Exact solutions for spinor field in the Schwarzschild metric

Sergey Keyzerov, Edward Rakhmetov, Igor Volobuev

НИИЯФ
МГУ

Introduction (historical part)

Despite the fact that quantum field theory in curved spacetime has been developing since the end of the 1960s [1, 2], the quantization of fields in the vicinity of black holes has not been sufficiently studied to perform correctly the canonical quantization procedure at all stages even for a real scalar field (see, for example, the discussion and cited literature in [3]).

Perhaps this is the reason why discussions concerning the spectrum of physical states of scalar (and the other) fields in the gravitational field of black holes still arise from time to time. For example, now and again it is asserted that at the energies lower than the field mass, the spectrum of physical states is discrete (see the discussion and cited literature in [4]), even if each state has an infinite norm. Moreover, until now the question remains if such physical states can be described in terms of known special functions and what is their normalization?

[1] D. G. Boulware, Phys. Rev. D 11, 1404 (1975).
[2] J. B. Hartle and S. W. Hawking, Phys. Rev. D 13, 2188 (1976).
[3] V. Egorov, M. Smolyakov, I. Volobuev, Phys. Rev. D 107, 025001 (2023); arXiv:2209.02067 [gr-qc].
[4] A. Zecca, Nuovo Cim. B 124, 1251 (2009).

Introduction (scalar and vector fields)

Until the beginning of the 21st century, classical scientific papers and monographs claimed that for the physical fields in the vicinity of black holes the solutions of the equations of motion are not expressed through known special functions.

It was Antonio Zecca [4], who first pointed out that in the simplest case of a real scalar field in the Schwarzschild spacetime, the radial part of such solutions can be described in terms of so called confluent Heun functions, which are solutions of the confluent Heun equation.

In the last decade, many papers have been presented in which solutions to the equations of motion for *scalar* [5] and *vector* [6] fields in the vicinity of black holes were expressed in one way or another through confluent Heun functions. The properties of these solutions have been well studied, in particular: the asymptotic behavior [7], completeness, physical (bounded) states and their spectrum [7,8].

[4] A. Zecca, Nuovo Cim. B 124, 1251 (2009).

[5] H.S. Vieira, V.B. Bezerra, Annals Phys. 373, 28 (2016); arXiv:1603.02233 [gr-qc].

[6] D. Philipp, V. Perlick, arXiv:1503.0810 (2015).

[7] Egorov V., Smolyakov., Volobuev I., Phys. Rev. D 107, № 2 (2023).

[8] Keizerov S., Rakhmetov E., Smolyakov M., Volobuev I., Physics of Particles and Nuclei, 56 (2), (2025).

Introduction (spinor field problems)

The situation with the spinor field looks completely different. Apparently, in the general case, the radial part of the Dirac equation in the Schwarzschild spacetime does not reduce [9, 12] to the Heun equation. For this reason, the properties of solutions of the Dirac equation in the vicinity of black holes are worse understood than the properties of solutions of the Klein-Gordon equation.

For example, there are still discussions about the incompleteness of fermion modes not only in the Schwarzschild spacetime, but even in the simplest case of «toy black hole» in the four-dimensional Rindler spacetime [10].

Discussions concerning the fermion physical states and their spectrum in the Schwarzschild [11] and Kerr-Newman [12] spacetime still arise from time to time also.

[9] D. N. Page, «Dirac equation around a charged, rotating black hole», Phys. Rev. D 14, 1509,(1976).

[10] A. M. Fedotov, N. B. Narozhny, V. D. Mur, E. G. Gelfer, «On the incompleteness of the Unruh fermion modes in the Minkowski space», JETP Letters, V. 89, P. 385–389, (2009).

[11] M.A. Vronsky, M.V. Gorbatenko, N.S. Kolesnikov, V.P. Neznamov, E.Yu. Popov, «Stationary Bound States of Dirac Particles in the Schwarzschild Gravitational Field», arXiv:1301.7595 [gr-qc], (2013).

[12] D. Batic, M. Sandoval, «The hypergeneralized Heun equation in QFT in curved space-times», arXiv:0805.4399 [gr-qc], (2008).

