Recent Progress in Exact Solutions of Einstein equations

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Plan

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Noether symmetries

- Killing equation $\mathcal{L}_K g_{\mu\nu} = 0 = K_{(\mu;\nu)}$ defines a symmetry of spacetime
- Two Killing vectors $K = K^\mu \partial_\mu$ either commute, or define the third Killing vector: $[K^1, K^2] = K^3$
- These commutators form Lie algebras (Jacoby and Leibnitz rules hold), giving rise to isometry group helping to solve Einstein equations
- Each Killing vector generates the integral $I = K_\mu \dot{x}^\mu$ of the geodesic equation $\dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0$, namely $dI/d\lambda = 0$. Sufficient number of integrals makes geodesic equations separable
- Also, a Killing vector transforms the covariant conservation equation $\nabla_\nu T^{\mu\nu} = 0$ into the current conservation (Noether theorem)

$$\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\, T^{\mu\nu} K_\mu \right) = 0$$

- A Killing tensor $K_{\mu\nu} = K_{(\mu\nu)}$ satisfying $K_{(\mu\nu;\tau)} = 0$ generates the geogesic integral $I = K_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ (similarly for higher rank K) but not the conserved current. How to use Killing tensors together with vectors?

Relativistic phase space

Starting with Polyakov action

$$S = -\frac{1}{2}\int \left(hg_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + m^2/h\right) d\lambda, \qquad P_\mu = -\frac{\delta S}{\delta \dot{x}^{\mu}} = h\dot{x}_\mu,$$

construct the geodesic Hamiltonian

$$H = -P_\mu \dot{x}^\mu - L = h\left(m^2 - g^{\mu\nu}P_\mu P_\nu\right)/2$$

and define Poisson brackets for two scalar functions $F(x, P)$, $G(x, P)$

$$\{F, G\} = \nabla_\mu F \, \frac{\partial G}{\partial P_\mu} - \nabla_\mu G \, \frac{\partial F}{\partial P_\mu}$$

satisfying Jacoby identity and Leibnitz rule. Then for any two integrals of motion I, J satisfting $\{I, H\} = 0$, $\{J, H\} = 0$, their Poisson bracket $K = \{I, J\}$ will be the third integral of motion: $\{K, H\} = 0$, so the Poisson brackets define the Lie algebra of motion integrals on the phase space.

Anti-Noether theorem

Let $K_1 = K_\nu(x)P^\nu$ be a motion integral linear in P. Since $\nabla_\mu H = 0$, then

$$\{K_1, H\} = 0 = -h\nabla_\mu K_\nu(x)P^\nu P^\mu,$$

implying the Killing equation for the coefficient function

$$K_{(\nu;\mu)} = 0,$$

(an inverse statement is known as Noether theorem). Similarly, for any symmetric tensor $K_{\nu_1\ldots\nu_n} = K_{(\nu_1\ldots\nu_n)}$ one can construct a P-polynomial $K_n = K_{\nu_1\ldots\nu_n}P^{\nu_1}\ldots P^{\nu_n}$. Then assuming K_n to be an integral of motion

$$\{K_n, H\} = 0 = -h\nabla_\mu K_{\nu_1\ldots\nu_n}(x)P^{\nu_1}\ldots P^{\nu_n}P^\mu,$$

one finds that $K_{\nu_1\ldots\nu_n}$ is a rank-n Killing tensor:

$$K_{(\nu_1\ldots\nu_n;\mu)} = 0,$$

Poisson algebra of motion integrals generates SN brackets

For any two integrals of motion I_n, J_m satisfting $\{I_n, H\} = 0$, $\{J_m, H\} = 0$, the Poisson bracket $K_{n+m-1} = -\{I_n, J_m\}$ will be the third integral of motion $\{K_{n+m-1}, H\} = 0$ defined with the rank $n + m - 1$ Killing tensor obtained by the *Shouten-Nijenhuis* (SN) bracket $K = [I, J]_{SN}$, or in component form

$$K_{\nu_1 \dots \nu_{n+m-1}} = nI_{\mu(\nu_1 \dots \nu_{n-1}} \nabla^\mu J_{\nu_n \dots \nu_{n+m-1})} - mJ_{\mu(\nu_1 \dots \nu_{m-1}} \nabla^\mu I_{\nu_m \dots \nu_{n+m-1})}$$

SN brackets form the Lie algebra generalising the Killing vectors algebra

- **Benenti theorem (1976):**
  In n dimensions a necessary and sufficient condition for separability of HJ equation is the existence of a closed commuting system of Schouten-Nijenhuis brackets of n Killing vectors and Killing tensors, with additional conditions for eigenvectors of the latter.
- For stationary axially symmetric (SAS) 4d spacetime with metric regarded as one (trivial) Killing tensor, one nontrivial KT is enough
- In higher dimensions the SN brackets may generate a tower of higher rank KT-s, since $[K_2, K_2']_{SN} = \tilde{K}_3$ and so on.

