Photon decay in crossed field with worldline instantons

Semiclassical Analysis Using Worldline Instantons

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Outline

- Non-perturbative $\gamma \to e^+ e^-$ decay in $\mathbf{E} \perp \mathbf{B}$
- Semiclassical approximation
- Euclidean path integrals
- Worldline instantons
- $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 - m^2 |\phi|^2$
- Euclidean scalar QED action:

$$S_E = \int d^4 x \left[-\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 + m^2 |\phi|^2 \right]$$

- Optical theorem approach:

Worldline Formalism for scalar QED

Scalar QED:

$$S_E = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 + m^2 |\phi|^2 \right), Z[A_\mu] = \int D\phi^* D\phi e^{-S_E[A_\mu, \phi^*, \phi]} = e^{-W[A_\mu]}.$$

where A_μ is classical EM field.

$$W[A_\mu] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 - \log\det\left(-D_\mu^2 + m^2\right) \right).$$

Schwinger proper time representation: (Schwinger, 1951)

$$W[A_\mu] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^2 + \int_0^\infty \frac{ds}{s} e^{-m^2 s} Tr\left(e^{sD_\mu^2}\right) \right).$$

Operator $(-D_\mu^2)$ can be interpreted as QM Hamiltonian.

$$Tr\left(e^{sD_\mu^2}\right) = \int d^4x \langle x_\mu | e^{-s(-D_\mu^2)} | x_\mu \rangle = \int_{p.b.c.} Dx_\mu e^{-\int_0^s dr \left(\frac{\dot{x}_\mu^2}{4} + ieA_\mu \dot{x}_\mu \right)}.$$

The Schwinger Effect with Worldline Instantons

Affleck, Alvarez, Manton, 1982

The Schwinger effect (vacuum pair production): $\Gamma = 2\,\mathrm{Im}\,W[A_\mu]$

$$\Gamma = 2\,\mathrm{Im}\int_0^\infty \frac{ds}{s} e^{-m^2 s}\int_{\mathrm{p.b.c.}} \mathcal{D}x_\mu\, e^{-\int_0^1 d\tau\left(\frac{\dot{x}_\mu^2}{4s} + ieA_\mu \dot{x}_\mu\right)}. \quad (1)$$

Take integrals over x_μ and s in the saddle point approximation

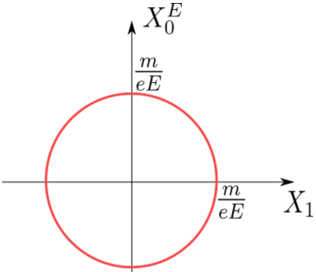
EOMs: $\frac{\ddot{x}_\mu}{2s} - eF_{\mu\nu}\dot{x}_\nu = 0$ and $-m^2 + \frac{\int_0^1 \dot{x}_\mu^2 d\tau}{4s^2} = 0.$

Uniform constant electric field E.

The leading order closed solution is a circle:

$$x_0^{cl} = \frac{m}{eE}\sin(2\pi\tau), \qquad x_1^{cl} = \frac{m}{eE}\cos(2\pi\tau),$$

$$x_2^{cl} = x_3^{cl} = 0, \qquad s = \frac{2\pi}{eE}.$$



$S[x_\mu^{\mathrm{cl}}] \propto$ area

The action on the solution x_μ^{cl} is $S[x_\mu^{\mathrm{cl}}] = \frac{\pi m^2}{eE}$.

Condition: $S[x_\mu^{\mathrm{cl}}] \gg 1, \quad \Gamma \propto e^{-S[x_\mu^{\mathrm{cl}}]}$.

The Schwinger Effect with Worldline Instantons

- Pre-exponential factor: Integrate over quadratic fluctuations δx_μ near classical solution: $x_\mu = x_\mu^{\rm cl} + \delta x_\mu$
- Zero mode: 4-volume VT
- Negative mode: $\delta x_\mu \propto x_\mu^{\rm cl}$ i in pre-exp factor $\rightarrow \Gamma = 2\,\mathrm{Im}\,W[A_\mu] \neq 0$

$$\Gamma = VT \cdot \frac{(eE)^2}{(2\pi)^3} e^{-\frac{\pi m^2}{eE}}$$

Other configurations of EM field

- **Constant uniform magnetic field**
  Solutions of EOMs: sh(ch) instead of sin(cos).
- **Crossed field:**
  $\sqrt{E^2 - H^2}$ instead of E into the classical solution

Worldline Formalism for photon decay in EM field

To compute Γ - the decay width, we use the optical theorem: the decay probability is related to the imaginary part of the polarization operator

$$\Gamma = \frac{1}{2\omega} \mathrm{Im}\, \Pi_{\mu\nu}(k) \epsilon^{\mu} \epsilon^{\nu}$$