Problem statement

The well-known Schwarzschild metric: $ds^2 = f_S dt^2 - f_S^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$, where $f_S \equiv 1 - r_0 / r$, and $r_0 = 2GM$ is the Schwarzschild radius. Here and further all quantities are made dimensionless using the Schwarzschild radius, then we have $f_S \equiv 1 - r^{-1}$.

Let Ψ be a spinor field with the mass *m* in four-dimensional curved spacetime and $\mu \equiv r_0 m$. Then the Dirac equation can be written in the following form

$$\left[i\hat{\gamma}^{\mu}\hat{\nabla}_{\mu} - \mu \right]\psi = 0,$$

where $\hat{\nabla}_{\mu}$ is covariant derivative operator for spinors:

$$\hat{\nabla}_{\alpha}\psi = (\partial_{\alpha} + 1/2 \cdot C_{\alpha(ab)}\hat{G}^{(ab)})\psi,$$

where $\hat{G}^{(ab)} \equiv \frac{1}{4}\hat{\gamma}^{[(a)}\hat{\gamma}^{(b)]}$ are the generators of the Lorentz group in the spinor representation, $C_{\mu(ab)}$ is the spin connection. Let $g_{\mu}^{(a)}$ be the local Lorentz frame fields or tetrad (also known as a vierbein), which is a set of orthonormal space time vector fields. Then the spin connection take the form

$$C_{\mu(ab)} \equiv g_{(a)}{}^{\nu}\nabla_{\mu}g_{(b)\nu}.$$

Here and further, Latin letters in round brackets denote the local Lorentz frame indices; Greek letters denote general coordinate indices.

Problem statement

Let's choose the tetrad in the diagonal form: $g_{\mu}^{(a)} = \sqrt{|g_{\mu\mu}|}\delta_{\mu}^{(a)}$.

Then, after calculating the Christoffel symbols and spin connections (see Appendix A), the Dirac equation takes the following form:

$$\left\{\frac{i}{\sqrt{f_S}}\hat{\gamma}^{(0)}\partial_0 + i\hat{\gamma}^{(1)}\sqrt{f_S}\left(\partial_r + \frac{1}{r} + \frac{f_S'}{4f_S}\right) + \frac{i}{r}\left[\hat{\gamma}^{(2)}\left(\partial_\theta + \frac{\operatorname{ctg}\theta}{2}\right) + \hat{\gamma}^{(3)}\frac{1}{\sin\theta}\partial_\varphi\right] - \mu\right\}\psi = 0.$$

Instead of the diagonal tetrad $g_{\mu}^{(a)}$ the coordinate reference frame of isotropic coordinates [13] is often used (see Appendix B). One can go to such a frame using a unitary transformation of the following form: $\psi = \hat{U}(\theta,\varphi)\psi'$, where

$$\hat{U} = \frac{1}{2}\sqrt{1+\sin\theta\cos\varphi+\cos\theta\sin\varphi}\,\hat{\mathrm{I}} + \frac{1}{2\sqrt{1+\sin\theta\cos\varphi+\cos\theta\sin\varphi}}\left[\begin{array}{l}(\sin\theta+\cos\varphi)\hat{\gamma}^{(2)}\hat{\gamma}^{(3)} + \\ +(\cos\theta+\sin\varphi)\hat{\gamma}^{(3)}\hat{\gamma}^{(1)} + \\ +(\cos\theta\cos\varphi-\sin\theta\sin\varphi)\hat{\gamma}^{(1)}\hat{\gamma}^{(2)}\end{array}\right]$$

[13] V. Egorov, M. Smolyakov, I. Volobuev, «Quantization of spinor field in the Schwarzschild spacetime and spin sums for solutions of the Dirac equation», Class. Quantum Grav. 41, 045002 (2024).