Benenti-Francaviglia ansatz

Metric parametrization ensuring existence of non-trivial Killing tensor was given by Benenti and Francaviglia (BF, 1979). It includes ten arbitrary functions, each depending on one variable $A_k(r)$, $B_k(y)$, $k = 1..5$.

- Metric is SAS, block diagonal (orthogonal transitivity), $x^\mu = (x^a, x^i)$, where $x^a = t, \varphi$ correspond to the subspace spanned by the Killing vectors $K^{(t)} = \partial_t$ and $K^{(\varphi)} = \partial_\varphi$ and $x^i = r, y$, belong to orthogonal two-dimensional space whose metric without loss of generality can be assumed diagonal. BF ansatz initially is written in terms of the contravariant metric tensor $g^{\mu\nu} = \left(g^{ab},\ g^{ij}\right)$:

$$g^{ab} = \Sigma^{-1} \begin{pmatrix} A_3 - B_3 & A_4 - B_4 \\ A_4 - B_4 & A_5 - B_5 \end{pmatrix}, \quad g^{ij} = -\Sigma^{-1} \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix}$$

- In order to ensure existence of an exact Killing tensor (EKT), the conformal factor $\Sigma = \Sigma(r, y)$ must be of the special form

$$\Sigma = A_1 + B_1.$$

BF Killing tensor

In terms of A, B, the BF Killing tensor reads $K^{\mu\nu} = \left(K^{ab},\ K^{ij}\right)$, where

$$\Sigma K^{ab} = \begin{pmatrix} A_1 B_3 + A_3 B_1 & A_1 B_4 + A_4 B_1 \\ A_1 B_4 + A_4 B_1 & A_1 B_5 + A_5 B_1 \end{pmatrix}, \quad \Sigma K^{ij} = \begin{pmatrix} -A_2 B_1 & 0 \\ 0 & A_1 B_2 \end{pmatrix}$$

The inverse metric and the Killing tensor have the following automorphism:

$$A_1 \leftrightarrow -B_1, \quad A_i \leftrightarrow B_i, \quad (i = 2..4), \quad g^{rr} \leftrightarrow -g^{yy}, \quad K^{rr} \leftrightarrow -K^{yy}.$$

For an arbitrary conformal factor Σ only a conformal Killing tensor exists. Two blocks of the covariant metric tensor $g_{\mu\nu} = (g_{ab},\ g_{ij})$: read

$$g_{ab} = \frac{\Sigma}{\mathcal{P}} \begin{pmatrix} A_5 - B_5 & -A_4 + B_4 \\ -A_4 + B_4 & A_3 - B_3 \end{pmatrix}, \qquad g_{ij} = -\Sigma \begin{pmatrix} A_2^{-1} & 0 \\ 0 & B_2^{-1} \end{pmatrix}$$

where

$$\mathcal{P} = (A_3 - B_3)(A_5 - B_5) - (A_4 - B_4)^2.$$

Slice reducibility of BF Killing tensor

BF metric parametrization suggests two fiberings of spacetime by hypersurfaces of constant r and constant y so that the BF Killing tensor has property to be reducible on them. This follows from representation:

$$K^{\mu\nu} = -A_1 g^{\mu\nu} - A_2 \delta^\mu_r \delta^\nu_r + \tilde{K}^{\mu\nu}_r, \quad \tilde{K}^{\mu\nu}_r = A_3 \delta^\mu_t \delta^\nu_t + 2A_4 \delta^{(\mu}_t \delta^{\nu)}_\varphi + A_5 \delta^\mu_\varphi \delta^\nu_\varphi.$$

The first term is trivial Killing tensor on the hypersurface $r =$const, since $A_1 = \mathrm{const}$ there. The second term is orthogonal to this surface and thus irrelevant, while the third term $\tilde{K}^{\mu\nu}_r$ is a reducible Killing tensor on this hypersurface. Similarly the BF Killing tensor can be presented as

$$K^{\mu\nu} = B_1 g^{\mu\nu} + B_2 \delta^\mu_y \delta^\nu_y + \tilde{K}^{\mu\nu}_y.$$

with the slice projection $\tilde{K}^{\mu\nu}_y = B_3 \delta^\mu_t \delta^\nu_t + 2B_4 \delta^{(\mu}_t \delta^{\nu)}_\varphi + B_5 \delta^\mu_\varphi \delta^\nu_\varphi$. Such a fibering is closely related to generalized photon surfaces separating domains of scattering and absorption of particles in formation of black hole shadows

Kobialko, Bogush and DG (2023-2024).

