Inflationary slow-roll parameters in Jordan frame for cosmological $F(R)$ gravity models

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Introduction

- Inflation remains the leading hypothesis describing the early-stage evolution of the Universe.
- Key verifiable predictions — properties of the CMB spectrum (such as spectral tilt index of scalar perturbations n_S, tensor-to-scalar power ratio r, etc.).
- One of the most popular models of inflation is an $F(R)$ model — Starobinsky's $R + R^2/6m^2$ model.
- Until recently, it was able to fit the observations very well, with the only parameter being the scalaron mass m — a very appealing feature!

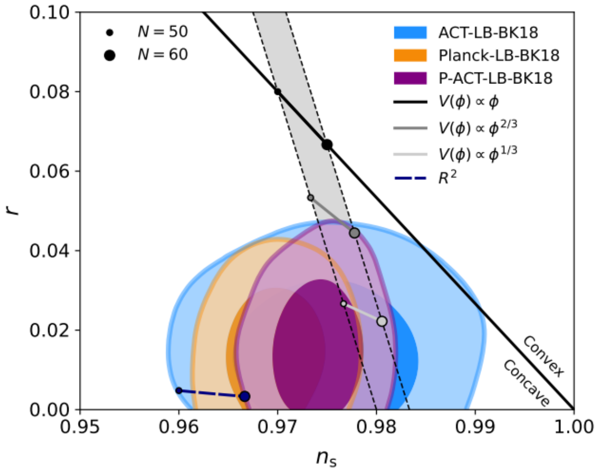
Introduction

- However, in light of the recent Atacama Cosmology Telescope (ACT) data release[1], Starobinsky's model is challenged.

$$\begin{aligned} n_S &= 0.965 \pm 0.004 \quad \text{(Planck-LB-BK18, 68\% CL)} \\ \rightarrow n_S &= 0.974 \pm 0.003 \quad \text{(Planck-ACT-LB-BK18, 68\% CL).} \end{aligned}$$

[1] T. Louis et al. (ACT), *The Atacama Cosmology Telescope: DR6 Power Spectra, Likelihoods and ΛCDM Parameters*, 2025. doi:10.48550/arXiv.2503.14452
E. Calabrese et al. (ACT), *The Atacama Cosmology Telescope: DR6 Constraints on Extended Cosmological Models*, 2025. doi:10.48550/arXiv.2503.14454

Introduction



Introduction

- Inflationary $F(R)$ models are most often studied not in the original (Jordan) frame, but in the Einstein frame:

$$
\begin{aligned}
S_F &= \int d^4x\sqrt{-g}\,F(R) \\
\to S_J &= \int d^4x\sqrt{-g}\,\left(U(\sigma)R - V_J(\sigma)\right) \\
\to S_E &= \int d^4x\sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V_E(\phi)\right).
\end{aligned}
$$

- But why not to study inflationary $F(R)$ gravity models directly in the original frame?

Introduction

- The goal of the presented work is twofold:
    1. Methodological goal: describe a procedure for calculating inflationary predictions of cosmological $F(R)$ gravity models in the Jordan frame;
    2. Research goal: apply the aforementioned procedure to some deformations of Starobinsky's model which might conform with the latest observations.

Initial model

- We start with the action

$$S_F = \int d^4x\sqrt{-g}F(R).$$

- Let's introduce a new action

$$S_J = \int d^4x\sqrt{-g}\left(U(\sigma)R - V(\sigma)\right),$$

where

$$U(\sigma) \equiv F_{,\sigma}(\sigma), \quad V(\sigma) \equiv U(\sigma)\sigma - F(\sigma).$$

- From S_J follows an equation

$$U_{,\sigma}(\sigma)\left(R - \sigma\right) = 0 \implies R = \sigma \, (\text{if } U_{,\sigma} \not\equiv 0),$$

and so the actions S_J and S_F are equivalent. We will work with S_J from now on.