Problem statement

Then, the Dirac equation takes the following form:

$$\hat{U}\left\{\frac{i}{\sqrt{f_S}}\hat{\gamma}^{(0)}\partial_0 + i\hat{\gamma}^{(q)}e^{r}_{(q)}\sqrt{f_S}\left(\partial_r + \frac{\sqrt{f_S}-1}{r\sqrt{f_S}} + \frac{f_S'}{4f_S}\right) + \frac{i}{r}\left[\hat{\gamma}^{(q)}e^{\theta}_{(q)}\partial_\theta + \hat{\gamma}^{(q)}e^{\varphi}_{(q)}\partial_\varphi\right] - \mu\right\}\psi' = 0.$$

Let us choose the following representation of gamma matrices:

$$\hat{\gamma}^{(0)} = \hat{\sigma}_3 \otimes \hat{I}, \quad \hat{\gamma}^{(q)} = i\hat{\sigma}_2 \otimes \hat{\sigma}^{(q)},$$

and let us write the ansatz of the solution in the following form:

$$\psi' = e^{-i\omega t}\hat{R}(r)\Omega(\theta,\varphi).$$

where

$$\hat{R}(r) \equiv \begin{pmatrix} A(r) & 0 \\ 0 & B(r) \end{pmatrix} \otimes \hat{I}, \qquad \Omega(\theta,\varphi) \equiv \begin{pmatrix} \Omega_{ljm} \\ \Omega_{l'jm} \end{pmatrix},$$

where Ω_{ljm} are the spherical spinors.

Problem statement

Substituting this ansatz into the Dirac equation, we have a system of two first-order differential equations:

$$\begin{cases} \partial_r A + \left(\dfrac{1}{r} + \dfrac{f_S'}{4 f_S}\right) A + \dfrac{\kappa}{r\sqrt{f_S}} A + i\left(\dfrac{\omega}{f_S} + \dfrac{\mu}{\sqrt{f_S}}\right) B = 0 \\ \partial_r B + \left(\dfrac{1}{r} + \dfrac{f_S'}{4 f_S}\right) B - \dfrac{\kappa}{r\sqrt{f_S}} B + i\left(\dfrac{\omega}{f_S} - \dfrac{\mu}{\sqrt{f_S}}\right) A = 0 \end{cases}$$

From this system it is easy to obtain second-order differential equations for functions $A(r)$ and $B(r)$. However, the coefficients of this equation turn out to be non-polynomial (contain radicals). As a rule, at this stage the study of the Dirac equation itself stops [3, 9], since equations with non-polynomial coefficients are studied relatively poorly and as a rule, their solutions are not expressed through any known special functions.

[9] D. N. Page, «Dirac equation around a charged, rotating black hole», Phys. Rev. D 14, 1509,(1976).

[13] V. Egorov, M. Smolyakov, I. Volobuev, «Quantization of spinor field in the Schwarzschild spacetime and spin sums for solutions of the Dirac equation», Class. Quantum Grav. 41, 045002 (2024).

Radial equation as a confluent equation

However, from the above system one can obtain second-order differential equations with polynomial coefficients. It's easy to show. Let us introduce new functions

$$F \equiv \frac{1}{2}\frac{\kappa + i\mu r}{r\sqrt{f_S}}\left(A+B\right), \quad Q \equiv \frac{1}{2}\left(A-B\right).$$

Then we have a system

$$\begin{cases} \partial_r F + \left(\dfrac{2}{r} + \dfrac{3 f_S'}{4 f_S} + \dfrac{i\omega}{f_S} - \dfrac{i\mu}{\kappa + i\mu r}\right) F + \dfrac{\kappa^2 + \mu^2 r^2}{r^2 f_S} Q = 0 \\ \partial_r Q + \left(\dfrac{1}{r} + \dfrac{f_S'}{4 f_S} - \dfrac{i\omega}{f_S}\right) Q + F = 0 \end{cases}$$

From here we obtain a second-order differential equation for the function *Q*(*r*):

$$\partial_r^2 Q + \left(\frac{3}{r} + \frac{f_S'}{f_S} - \frac{i\mu}{\kappa + i\mu r}\right)\partial_r Q + \left\{\left(\frac{1}{r} + \frac{f_S'}{4 f_S} - \frac{i\omega}{f_S}\right)' + \left(\frac{2}{r} + \frac{3 f_S'}{4 f_S} + \frac{i\omega}{f_S} - \frac{i\mu}{\kappa + i\mu r}\right)\left(\frac{1}{r} + \frac{f_S'}{4 f_S} - \frac{i\omega}{f_S}\right) - \frac{\kappa^2 + \mu^2 r^2}{r^2 f_S}\right\} Q = 0$$

