partial_\theta & \partial_\varphi\end{pmatrix}Y(j,m;\theta,\varphi)$$

$$Y_q^{(0)}(j,m;\theta,\varphi)=\frac{i}{\sqrt{j(j+1)}}\begin{pmatrix}0 & -\frac{1}{\sin\theta}\partial_\varphi & \sin\theta\partial_\theta\end{pmatrix}Y(j,m;\theta,\varphi)$$

$$Y_q^{(-1)}(j,m;\theta,\varphi)=\begin{pmatrix}1 & 0 & 0\end{pmatrix}Y(j,m;\theta,\varphi)$$

Substituting these expressions into the equations of motion and using the fact that $Y_q^{(\lambda)}$ are eigenfunctions of the angular part of the d'Alembertian, which coincides with the same part of the d'Alembertian in flat space

$$\Delta_\Omega Y_q^{(\lambda)}(j,m;\theta,\varphi)=-j(j+1)Y_q^{(\lambda)}(j,m;\theta,\varphi)$$

we obtain the equations for the radial functions.

Equations for radial functions

$$\frac{r-1}{r}\partial_r\left(r^2F_{(-1)}\right)-\sqrt{j\left(j+1\right)}F_{(1)}=0$$

$$\left[\frac{\omega^2 r}{r-1}-\frac{j\left(j-1\right)}{r^2}\right]F_{(-1)}-\frac{\sqrt{j\left(j+1\right)}}{r^2}\partial_r F_{(1)}=0$$

$$\frac{1}{\sqrt{j\left(j+1\right)}}\left[\frac{\omega^2 r}{r-1}+\partial_r\frac{r-1}{r}\partial_r\right]F_{(1)}-\partial_r\frac{r-1}{r}F_{(-1)}=0$$

$$\left[\partial_r\frac{r-1}{r}\partial_r+\frac{\omega^2 r}{r-1}-\frac{j\left(j+1\right)}{r^2}\right]F_{(0)}=0$$

Only two of the first three equations are independent. In addition, in one of them, the unknown function $F_{(1)}$ can be eliminated, which gives an equation similar to the fourth one

$$\left[\partial_r\frac{r-1}{r}\partial_r+\frac{\omega^2 r}{r-1}-\frac{j\left(j-1\right)}{r^2}\right]\left(r^2F_{(-1)}\right)=0$$

Reduction to the form of the confluent Heun equation

Let us consider the equation

$$\left[\partial_r \frac{r-1}{r}\partial_r + \frac{\omega^2 r}{r-1} - \frac{j(j-1)}{r^2}\right]P = 0$$

Using the substitution

$$P(r) = (r-1)^{i\omega} e^{i\omega r} G(r)$$

we can bring it to the form of the confluent Heun equation

$$\left\{r(r-1)\partial_r^2 + \left[-(r-1) + (1+i2\omega)r + i2\omega r(r-1)\right]\partial_r - \left(2\omega^2 + j(j+1) + i\omega\right)\right\}G = 0$$

One of the solutions of this equation has the form

$$HeunC\left(2\omega^2 + j(j+1) + i\omega, 0, 1+i2\omega, -1, i2\omega, 1-r\right)$$

Formally, the second solution is expressed as $e^{-i2\omega r} HeunC(\ldots)$. However, it is more convenient to do it in a different way. For the first solution, we get $P(r)$ in the form

$$P(r) = e^{i\omega r}(r-1)^{i\omega} HeunC\left(2\omega^2 + j(j+1) + i\omega, 0, 1+i2\omega, -1, i2\omega, 1-r\right)$$

And then we take the complex-conjugate function P^* as the second solution.

Radial functions

Thus, we obtained solutions for the radial functions in the following form:

$$F_{(0)}(r) = C_{(0),+} e^{i\omega r} (r-1)^{i\omega} HeunC\left(2\omega^2 + j(j+1) + i\omega, 0, 1+i2\omega, -1, i2\omega, 1-r\right) +$$
$$+ C_{(0),-} e^{-i\omega r} (r-1)^{-i\omega} HeunC\left(2\omega^2 + j(j+1) - i\omega, 0, 1-i2\omega, -1, -i2\omega, 1-r\right)$$

$$F_{(1)}(r) = C_{(1),+} \frac{r-1}{r} \partial_r \left[e^{i\omega r} (r-1)^{i\omega} HeunC\left(2\omega^2 + j(j+1) + i\omega, 0, 1+i2\omega, -1, i2\omega, 1-r\right)\right] +$$
$$+ C_{(1),-} \frac{r-1}{r} \partial_r \left[e^{-i\omega r} (r-1)^{-i\omega} HeunC\left(2\omega^2 + j(j+1) - i\omega, 0, 1-i2\omega, -1, -i2\omega, 1-r\right)\right]$$

$$F_{(-1)}(r) = \sqrt{j(j+1)}\, C_{(1),+} e^{i\omega r} (r-1)^{i\omega} HeunC\left(2\omega^2 + j(j+1) + i\omega, 0, 1+i2\omega, -1, i2\omega, 1-r\right) +$$
$$+ \sqrt{j(j+1)}\, C_{(1),-} e^{-i\omega r} (r-1)^{-i\omega} HeunC\left(2\omega^2 + j(j+1) - i\omega, 0, 1-i2\omega, -1, -i2\omega, 1-r\right)$$

Complex constants $C_{(0),\pm}$ and $C_{(1),\pm}$ are determined from the orthonormality conditions