Equations of the model

- The metric equations are

$$U(\sigma)\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)=\nabla_\mu\nabla_\nu U-g_{\mu\nu}\Box U-\frac{1}{2}V(\sigma)g_{\mu\nu}.$$

- The field equation is

$$RU_{,\sigma}-V_{,\sigma}=0.$$

Equations in the FLRW metric

- We introduce the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) ansatz for metric tensor, with the line element being

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2).$$

- With this ansatz, the equations of the model are

$$3UH^2 = -3H\dot{U} + \frac{1}{2}V,$$
$$-U\left(2\dot{H} + 3H^2\right) = \ddot{U} + 2H\dot{U} - \frac{1}{2}V,$$
$$6(\dot{H} + 2H^2)U_{,\sigma} = V_{,\sigma}.$$

Here, the "dots" represent derivatives over t, and $H \equiv \dot{a}/a$.

Equations in the FLRW metric

- By combining the equations of the model, we obtain the following:

$$\ddot{U} + 3H\dot{U} + W_{,U} = 0,$$

where

$$W_{,U} \equiv \frac{1}{3}\left(V_{,U}U - 2V\right) \quad \left(V_{,U} = \frac{V_{,\sigma}}{U_{,\sigma}}\right).$$

- This result resembles the standard field equation of single-field models of inflation:

$$\ddot{\phi} + 3H\dot{\phi} + V_{E,\phi} = 0.$$

Stability analysis of the field equation

- Firsly, using the $(0,0)$ metric equation, we express H in terms of U and $\dot{U}$:

$$H(U,\dot{U}) = \sqrt{\left(\frac{\dot{U}}{2U}\right)^2 + \frac{V}{6U}} - \frac{\dot{U}}{2U}.$$

  Note that we take only the solution which is relevant for the inflation.

- Then, we can rewrite the previously obtained equation for U as a dynamical system:

$$\begin{cases} \dot{q} = p \\ \dot{p} = -3H(q,p)p - W_{,q} \end{cases}.$$

  Here $q \equiv U - U|_{\sigma=0}$.

Stability analysis of the field equation

- Let's introduce a candidate function

$$V_{\text{Lyap.}}(q,p) = \frac{p^2}{2} + W(q),$$

where

$$W(q) = \int_0^q W_{,q'}(q')dq'.$$

- The total time derivative of $V_{\text{Lyap.}}$ is

$$\dot{V}_{\text{Lyap.}} = \frac{\partial V_{\text{Lyap.}}}{\partial q}\dot{q} + \frac{\partial V_{\text{Lyap.}}}{\partial p}\dot{p} = -3Hp^2.$$

Stability analysis of the field equation

- Now we use Lyapunov's second method for stability: if $V_{\text{Lyap.}}$ is such that
  - $V_{\text{Lyap.}}(q,p)=0 \iff (q,p)=(0,0)$,
  - $V_{\text{Lyap.}}(q,p)>0 \iff (q,p)\neq(0,0)$,
  - $\dot{V}_{\text{Lyap.}}\leq 0$ for $(q,p)\neq(0,0)$,

  then $V_{\text{Lyap.}}$ is the Lyapunov function of the dynamical system, and $(q,p)=(0,0)$ is a stable point of the system.
- Moreover, since our system is autonomous, if we can additionally show that the set $\left\{(q,p):(q,p)\neq(0,0)\wedge\dot{V}_{\text{Lyap.}}(q,p)=0\right\}$ is *not* an invariant set under the action of the dynamical system, then $(q,p)=(0,0)$ is an *asymptotically* stable point (the Barbashin-Krasovskii-LaSalle principle).

Stability analysis of the field equation

- All the conditions mentioned are satisfied (at least for $U > 0$; we don't consider values of $U \leq 0$ due to them being unphysical), provided that
  1. $V(q) \geq 0$,
  2. $V(q) = 0 \iff q = 0$[2],
  3. $W(q) > 0 \iff q \neq 0$,
  4. $W_{,q}(q) = 0 \iff q = 0$.
- To conclude: if V and W satisfy these conditions, we have our inflationary attractor!