Radial equation as a confluent equation

After the substitution of the form

$$Q(r) = r^{-1} f_S^{-1/4} e^{i\omega\int\frac{dr}{f_S}} \tilde{Q}(r)$$

we get some differential equation with polynomial coefficients

$$P_0(r)\partial_r^2\tilde{Q} + P_1(r)\partial_r\tilde{Q} + P_2(r)\tilde{Q} = 0,$$

where

$$P_0(r) \equiv r(r-1)(r - i\kappa/\mu),$$

$$P_1(r) \equiv \left(-\frac{1}{2} + r + i2\omega r^2\right)(r - i\kappa/\mu) - r(r-1),$$

$$P_2(r) \equiv -\left(\kappa^2 + \mu^2 r^2\right)(r - i\kappa/\mu).$$

As we can see, the equation has four singular points:

$$r_1 = 0, \quad r_2 = 1, \quad r_3 = i\kappa/\mu, \quad r_4 = \infty.$$

As will be shown below, the first three of them are Fuchsian singular points, and the last one is non-Fuchsian.

Radial equation as a confluent equation

At these singular points we have the following multiplicities of zeros for the relations P_1 / P_0 and P_2 / P_0

$$K_{P_0/P_1}\left(r_{1,2,3}\right)=1, \quad K_{P_0/P_1}\left(r_4\right)=2.$$

$$K_{P_0/P_2}\left(r_{1,2}\right)=1, \quad K_{P_0/P_2}\left(r_3\right)=-1, \quad K_{P_0/P_2}\left(r_4\right)=4.$$

So-called s-rank $R(r_j)$ of a singular point can be defined as follows:

$$R\left(r_j\right)\equiv \max\left\{K_{P_0/P_1}(r_j), \frac{1}{2}K_{P_0/P_2}(r_j)\right\}.$$

Then $R\left(r_{1,2,3}\right)=1, \quad R\left(r_4\right)=2.$

The set of the s-ranks of singular points of second-order differential equation with polynomial coefficients constitutes its s-multisymbol. Equations characterized by the same s-multisymbol belong to the same type of equations. The equation under consideration has an s-multisymbol of the form (1,1,1,2). Therefore, it can be classified as a confluent equation [14].

[14] S. Slavyanov, W. Lay, «Special Functions: Unified Theory based on Singularities», Oxford University Press, 2000.

Massless radial equation as a confluent Heun equation

If the mass of the spinor field is zero, then the equation under consideration degenerates into the confluent Heun equation:

$$r\left(r-1\right)\partial_r^2\tilde{Q}+\left[\frac{1}{2}\left(r-1\right)+\frac{1+i4\omega}{2}r+i2\omega r\left(r-1\right)\right]\partial_r\tilde{Q}-\kappa^2\tilde{Q}=0.$$

Because its s-multisymbol is (1,1,2).

Thus, its solutions are the well-known confluent Heun functions [14] :

$$\tilde{Q}_1\left(r\right)=HeunC\left(\kappa^2,0,\frac{1}{2},\frac{1}{2}+i2\omega,i2\omega;r\right),$$

$$\tilde{Q}_2\left(r\right)=\sqrt{r}\,HeunC\left(\kappa^2+\frac{1}{4},i\omega,\frac{3}{2},\frac{1}{2}+i2\omega,i2\omega;r\right).$$

[14] S. Slavyanov, W. Lay, «Special Functions: Unified Theory based on Singularities», Oxford University Press, 2000.

Massless radial equation as a confluent Heun equation

Finally, for the radial part of the massless Dirac equation we have the following solutions:

$$\hat{R}(r) \equiv \begin{pmatrix} A(r) & 0 \\ 0 & B(r) \end{pmatrix} \otimes \hat{\mathrm{I}},$$

where

$$A(r) = Ce^{i\omega r}(r-1)^{i\omega} \frac{1}{\sqrt{r}} \left\{ \frac{\sqrt[4]{r(r-1)}}{\kappa + i\mu r} \partial_r \tilde{Q} - \frac{1}{\sqrt[4]{r(r-1)}} \tilde{Q} \right\},$$

$$B(r) = Ce^{i\omega r}(r-1)^{i\omega} \frac{1}{\sqrt{r}} \left\{ \frac{\sqrt[4]{r(r-1)}}{\kappa + i\mu r} \partial_r \tilde{Q} + \frac{1}{\sqrt[4]{r(r-1)}} \tilde{Q} \right\},$$

where $\tilde{Q}_{1,2}(r)$ are the well-known confluent Heun functions:

$$\tilde{Q}(r)_1 = HeunC\left(\kappa^2, 0, \frac{1}{2}, \frac{1}{2} + i2\omega, i2\omega; r \right),$$

$$\tilde{Q}(r)_2 = \sqrt{r}\, HeunC\left(\kappa^2 + \frac{1}{4}, i\omega, \frac{3}{2}, \frac{1}{2} + i2\omega, i2\omega; r \right).$$