- $\Pi_{\mu\nu}$ is the virtual $e^+ e^-$ loop contribution
- ϵ^{μ} is the real photon polarization vector

Polarization Operator

$\Pi_{\mu\nu}(k)$ can be defined as the Fourier transform of two EM currents:
$\Pi_{\mu\nu}(k) = \int d^4 y\, e^{iky} \langle j_{\mu}(y/2) j_{\nu}(-y/2) \rangle$, where the EM current is defined as:
$j_{\mu} = -ie(\varphi^* D_{\mu}\varphi - \varphi D_{\mu}\varphi^*)$

$$\Gamma_{\gamma \to e^+ e^-} = \frac{e^2}{2\omega} \mathrm{Im} \int_0^{\infty} \frac{dT}{T} \int_{p,b,c} Dx_{\mu} \oint d\tau_1 \varepsilon_{\mu}(k) \dot{x}_{\mu}(\tau_1) \oint d\tau_2 \varepsilon^*_{\mu}(k) \dot{x}_{\mu}(\tau_2) e^{-S_m[x_{\mu};\tau_1,\tau_2]}$$

$$S_m = m^2 T + \oint d\tau \left(\frac{\dot{x}_{\mu}^2}{4s} + ieA_{\mu}\dot{x}_{\mu} \right) + ik_{\mu}x_{\mu}(\tau_1) - ik_{\mu}x_{\mu}(\tau_2)$$

General Case: Crossed Fields Setup

Field Configuration:

$$\mathbf{E} = (0, E, 0), \quad \mathbf{B} = (H, 0, 0)$$

Photon Momentum:

$$k_\mu = (\omega, \mathbf{k})$$

$$\mathbf{k} = \omega(\cos\theta\cos\phi, \sin\phi\cos\theta, \sin\theta)$$

Decay Condition:

$$\omega \geq \frac{2m}{\sqrt{\sin^2\theta\left(\cos^2\phi + \left(\frac{E}{H}\right)^2\sin^2\phi\right)}}$$

Field Tensor:

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & -H \\ 0 & 0 & H & 0 \end{pmatrix}$$

Equations of Motion

Variational Principle:

$$\delta S = 0, \quad S = m^2 T + \int_0^1 d\tau \left(\frac{\dot{x}_\mu^2}{4T} + ieA_\mu \dot{x}_\mu \right)$$

Time Component:

$$\frac{\ddot{x}_0}{2T} - eE\dot{x}_2 = \omega \left(\delta(\tau - \tau_1) - \delta(\tau - \tau_2) \right)$$

$$\frac{\ddot{x}_1}{2T} = -\mathcal{K}_\delta$$

Spatial Components:

$$\frac{\ddot{x}_3}{2T} - ieH\dot{x}_2 = -\mathcal{M}_\delta$$

$$\frac{\ddot{x}_2}{2T} + eE\dot{x}_0 + ieH\dot{x}_3 = -\mathcal{S}_\delta$$

Solution Strategy:

- Piecewise solutions in $\tau \in [0, 1/2]$ and $[1/2, 1]$
- Hyperbolic functions for $H > E$
- Trigonometric functions for $E > H$

Special Cases

Pure Magnetic Field ($E \to 0$)

- Decay rate: $\Gamma \sim \exp\left(-\frac{8m^3}{3\omega eH\sin\phi}\right)$
- Trajectories: Hyperbolic functions

Pure Electric Field ($H \to 0$)

- Decay rate: $\Gamma \sim \exp\left(-\frac{8m^3}{3\omega eE\sin\theta}\right)$
- Trajectories: Trigonometric functions

Orthogonal Case ($\theta = 90^\circ, \phi = 90^\circ$) $H > E$

- $\mathbf{k} \perp \mathbf{E}, \mathbf{k} \perp \mathbf{H}$
- Trajectories: $E > H$ Trigonometric, $H > E$ Hyperbolic
- Action: $S = \dfrac{8m^3}{3\omega\sqrt{|H^2 - E^2|}}$. $\Gamma \sim \exp(-S)$

Worldline Instantons for Photon Decay $\gamma \to e^+e^-$

Ext. photon $k_\mu = (\omega, 0, 0, \omega)$

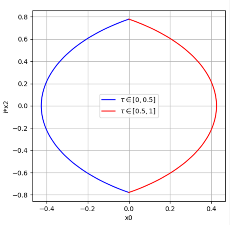
Electric field $(0, E, 0)$
Magnetic field $(H, 0, 0)$.