Further symmetries: Null shear-free congruences

Slice-reducibility can be further enforced by assumptions

$$A_4 = \sqrt{A_3 A_5}, \qquad B_4 = \sqrt{B_3 B_5},$$

in which case the reducible part $\tilde{K}_r^{\mu\nu} = \left(\sqrt{A_3}\delta_t^\mu + \sqrt{A_5}\delta_\varphi^\mu\right)\left(\sqrt{A_3}\delta_t^\nu + \sqrt{A_5}\delta_\varphi^\nu\right)$ factorizes, and the same holds for $\tilde{K}_y^{\mu\nu}$. It turns out that space-time acquires further symmetry, namely, existence of null shear-free geodesic congruence. Such property in the vacuum case is the sign of Petrov type D.

But our constrained BF metrics generically still belongs to non-algebraically special type I. To see this, one pass to Newman-Penrose null tetrad representing an inverse metric as $\quad g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu \bar{m}^\nu - \bar{m}^\mu m^\nu$:

$$l = \frac{1}{\sqrt{2\Sigma}}\left(\sqrt{A_3}\partial_t + \sqrt{A_5}\partial_\varphi - \sqrt{A_2}\partial_r\right), \quad m = \frac{1}{\sqrt{2\Sigma}}\left(\sqrt{B_3}\,\partial_t + \sqrt{B_5}\partial_\varphi - i\sqrt{B_2}\partial_y\right),$$

$$n = \frac{1}{\sqrt{2\Sigma}}\left(\sqrt{A_3}\partial_t + \sqrt{A_5}\partial_\varphi + \sqrt{A_2}\,\partial_r\right), \quad \bar{m} = \frac{1}{\sqrt{2\Sigma}}\left(\sqrt{B_3}\,\partial_t + \sqrt{B_5}\partial_\varphi + i\sqrt{B_2}\partial_y\right).$$

Newman-Penrose basics

Recall the definitions of the NP projections of the covariant derivatives

$$D = l^\mu \nabla_\mu, \quad \Delta = n^\mu \nabla_\mu, \quad \delta = m^\mu \nabla_\mu, \quad \bar{\delta} = \bar{m}^\mu \nabla_\mu,$$

and the action of D, Δ on the vectors l^μ, n^μ

$$Dl^\mu = (\epsilon + \bar{\epsilon}) l^\mu - \bar{\kappa} m^\mu - \kappa \bar{m}^\mu$$
$$\Delta n^\mu = -(\gamma + \bar{\gamma}) n^\mu + \nu m^\mu + \bar{\nu} \bar{m}^\mu.$$

Consider null congruences aligned with l^μ, n^μ. If $\kappa = 0 = \nu$ they are *geodesic*, with ϵ, γ being measure of non-affinity. Another important quantity of null congruences is *shear*, which is defined for them as

$$\sigma = -m^\mu \delta\, l_\mu, \qquad \bar{\lambda} = m^\mu \delta\, n_\mu$$

respectively. Calculating the spin coefficients for our tetrad, we find:

$$\kappa = \nu = 0, \qquad \sigma = \lambda = 0,$$

which means that both congruences are *geodesic* and *shearfree*. Other spin coefficients are generically non-zero and pairwise equal:

$$\mu = \rho, \quad \tau = \pi, \quad \epsilon = \gamma, \quad \alpha = \beta.$$

Weyl projections

Such properties are typical for Petrov type D. To establish Petrov type in our case, we calculate the NP projections of the Weyl tensor:

$$
\begin{aligned}
\Psi_0 &= -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta, \\
\Psi_1 &= -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta, \\
\Psi_2 &= -C_{\alpha\beta\gamma\delta} \left(l^\alpha n^\beta l^\gamma n^\delta - l^\alpha n^\beta m^\gamma \bar{m}^\delta \right) /2, \\
\Psi_3 &= -C_{\alpha\beta\gamma\delta} n^\alpha l^\beta n^\gamma m^\delta, \\
\Psi_4 &= -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta,
\end{aligned}
$$

From the computer assisted calculations one finds that two of them are zero,

$$\Psi_0 = 0 = \Psi_4,$$

while the others are rather cumbersome in terms of BF coefficients. Still one can extract the following relation between the other two:

$$\Psi_1 = \Psi_3,$$

reflecting obvious symmetry of the tetrad under $A \leftrightarrow B$.