[2] This condition isn't necessary for the existence of an attractor, but it guarantees the general relativity limit for our $F(R)$ model.

Slow-roll parameters

- Let's introduce the slow-roll parameters

$$\epsilon_H = -\frac{\dot{H}}{H^2}, \quad \epsilon_U = -\frac{\dot{U}}{2HU}, \quad \eta_H = \frac{\dot{\epsilon}_H}{H\epsilon_H}, \quad \eta_U = \frac{\dot{\epsilon}_U}{H\epsilon_U}.$$

- Now we rewrite the metric equations of the model in terms of these parameters:

$$3H^2 U\left(1 - 2\epsilon_U\right) = \frac{1}{2}V,$$

$$\epsilon_H - \epsilon_U = \epsilon_U\left(\epsilon_H + 2\epsilon_U - \eta_U\right).$$

Slow-roll parameters

- Now we derive something like the traditional potential slow-roll parameters of the single-field inflationary models. We start with

$$\ddot{U} + 3H\dot{U} + \frac{1}{3}\left(V_{,U}U - 2V\right) = 0.$$

- Under the attractor approximation $(\ddot{U}/(H\dot{U}) \sim 0)^3$, we get

$$-\dot{U} = \frac{V_{,U}U - 2V}{9H}$$

$$\Longrightarrow \quad -\frac{\dot{U}}{2HU} = \epsilon_U = \frac{1}{3}\frac{V_{,U}U - 2V}{6H^2U} = \frac{1}{3}\frac{V_{,U}U - 2V}{V}(1 - 2\epsilon_U).$$

[3] Note that the attractor approximation follows from slow-roll conditions $(\epsilon, \eta \ll 1)$, since $\eta_U = \ddot{U}/(H\dot{U}) + \epsilon_H + 2\epsilon_U$.

Slow-roll parameters

- Solving for ϵ_U, we get

$$\epsilon_U = \frac{V_{,U}U - 2V}{2V_{,U}U - V} = \frac{V_{,\sigma}U - 2U_{,\sigma}V}{2V_{,\sigma}U - U_{,\sigma}V}.$$

- After obtaining the expression for ϵ_U, the others follow; for η_U we have, by definition,

$$\eta_U = \frac{\dot{\epsilon}_U}{H\epsilon_U} = \frac{\epsilon_{U,U}\dot{U}}{H\epsilon_U} = -2U\epsilon_{U,U} = -2\frac{U}{U_{,\sigma}}\epsilon_{U,\sigma},$$

and we can express ϵ_H in terms of ϵ_U and η_U as

$$\epsilon_H = \epsilon_U \frac{1 + 2\epsilon_U - \eta_U}{1 - \epsilon_U}.$$

Number of e-folds N

- For the number of e-folds since some time t to the end of inflation (at time t_e), we have, by definition

$$N(t) = -\ln \frac{a(t)}{a(t_e)} = -\int_{t_e}^{t} H(\tau) d\tau.$$

- We can express N as a function of σ:

$$N(\sigma) = -\int_{\sigma_e}^{\sigma} \frac{H(t(\sigma'))}{\dot{U}(\sigma')} U_{,\sigma'}(\sigma') d\sigma' = \frac{1}{2}\int_{\sigma_e}^{\sigma} \frac{U_{,\sigma'}(\sigma')}{U(\sigma')\epsilon_U(\sigma')} d\sigma'.$$

CMB observables in terms of slow-roll parameters

- The results for n_S and r in terms of slow-roll parameters differ from the usual ones in the Einstein frame. For brevity, we do not present the derivation here. Up to the second order in slow-roll parameters, they are

$$n_S = 1 - 2\eta_U - 6\epsilon_U^2 - 2\epsilon_U\eta_U, \quad r = 48\epsilon_U^2.$$

- This result explicitly shows a well-known feature of inflationary $F(R)$ models: tensor-to-scalar ratio r is $o(1 - n_S)$ (if ϵ and η are of the same order of magnitude).