Asymptotics

The asymptotic behavior was obtained by constructing solutions in the form of infinite power series using the Frobenius-Thomé method:

On the horizon:

$$A_1 \sim -(r-1)^{-\frac{1}{4}+i\omega}, \quad B_1 \sim (r-1)^{-\frac{1}{4}+i\omega}.$$

$$A_2 \sim (r-1)^{-\frac{1}{4}-\frac{\mu^3\kappa}{\omega(\mu^2+\kappa^2)}-i\left(\omega-\frac{\mu^4}{\omega(\mu^2+\kappa^2)}\right)}, \quad B_2 \sim (r-1)^{-\frac{1}{4}-\frac{\mu^3\kappa}{\omega(\mu^2+\kappa^2)}-i\left(\omega-\frac{\mu^4}{\omega(\mu^2+\kappa^2)}\right)}.$$

At the infinity

$$A_1 \sim \left(-\frac{\mu}{2\omega}-\frac{\omega}{\mu}-1\right)e^{i\frac{\sqrt{\omega^2-\mu^2}}{2\omega}r}\, r^{-\frac{3\kappa}{4\mu}\omega-1-i\omega}, B_1 \sim \left(-\frac{\mu}{2\omega}-\frac{\omega}{\mu}+1\right)e^{i\frac{\sqrt{\omega^2-\mu^2}}{2\omega}r}\, r^{-\frac{3\kappa}{4\mu}\omega-1-i\omega}.$$

$$A_2 \sim \left(-\frac{\mu}{2\omega}-\frac{\omega}{\mu}-1\right)e^{-i\frac{\sqrt{\omega^2-\mu^2}}{2\omega}r}\, r^{-\frac{3\kappa}{4\mu}\omega-1+i\omega}, \quad B_2 \sim \left(-\frac{\mu}{2\omega}-\frac{\omega}{\mu}+1\right)e^{-i\frac{\sqrt{\omega^2-\mu^2}}{2\omega}r}\, r^{-\frac{3\kappa}{4\mu}\omega-1+i\omega}.$$

the last expression is in good agreement with the results of the paper [13] .

[13] V. Egorov, M. Smolyakov, I. Volobuev, «Quantization of spinor field in the Schwarzschild spacetime and spin sums for solutions of the Dirac equation», Class. Quantum Grav. 41, 045002 (2024).

Two-fold degeneracy of the massless states

The states of infinite motion for the scalar field in the Schwarzschild metric prove to be doubly degenerate [15]. This is a topological effect, that is absent in Minkowski spacetime [3].

Since the structure of solutions of the massless Dirac equation is similar to the structure of solutions of the Klein-Gordon equation, then for massless spinor fields we also have a similar two-fold degeneracy for the states of the continuous spectrum

[3] V. Egorov, M. Smolyakov, I. Volobuev, «Doubling of physical states in the quantum scalar field theory for a remote observer in the Schwarzschild spacetime» Phys. Rev. D 107, 025001 (2023); arXiv:2209.02067 [gr-qc].

[15] J. Barranco, A. Bernal, J. C. Degollado, A. Diez-Tejedor, M. Megevand, M. Alcubierre, D. Nunez, O. Sarbach, «Are black holes a serious threat to scalar field dark matter models?» Phys. Rev. D 84, 083008 (2011); arXiv:1108.0931 [gr-qc].