Invariants

Vanishing of Ψ_0 and Ψ_4 means that the real vectors l^μ, n^μ are two distinct principal null directions of the constrained BF metric. Also this means that our tetrad is not canonical for determination of the Petrov type. We therefore proceed by computing the values of the quadratic and cubic curvature invariants of the Weyl tensor

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \qquad J = \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}.$$

As is known, in order for a metric to be algebraically special, the following relationship between invariants must be satisfied:

$$I^3 = 27J^2.$$

With our results one finds that the constrained BF metrics are algebraically special if

$$\Psi_1^2 = k\Psi_2^2, \quad \text{with} \quad k = 9/16, \ \text{or} \ 0.$$

- If this does not hold, the metric is of the Petrov type I.
- If holds with $k = 0$ and $\Psi_2 \neq 0$ then the metric type is D.
- Other algebraically special types are not possible , for example if one assumes type II, for which $\Psi_0 = 0 = \Psi_1$, one immediately finds that the Weyl tensor completely vanishes, i.e. the metric is of type O.
- In view of the Goldberg-Sachs theorem, for type I spacetime, admitting a null geodesic shear-free congruence, the Ricci tensor should be non-zero. This is the case for supergravity black holes.
- Thus our class I_B (eight A, B functions) consists of non-vacuum metrics, admitting a Killing tensor and a pair of null geodesic shear-free congruences. These properties are close to properties of D type, they will ensure separability of the Hamilton-Jacobi equation.
- Special feature of type I_B is that its algebraically special subsector is only type D

Next symmetry: Klein-Gordon separability

Generic type I_B class of metrics still does not quarantee separability of the wave equations. Consider the Klein-Gordon equation for a real scalar field ϕ:

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = -\mu^2\phi.$$

Crucial for separability is the determinant of the metric, which with the first two constraints reads:

$$\sqrt{-g} = \frac{\Sigma^2}{\sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{A_5B_3}\right)}.$$

Consideration of the inverse metric clearly shows that the separability condition is

$$\sqrt{-g} = \Sigma.$$

This leads to the third restriction on the Benenti coefficient functions:

$$\Sigma = \sqrt{A_2B_2}\left(\sqrt{A_3B_5} - \sqrt{B_3A_5}\right),$$

which can be rewritten as $A_1 + B_1 = bA_{23} - aB_{23}$, introducing $A_{23} = \sqrt{A_2A_3}$, $B_{23} = \sqrt{B_2B_3}$ in the gauge $A_2A_5 = a^2$, $B_2B_5 = b^2$ (const).

Secondary constraints

Differentiating this with respect to the appropriate arguments, one can find useful differential relations following from the third constraint:

$$A_1' = bA_{23}', \qquad B_1' = -aB_{23}'.$$

(Note, that in these relations primes can not be omitted!)
With the third constraint the separability of KG equation is easily shown:

$$\text{with} \qquad \phi(x^\mu) = \mathrm{e}^{-i\omega t + im\varphi} R(r) Y(y), \qquad \text{one gets :}$$

$$\frac{((A_2)'R)'}{R} + \frac{((B_2)'Y)'}{Y} + U(r) - V(y) = 0,$$

where primes denote derivatives with respect to r, y, and

$$U(r) = (\omega\sqrt{A_3} - m\sqrt{A_5})^2 - \mu^2 A_1,$$
$$V(y) = (\omega\sqrt{B_3} - m\sqrt{B_5})^2 + \mu^2 B_1.$$

This second constraint effectively reduces the number of arbitrary functions to seven, two of which A_2, B_2 can still be fixed using gauge freedom, so the number of essentially independent functions is five.

Carter's operator and commutativity

Klein-Gordon separability can be also explored with the help of Carter's second order differential operator associated with the Killing tensor:

$$\hat{K} = \nabla_\mu K^{\mu\nu} \nabla_\nu,$$

which must commute with D'Alembert operator. This commutator was elaborated by Carter:

$$[\Box, \hat{K}]\,\phi = \frac{4}{3} \nabla_\alpha (K_\sigma{}^{[\alpha} R^{\beta]\sigma}) \nabla_\beta\, \phi.$$

Projecting tensor at the right hand side onto the NP tetrad, we obtain

$$K_\sigma{}^{[\alpha} R^{\beta]\sigma} = 2(K_{ln} + K_{m\bar{m}})(n^\beta(\bar{m}^\alpha \Phi_{01} + m^\alpha \Phi_{10}) - n^\alpha(\bar{m}^\beta \Phi_{01} + m^\beta \Phi_{10}) + \\ + (l^\beta \bar{m}^\alpha - l^\alpha \bar{m}^\beta)\Phi_{12} + (l^\beta m^\alpha - l^\alpha m^\beta)\Phi_{21}).$$