CMB observables in terms of slow-roll parameters

- Let's check that the obtained expressions for the CMB observables do conform with the usual ones, expressed in terms of the traditional Einstein-frame slow-roll parameters. It can be shown that

$$\epsilon_V = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{V_{E,\phi}}{V_E}\right)^2 \approx 3\epsilon_U^2,$$

$$\eta_V = M_{\mathrm{Pl}}^2 \frac{V_{E,\phi\phi}}{V_E} \approx -\eta_U + 6\epsilon_U^2 - 2\epsilon_U\eta_U.$$

- Then, expressing ϵ_U and η_U in terms of ϵ_V and η_V, one obtains (in the lowest order)

$$n_S = 1 - 6\epsilon_V + 2\eta_V, \quad r = 16\epsilon_V,$$

as expected.

Sanity check: applying the approximations to Starobinsky's model

- In Starobinsky's model, we have

$$U(\sigma) = \frac{M_{\rm Pl}^2}{2}\left(1 + \frac{\sigma}{3m^2}\right), \quad V(\sigma) = \frac{M_{\rm Pl}^2}{2}\frac{\sigma^2}{6m^2} = \frac{3}{4}M_{\rm Pl}^2 m^2 (u-1)^2,$$

where $u \equiv 2U/M_{\rm Pl}^2$.

- Our approximate slow-roll parameters are

$$\epsilon_U = \frac{2}{3u+1}, \quad \eta_U = \frac{12u}{(3u+1)^2}, \quad \epsilon_H = \frac{2}{3u-1}\left(1 + \frac{4}{(3u+1)^2}\right).$$

Sanity check: applying the approximations to Starobinsky's model

- for N, we obtain

$$N = \frac{1}{2}\int_{u_e}^{u} \frac{du'}{u'\epsilon_U(u')} = \frac{3}{4}(u - u_e) + \frac{1}{4}\ln\frac{u}{u_e} \approx \frac{3}{4}u,$$

  from which we get

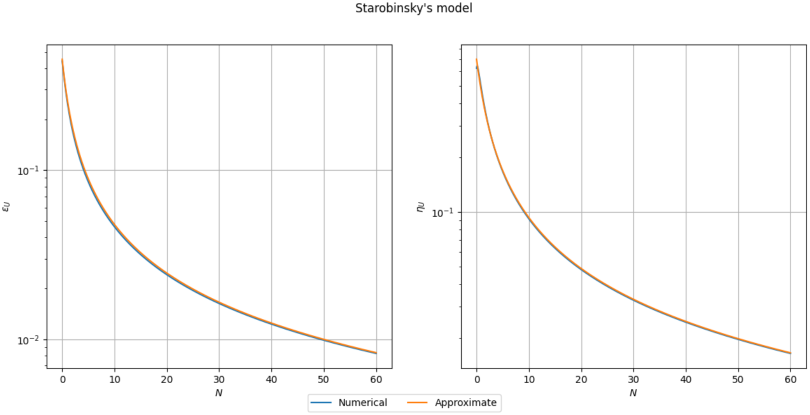
$$u \approx \frac{4}{3}N, \quad \epsilon_U \approx \frac{2}{3u} \approx \frac{1}{2N}, \quad \eta_U \approx \frac{4}{3u} \approx \frac{1}{N}.$$

- Finally, we obtain the CMB observables[4]:

$$n_S \approx 1 - 2/N, \quad r \approx 12/N^2.$$

[4] The numbers of e-folds in the Einstein frame (N_E) and in the Jordan frame (N_J) differ only by $o(N)$; we consider them to be the same.