Summary

1. Solutions of the Dirac equation in the Schwarzschild black hole metric are obtained.
2. In the case of a massive fermion field, the radial part of the Dirac equation in the Schwarzschild metric is a confluent equation with four singular points and, apparently, its solutions cannot be expressed in terms of known special functions.
3. In contrast, for a massless spinor field, the radial part of the Dirac equation in the Schwarzschild metric reduces to the confluent Heun equation and therefore its solutions can be expressed in terms of confluent Heun functions.
4. The asymptotic behavior of the solutions of the radial part of the Dirac equation was obtained by constructing solutions in the form of infinite power series using the Frobenius-Thomé method.
5. Since the structure of solutions of the massless Dirac equation is similar to the structure of solutions of the Klein-Gordon equation, then for massless spinor fields we also have a similar two-fold degeneracy for the states of the continuous spectrum.

Thank you for your attention

Appendix A: spin connections

The diagonal tetrad

$$g^{(0)}_{\mu} = \begin{pmatrix}\sqrt{f_S} & 0 & 0 & 0\end{pmatrix}, \quad g^{(1)}_{\mu} = \begin{pmatrix}0 & \dfrac{1}{\sqrt{f_S}} & 0 & 0\end{pmatrix}, \quad g^{(2)}_{\mu} = \begin{pmatrix}0 & 0 & r & 0\end{pmatrix}, \quad g^{(3)}_{\mu} = \begin{pmatrix}0 & 0 & 0 & r\sin\theta\end{pmatrix}.$$

$$g_{(0)\mu} = \begin{pmatrix}\sqrt{f_S} & 0 & 0 & 0\end{pmatrix}, \quad g_{(1)\mu} = \begin{pmatrix}0 & -\dfrac{1}{\sqrt{f_S}} & 0 & 0\end{pmatrix}, \quad g_{(2)\mu} = \begin{pmatrix}0 & 0 & -r & 0\end{pmatrix}, \quad g_{(3)\mu} = \begin{pmatrix}0 & 0 & 0 & -r\sin\theta\end{pmatrix}.$$

Christoffel symbols:

$$\Gamma_{001} = -\Gamma_{100} = \frac{1}{2}\partial_r f_S, \quad \Gamma_{111} = -\frac{1}{2} f_S^{-2}\partial_r f_S, \quad \Gamma_{122} = -\Gamma_{221} = -r^{-1} g_{22} = r,$$

$$\Gamma_{133} = -\Gamma_{331} = -r^{-1} g_{33} = r\sin^2\theta, \quad \Gamma_{233} = -\Gamma_{332} = -\operatorname{ctg}\theta\, g_{33} = r^2\cos\theta\sin\theta.$$

$$\Gamma^0_{01} = f_S^{-1}\frac{1}{2}\partial_r f_S, \quad \Gamma^1_{00} = f_S\frac{1}{2}\partial_r f_S, \quad \Gamma^1_{11} = \frac{1}{2} f_S^{-1}\partial_r f_S, \quad \Gamma^1_{22} = f_S r^{-1} g_{22} = -f_S r, \quad \Gamma^2_{21} = -r^{-3} g_{22} = r^{-1},$$

$$\Gamma^1_{33} = f_S r^{-1} g_{33} = -f_S r\sin^2\theta, \qquad \Gamma^3_{31} = -r^{-3}\sin^{-2}\theta\, g_{33} = r^{-1},$$

$$\Gamma^2_{33} = r^{-2}\operatorname{ctg}\theta\, g_{33} = -\cos\theta\sin\theta, \quad \Gamma^3_{32} = -r^{-2}\sin^{-3}\theta\cos\theta\, g_{33} = \operatorname{ctg}\theta.$$

Spin connections:

$$C_{(001)} = \frac{f_S'}{2\sqrt{f_S}}, \quad C_{(212)} = \frac{\sqrt{f_S}}{r}, \quad C_{(313)} = \frac{\sqrt{f_S}}{r}, \quad C_{(323)} = \frac{\operatorname{ctg}\theta}{r}.$$

Appendix B: Spherical Spinors

Реперные векторы $e^{a}_{(q)}$ совпадают с декартовыми реперными векторами h^{a}_{q} в сферических координатах:

$$h^{r}_{q} = \left(\sin\theta\cos\varphi \quad \sin\theta\sin\varphi \quad \cos\theta\right)$$

$$h^{\theta}_{q} = \left(\cos\theta\cos\varphi \quad \cos\theta\sin\varphi \quad -\sin\theta\right)$$

$$h^{\varphi}_{q} = \frac{1}{\sin\theta}\left(-\sin\varphi \quad \cos\varphi \quad 0\right)$$