So a sufficient condition for commutativity is vanishing of two Ricci scalars

$$\Phi_{01} = R_{\mu\nu} l^\mu m^\nu / 2 = \overline{\Phi_{10}}, \quad \Phi_{12} = R_{\mu\nu} n^\mu m^\nu / 2 = \overline{\Phi_{21}}.$$

Key feature: polynomial structure of BF coefficients

Computing the above projections of the Ricci tensors with account for the third constraint and equating them to zero one obtains the differential condition:

$$aA_{23}'' + bB_{23}'' = 0,$$

where primes denotes derivatives over respective arguments. With account for previously found relations, this can be also rewritten as

$$a^2 A_1'' - b^2 B_1'' = 0.$$

Since one term is a function of r, while the other is a function of y, each of them must be constant. Other speaking, A_1 and B_1 must be at most quadratic polynoms of respective arguments.

- This feature is crucial for integrability of non-linear Einstein equations, which are essentially non-linear, so a simple separation of variables is impossible.

After imposing three constraints on BF functions we can rewrite the spacetime metric as follows

$$ds^2 = \frac{A_2 B_2}{\Sigma}\left(\sqrt{B_5}dt - \sqrt{B_3}d\varphi\right)^2 - \frac{A_2 B_2}{\Sigma}\left(\sqrt{A_5}dt - \sqrt{A_3}d\varphi\right)^2 - \frac{\Sigma}{A_2}dr^2 - \frac{\Sigma}{B_2}dy^2,$$

where

$$\Sigma = \sqrt{A_2 B_2}(\sqrt{A_3 B_5} - \sqrt{A_5 B_3}).$$

This is exactly the Carter ansatz of 1968 up to a signature convention and notation. Carter showed that for these metrics the vacuum and electrovacuum (with the corresponding Maxwell form) Einstein equations are solvable analytically and lead to several families of solutions, among which were the Kerr and Kerr-Newman black holes belonging to the Petrov type D. Now we checked that our parameterization admits class I_B of general type I and is applicable to morefeneral sources (but not all). We checked this explicitly on $\mathcal{N} = 4$ supergravity.

Killing-Yano symmetry as criterium of type D

The Killing-Yano tensor $Y_{\mu\nu} = -Y_{\nu\mu}$ satisfying the equation

$$\nabla_{(\alpha} Y_{\mu)\nu} = 0,$$

can be regarded as a "square root" of the Killing tensor:

$$Y_\mu{}^\alpha Y_{\alpha\nu} = K_{\mu\nu}.$$

Since we know the Killing tensor independently of Petrov type of the metric, we can consider these equations as further constraint on BF parameterization which prescribe the metric to be of type D.
In terms of the constrained tetrad the Killing tensor has only two non-vanishing NP projections exactly as in the case of the type D :

$$K_{ln} = B_1, \qquad\qquad K_{m\bar{m}} = A_1.$$

So projecting KJ splitting of KT on the NP tetrad, one obtains:

$$Y^2_{ln} = B_1, \qquad\qquad Y^2_{m\bar{m}} = -A_1,$$

D subclass

Extracting roots from these equations is somewhat subtle and demands further analysis. The result is

$$Y_{ln} = \sqrt{B_1}, \qquad Y_{m\bar{m}} = -i\sqrt{A_1}.$$

Now we have to satisfy the KY equation for consistency . Omitting details we arrive at the following relation for the metric to be D type:

$$A_1 = (br + c_1)^2, \qquad B_1 = (ay + d_1)^2,$$
$$A_{23} = br^2 + 2\, c_1 r + c_2, \qquad B_{23} = -(ay^2 + 2\, d_1 y + d_2),$$

with conditions$b\, c_2 + a\, d_2 = c_1^2 + d_1^2$. With this one finds

$$8\Sigma^3\Psi_{1,3} = -\sqrt{A_2 B_2}\left\{\Sigma(aA''_{23} - bB''_{23}) - ab({A'_{23}}^2 + {B'_{23}}^2)\right\} = 0,$$

So only $\Psi 2 \neq 0$. In the static case $a = 0$, $B_{23} = 0$, we have $\Psi_{1,3} = 0$ so the static solution is always algebraically special.