Asymptotic behavior of radial functions

Near the horizon, radial functions oscillate infinitely rapidly with a finite amplitude.

$$F_{(0)}(r) = C_{(0),+}(r-1)^{i\omega} + C_{(0),-}(r-1)^{-i\omega}$$

$$F_{(1)}(r) = i\omega C_{(1),+}(r-1)^{i\omega} - i\omega C_{(1),-}(r-1)^{-i\omega}$$

$$F_{(-1)}(r) = \sqrt{j(j+1)}C_{(1),+}(r-1)^{i\omega} + \sqrt{j(j+1)}C_{(1),-}(r-1)^{-i\omega}$$

In the infinitely distant zone, solutions oscillate with decreasing amplitude.

$$F_{(0)}(r) = B_{(0),+}e^{i\omega r}(r-1)^{i\omega-1} + B_{(0),-}e^{-i\omega r}(r-1)^{-i\omega-1}$$

$$F_{(1)}(r) = i\omega\left[B_{(1),+}e^{i\omega r}(r-1)^{i\omega-1} - B_{(1),-}e^{-i\omega r}(r-1)^{-i\omega-1}\right]$$

$$F_{(-1)}(r) = \sqrt{j(j+1)}\left[B_{(1),+}e^{i\omega r}(r-1)^{i\omega-1} + B_{(1),-}e^{-i\omega r}(r-1)^{-i\omega-1}\right]$$

As we see, all solutions are regular at the horizon and at the infinitely distant point for all energy values. First, this means that the energy spectrum is continuous. Their asymptotic behavior allows us to introduce normalization to the delta function. In addition, it should be noted that for each polarization there are two independent solutions, which implies the presence of a twofold degeneracy of quantum states.

Solution for the vector potential

$$A_q(\omega, j, m, 0; x) =$$

$$= \sum_{l=1}^{2} c(\omega, j, m, 0, l) \frac{i}{\sqrt{j(j+1)}} e^{-i\omega t} F_{(0)l}(r) \begin{pmatrix} 0 & -\frac{1}{\sin\theta}\partial_\varphi & \sin\theta\partial_\theta \end{pmatrix} Y(j, m; \theta, \varphi)$$

$$A_q(\omega, j, m, 1; x) =$$

$$= \sum_{l=1}^{2} c(\omega, j, m, 1, l) e^{-i\omega t} \begin{Bmatrix} F_{(-1)l}(r) \\ \frac{1}{j(j+1)} \frac{r-1}{r} \partial_r \left(r^2 F_{(-1)l}\right) \partial_\theta \\ \frac{1}{j(j+1)} \frac{r-1}{r} \partial_r \left(r^2 F_{(-1)l}\right) \partial_\varphi \end{Bmatrix} Y(j, m; \theta, \varphi)$$

Normalization coefficients

Using the orthogonality property of spherical vectors, as well as the asymptotic behavior of radial functions on the horizon and at the point at infinity, we can assert that for the scalar product

$$\left(A\cdot A'\right)\equiv\int_1^\infty\frac{dr}{r-1}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi g^{\mu\nu}\left\{A_\mu^*\dot{A}'_\nu-\dot{A}_\mu^*\dot{A}_\nu\right\}$$

we get

$$\left(A(\omega,j,m,\lambda)\cdot A(\omega',j',m',\lambda')\right)=$$

$$=\int_1^\infty\frac{dr}{r-1}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi\left\{A_\mu^*\left(\omega,j,m,\lambda;r,\theta,\varphi\right)\dot{A}_\nu\left(\omega',j',m',\lambda';r,\theta,\varphi\right)=\right.$$

$$\left.-\dot{A}_\mu^*\left(\omega,j,m,\lambda;r,\theta,\varphi\right)A'_\nu\left(\omega',j',m',\lambda';r,\theta,\varphi\right)\right\}=\delta\left(\omega-\omega'\right)\delta_{ll'}\delta_{mm'}\delta_{\lambda\lambda'}\frac{c\left(\omega,j,m,\lambda,l\right)^2}{2j\left(j+1\right)}$$

Therefore, to obtain normalization $\frac{1}{2\omega}\delta\left(\omega-\omega'\right)\delta_{ll'}\delta_{mm'}\delta_{\lambda\lambda'}$ we have to set

$$c\left(\omega,j,m,\lambda,l\right)=\sqrt{\frac{j\left(j+1\right)}{\omega}}$$

Conclusion

We have obtained exact solutions for electromagnetic field in the Schwarzschild metric.

It is shown that there exists a gauge in which the temporal component of the vector potential is identically zero. It is also shown that there exists a gauge compatible with the previous property, which tends to the Poincaré gauge at the horizon and to the Coulomb gauge at infinity. In this gauge, it is possible to rewrite the system of equations in such a way that the resulting physical equations are reduced to confluent Heun equations.

The angular part of the solutions is expressed through spherical vectors.

There are two independent polarizations of the vector field. In the chosen gauge, one of the polarizations has no longitudinal part, the other one has but its form is uniquely expressed through the transverse one.

The radial functions can be expressed through confluent Heun functions, or through series by the Frobenius method (near the horizon) or the Frobenius-Thomé method at an infinitely distant point.

For each set of parameters (energy and momentum) there are two solutions, and both solutions are regular at the horizon and at the infinitely distant point. Therefore, two orthonormal combinations can be constructed from them. Thus, in the case of electromagnetic field, as well as in the case of a scalar field in the Schwarzschild metric, there is a twofold degeneracy of quantum states.

Thank you for your attention!

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