Classical solution —
two arcs of circle (electric)
two hyperbolas (magnetic)
connected at $\tau_1 = 0, \tau_2 = 0.5$

$$\Gamma \propto e^{-S}.$$

In the limit $\omega \gg 2m$

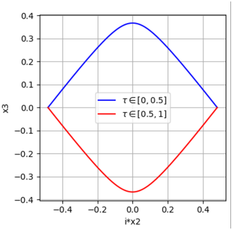
Electric field



$$S = \frac{8m^3}{3\omega eE}$$

Monin, Voloshin, 2010

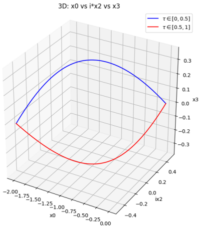
Magnetic field



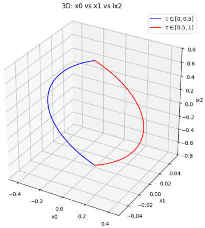
$$S = \frac{8m^3}{3\omega eH}$$

Satunin 2013

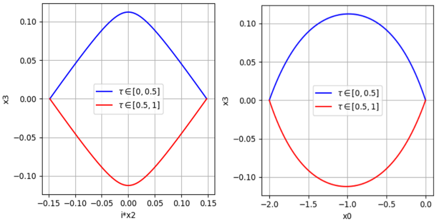
Electric field 3D



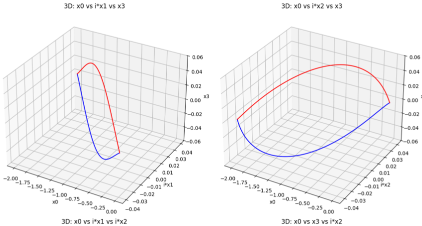
Magnetic field



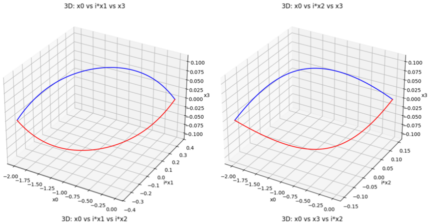
Crossed Fields 2D ($H > E$)



Crossed Fields 3D ($E = 2H$)



Crossed Fields 3D ($H > E$)



Electromagnetic Field Invariants for Crossed Fields ($\mathbf{E} \perp \mathbf{B}$)

Fundamental invariants of EM Field

- $\mathbf{E} \cdot \mathbf{B} = 0$ (orthogonal fields)
- The invariants simplify to:

$$\mathcal{F} = F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$$

$$\mathcal{G} = F_{\mu\nu} \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = 0$$

Other invariants

$$F_{\mu\nu} F_{\nu\lambda} k^\mu k^\lambda \neq 0$$

$$F_{\mu\nu} \tilde{F}_{\nu\lambda} k^\mu k^\lambda = 0$$

$$F_{\alpha\beta} F_{\beta\gamma} F_{\gamma\delta} k^\alpha k^\delta = 0$$

Special Cases $F_{\mu\nu}F_{\nu\lambda}k^{\mu}k^{\lambda}$

Pure Magnetic Field (E $\to$ 0)

$$F_{\mu\nu}F_{\nu\lambda}k^{\mu}k^{\lambda} = -\omega^2 H^2$$

Pure Electric Field (H $\to$ 0)

$$F_{\mu\nu}F_{\nu\lambda}k^{\mu}k^{\lambda} = -\omega^2 E^2$$

Orthogonal Case $(\theta = 90^\circ, \phi = 90^\circ)$ $H > E$

$$F_{\mu\nu}F_{\nu\lambda}k^{\mu}k^{\lambda} = -\omega^2 (H^2 - E^2)$$

Astrophysical Applications

Relevant Objects:

- **Magnetars**: $B \sim 10^{11}$ T
- **Accretion Disks**: Turbulent EM fields
- **Gamma-ray Bursts**: Ultra-relativistic photons

Critical Fields:

$$B_c = 4.41 \times 10^9 \text{ T}$$

$$E_c = 1.32 \times 10^{18} \text{ V/m}$$

Detection Challenges:

- High matter density in accretion disks
- Limited telescope resolution
- Exponential suppression of probability

Conclusion

- **Achievements**:
  - Developed instanton-based framework for photon decay
  - Solved equations of motion for crossed fields
  - Recovered known limits ($E \to 0$, $H \to 0$)
  - Analyzed astrophysical implications
- **Key Insight**: Decay rate controlled by effective action:

$$\Gamma \sim \exp\left(-S_{\text{eff}}\right), \quad S_{\text{eff}} = \frac{8m^3}{3\omega e\sqrt{|H^2 - E^2|}\,\sin\theta_{\text{eff}}}$$

Worldline instantons provide powerful tool for strong-field QED