Einstein-Maxwell-dilaton-axion action

This is truncated $\mathcal{N} = 4, D = 4$ supergravity

$$S = -\frac{1}{16\pi}\int\left(R + \frac{2\nabla z\nabla\bar{z}}{(z-\bar{z})^2} - (iz\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + c.c.)\right)\sqrt{-g}\,d^4x$$

Where a complex axidilaton is $z = \kappa + ie^{-2\phi}$ and self-dual Maxwell tensor is $\mathcal{F}^{\mu\nu} = \frac{1}{2}\left(F^{\mu\nu} + i\tilde{F}^{\mu\nu}\right)$. The metric is given by the Carter's ansatz

$$ds^2 = \frac{A_2}{\Sigma}(dt - Bd\varphi)^2 - \frac{B_2}{\Sigma}(adt - Ad\varphi)^2 - \frac{\Sigma}{A_2}dr^2 - \frac{\Sigma}{B_2}dy^2,$$

where we renamed $A_{23} = A$ and $B_{23} = B$ and set $b = 1$ by rescaling of t. Here a is a real constant, A, A_2 depend only on r, B, B_2 depend only on y and $\Sigma = A - aB$, ensuring integrability of the Klein-Gordon equation. We use the orthogonal tetrad $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$, $a, b = 1, 2, 3, 4$, with
$e^1 = \alpha(bdt - Bd\varphi),\ e^2 = \beta(adt - Ad\varphi),\ e^3 = \alpha^{-1}dr,\ e^4 = \beta^{-1}dy,$
where $\alpha = \sqrt{A_2/\Sigma},\ \beta = \sqrt{B_2/\Sigma}$.

Einstein equations

The Einstein equations can be presented in tetrad form as

$$R_{ab} = T^{sc}_{ab} + \frac{(z-\bar{z})}{2i} T^{em}_{ab},$$

where the reduced scalar term without trace part is equal to

$$T^{sc}_{ab} = -\frac{1}{(z-\bar{z})^2}(z_{,a}\bar{z}_{,b} + z_{,b}\bar{z}_{a}),$$

while the Maxwell term is standard:

$$T^{em}_{ab} = 2\left(F_{ac}F^{c}_{b} + \frac{\eta_{ab}}{4}F_{cd}F^{cd}\right).$$

The reduced scalar stress tensor has nonzero components only in the $3,4$ sector (or r, y in the coordinate basis).redenoting tetrad indices as $m = 1,2$ and $j = 3,4$ we have zero the following non-diagonal components

$$R_{mj} = 0, \qquad R_{34} = 0.$$

The right hand side of the R_{mj} Einstein equation is zero. The T_{34} component leads to pure axidilaton equation $\quad z_{,r}\bar{z}_{,y} + z_{,y}\bar{z}_{,r} = 0.$

Scalar equations

The $T_{34} = 0$ component then leads to pure axidilaton equation $z_{,r}\bar{z}_{,y} + z_{,y}\bar{z}_{,r} = 0$. The remaining non-diagonal component R_{12} is quite simple:

$$R_{12} = \frac{\sqrt{A_2 B_2}}{2\Sigma^2}\left(aA'' + B''\right),$$

while the right hand side of the corresponding Einstein equation is zero. This leads to the constraint $aA'' + B'' = 0$. Since the first term depend on r, while the second on y this means that A ANd B are both polynomials of the second order. Using this condition and the list of non-zero Ricci components we obtain for diagonal components of Einstein equations the identities $R_{11} + R_{33} = \alpha^2\mathcal{R},\; R_{22} - R_{44} = -a^2\beta^2\mathcal{R},$ where $\mathcal{R} = \left[(A')^2 + (B')^2 - \Sigma A''\right]/2\Sigma^2$. The right hand sides for the both combinations of the Ricci tensors are free from the Maxwell terms, while substituting the scalar reduced stress tensor we derive the second separate axidilaton equation:

$$a^2 z_{,r}\bar{z}_{,r} - z_{,y}\bar{z}_{,y} = 0.$$

Axidilaton and Maxwell

From axidilaton equations it follows that z is a holomorphic or antiholomorphic function of $w = r + iay$, except at the poles. Such a function must be a fractional-linear transformation

$$z = \frac{c_1(r + iay) + c_2}{c_3(r + iay) + c_4},$$

where all c_i are arbitrary complex constants, and the constraint $c_1 c_4 - c_2 c_3 \neq 0$ must be satisfied for invertibility. If $c_1 \neq 0$ and $c_3 \neq 0$, the axidilaton field will tend to a non-zero constant $z_\infty = c_1/c_3$ ar infinity.

The Maxwell one form compatible with the metric ansatz is given by two scalars $R(r)$ and $Y(y)$

$$A_{[1]} = \frac{R}{\alpha\Sigma} e^1 + \frac{Y}{\beta\Sigma} e^2,$$

for which on derives from the Maxwell equations one more separable linear equation

$$aR'' - Y'' = 0,$$

from which one establish their polynomial structure too.

Full integrability

Correspondingly there are two non-zero components of the Maxwell tensor

$$F_{13} = -\tilde{F}_{24} = \frac{A'(R + aY) - \Sigma R'}{\Sigma^2}, \quad F_{24} = \tilde{F}_{13} = -\frac{B'(R + aY) + \Sigma Y'}{\Sigma^2}$$

Then the Maxwell energy-momentum tensor can be presented as follows:

$$T^{em}_{11} = T^{em}_{22} = -T^{em}_{33} = T^{em}_{44} = F^2_{13} + F^2_{24}. \quad (1)$$

With the special form of the source terms in Einstein equation one further obtain the linear eqiation

$$A''_2 + B''_2 = 0,$$

completing the proof of the plynomial structure of all relevent coeffitient functions. The remaining equations of the system is then solved as the algebraic system for coeffitients of the polynomials.

EMDA Nutty dyon

$$ds^2 = \frac{\Delta_r - a^2\sin^2\theta}{\Sigma}(dt - w d\varphi)^2 - \Sigma\left(\frac{dr^2}{\Delta_r} + d\theta^2 + \frac{\Delta_r \sin^2\theta}{\Delta_r - a^2\sin^2\theta} d\varphi^2\right),$$

$$\Delta_r = (r - r_0)(r - 2M) + a^2 - (N - N_-)^2, \; \Sigma = r(r - r_-) + (a\cos\theta + N)^2 - N_-^2,$$

$$w = \frac{2}{a^2\sin^2\theta - \Delta_r}\left[N\Delta_r\cos\theta + a\sin^2\theta(M(r - r_-) + N(N - N_-))\right],$$

$$r_- = \frac{M|Q - iP|^2}{|M + iN|^2}, \quad N_- = \frac{N|Q - iP|^2}{2|M + iN|^2},$$ Derived in 1994 by DG and Kechkin, (PRD 50, 7394 (1994)) with the help of dimensional reduction and Harrison transformations, it is now reproduced by direct integration of Einstein equations (DG and R.Karsanov, PRD, 111, 104011 (2025)). Its BF form reads:

$$A_1 = r(r - r_-), \quad B_1 = (a\,y + N)^2 - N_-^2, \quad A_2 = \Delta_r, \quad B_2 = 1 - y^2,$$
$$A = r(r - r_-) + a^2 + N^2 - N_-^2, \qquad B = a(1 - y^2) - 2Ny,$$

where $y = \cos\theta$, polynomial structure is obvious, and the constraint holds

$$A_1 + B_1 = A - aB$$

Petrov type and uniqueness

Two non-zero NP projections of the Killing tensor are

$$K_{ln} = (ay + N)^2 - N_-^2, \qquad K_{m\bar{m}} = r(r - r_-).$$

The metric is a non-vacuum Petrov type I_B, with the following set of non-zero Weyl scalars:

$$\Psi_1 = \Psi_3 = \frac{a(4N_-^2 + r_-^2)\sin\theta\sqrt{\Delta_r}}{8\Sigma^3}, \qquad 12\Sigma^3\Psi_2 \neq 0 \ \ \text{(cumbersome)}$$

- Our general ALF solution contains two more parameters (not shown): conical defect (cosmic string), and the parameter fixing the Misner string. Also asymptotic value of axidilaton was fixed. Up to this, the solution is found to be unique ALF solution of EMDA theory. Previous uniqueness proofs were based on sigma-model picture, now it is proved by direct integration of Einstein equations.
- But ALF is not the only admissible asymptotic for EMDA black holes. Other possibility is Linear dilaton asymptotic, which is neither AF, nor ADS (*G. Clément, DG and S. Leygnac, PRD 67 024012, 2003*) . LDB is an exact solution of the heterotic string theory. This gives rise to holographic duality with the so called 6d little string theory.

This is obtained in the degenerate case of the axidilaton fraction with $c_1 = 0$:

$$z = \frac{c_2}{c_3(r - iay) + c_4}.$$

The solution reads

$$ds^2 = \frac{\Delta - a^2 \sin^2\theta}{\Gamma}(dt - \omega d\varphi)^2 - \Gamma\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta \sin^2\theta}{\Delta - a^2\sin^2\theta}\, d\varphi^2\right),$$

$$e^{2\phi} = \frac{r^2 + (N + a\cos\theta)^2}{\Gamma}, \qquad \kappa = \frac{r_0}{M}\,\frac{N(r - M) - aM\cos\theta}{r^2 + (N + a\cos\theta)^2},$$

$$v = \frac{r^2 + (N + a\cos\theta)^2}{\Gamma}, \qquad u = \frac{r_0}{M}\,\frac{N(r - M) - aM\cos\theta}{\Gamma},$$

$$\Delta = r^2 - 2Mr + a^2 - N^2, \quad \Gamma = \frac{r_0}{M}(Mr + N^2 + aN\cos\theta),$$

$$\omega = -\frac{r_0}{M}\,\frac{N\Delta\cos\theta + a(Mr + N^2)\sin^2\theta}{\Delta - a^2\sin^2\theta}.$$

where g_{tt}, $g_{\theta\theta}$ are linearly growing as $r \to \infty$, as well as the dilaton exponential and the electric potential v

Generalized Lense - Thirring metrics

The above solution was recently interpreted as an exact solution representing *generalized Lense-Thirring* (Visser (2020), Kubiznak et al. (2021-2025)...). Original LT metric can be obtained linearizing Kerr as

$$ds^2 = fdt^2 - 2a\sin^2\theta(f-1)dtd\varphi - dr^2/f - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

with $f = 1 - 2m/r$. This is only an appromimate solution of vacuum Einstein equations, which inherits Kerr KT also as approximate quantity:

$$K = 2a\partial_t\partial_\varphi + \partial_\theta^2 + \partial_\varphi^2/\sin^2\theta.$$

But if one performs squaring, keeping an approximate character of solution

$$ds^2 = fdt^2 - dr^2/f - r^2\sin^2\theta\left(d\varphi - adt(f-1)/r^2\right)^2 - r^2d\theta^2,$$

the above Killing tensor becomes exact. This metric was called *generalized LT*, it has a sequence of multidimensional GLT off-shell metrics possessing not only rank two KT tensor, but a tower of higher rank KT tensors obtained via the SN algebra. Our RLDBH can be put in this form, presenting the only known so far *exact* non-vacuum solution of GLT type

Gauged EMDA

See the separate talk by R. Karsanov. The action is

$$S = \frac{1}{16\pi}\int\left(-R+2\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}e^{4\phi}\partial_\mu\kappa\partial^\mu\kappa+\frac{1}{l^2}V-e^{-2\phi}F_{\mu\nu}F^{\mu\nu}-\kappa F_{\mu\nu}\tilde{F}^{\mu\nu}\right)\sqrt{}$$

where a potential terms is present

$$V = 4 + e^{-2\phi} + e^{2\phi}\left(\kappa^2 + 1\right).$$

In this case the BF functions A_2, B_2 become the fourth order polynomials

$$A_2 = a_0 - 2a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4,$$

$$B_2 = b_0 + 2b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4.$$

- Solutions acquire AdS asymptotic
- Topologocal solutions with plane and hyperbolic horizons are possible
- This is the first analytic derivation for them

General bosonic D=4 supergravity action

General bosonic part of extended 4d sugras is a tensor-vector-scalar theory:

$$S=\int d^4x\left[\left(R-\frac{1}{2}f_{AB}\partial_\mu\Psi^A\partial^\mu\Psi^B+V(\Psi)-\frac{1}{2}K_{IJ}F^I_{\mu\nu}F^{J\mu\nu}\right)\sqrt{-g}-\frac{1}{2}H_{IJ}F^I_{\mu\nu}F^J_{\lambda\tau}\epsilon^{\mu\nu\lambda\tau}\right]$$

The scalar moduli parametrize a four-dimensional coset (e.g. $U(8)/E_{7(7)}$ for $\mathcal{N}=8, D=4$ supergravity) with an associated target metric $f_{AB}(\Psi)$.

- Vector fields transform under the same global symmetry implemented by real symmetric matrices K_{IJ}, H_{IJ} depending on scalar fields Ψ^A. For pure $\mathcal{N}=2$ sugra there are no scalar fields.
- ALF black holes for this general action were constructed (Cvetic, Pope, Chow, Compere...) using Harrison transformations within an appropriate three-dimensional sigma-model (the corresponding global group being SO(4,4) up to further dualities in the space of charges). The general metric can be also presented in polynomial form, though not all the metric admit the constrained BF representation used here.

Conclusions

- A class of metrics admitting:
  - two commuting Killing vectors,
  - a second rank non-trivial Killing tensor,
  - two null geodesic shear-free congruences,
  - ensuring separability of the Klein-Gordon equation

  is found by refining BF ansatz for metrics possessing a Killing tensor.
- The corresponding Petrov type is determined as:
  - a sector I_B inside type I such that
  - its algebraically special subclass is only type D
- Direct integrability of $\mathcal{N} = 4$ four-dimensional supergravity on this class of metrics is established and the black hole solutions are reproduced and generalized
- Gauged supergravity black holes solutions with AdS asymptotics were analytically obtained for the first time
- The method looks promising to further extensions

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Thank you for attention!