Sanity check: applying the approximations to Starobinsky's model



Deformation of Starobinsky's model

- Let's take the following one-parametric deformation of Starobinsky's model (which is recovered in the limit $\delta \to 0$):

$$U(\sigma) = \frac{M_{\rm Pl}^2}{2}\left(1 + \frac{1}{\delta}\,\mathrm{arcsinh}\,\frac{\delta\sigma}{3m^2}\right),$$

$$V(\sigma) = \frac{3M_{\rm Pl}^2 m^2}{2\delta^2}\left(\sqrt{\left(\frac{\delta\sigma}{3m^2}\right)^2 + 1} - 1\right).$$

- The corresponding F function is

$$F(\sigma) = \frac{M_{\rm Pl}^2}{2}\left(\sigma + \frac{\sigma}{\delta}\,\mathrm{arcsinh}\,\frac{\delta\sigma}{3m^2} - \frac{3m^2}{\delta^2}\left(\sqrt{\left(\frac{\delta\sigma}{3m^2}\right)^2 + 1} - 1\right)\right).$$

Deformation of Starobinsky's model

- In terms of $u = 2U/M_{\rm Pl}^2$, potential V reads

$$V(u) = \frac{3M_{\rm Pl}^2}{2\delta^2}\left(\cosh\left(\delta(u-1)\right)-1\right) = \frac{3M_{\rm Pl}^2}{\delta^2}\sinh^2\left(\frac{\delta}{2}(u-1)\right).$$

- For ϵ_U, we obtain

$$\epsilon_U = \frac{\delta u - 2\tanh\left(\frac{\delta}{2}(u-1)\right)}{2\delta u - \tanh\left(\frac{\delta}{2}(u-1)\right)}.$$

- Assuming $\delta(u-1) \ll 1$, we get

$$\epsilon_U \approx 2\frac{1+\frac{\delta^2}{12}(u-1)^3}{3u+1+\frac{\delta^2}{12}(u-1)^3}.$$

Deformation of Starobinsky's model

- For N we get, under a further assumption $u_e \ll u \ll (\delta^2/12)^{1/3}$,

$$N \approx \frac{3}{4} u \left(1 - \frac{\delta^2}{48} u^3\right),$$

and so

$$u \approx \frac{4}{3} N \left(1 + \frac{\delta^2}{48} \left(\frac{4}{3} N\right)^3\right).$$

- This result leads us to

$$\epsilon_U \approx \frac{1}{2N} \left(1 + \frac{4\delta^2}{27} N^3\right), \quad \eta_U \approx \frac{1}{N} \left(1 - \frac{4\delta^2}{9} N^3\right).$$

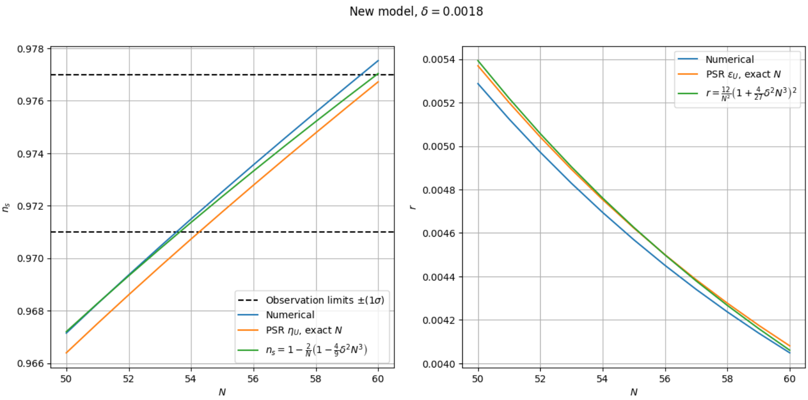
Deformation of Starobinsky's model

- Finally, we obtain the CMB observables:

$$n_S \approx 1 - \frac{2}{N}\left(1 - \frac{4\delta^2}{9}N^3\right), \quad r \approx \frac{12}{N^2}\left(1 + \frac{4\delta^2}{27}N^3\right)^2.$$

- We see that Starobinsky's model is recovered if we set $\delta = 0$. Moreover, n_S seems to be slightly larger for our model, and that's exactly what we wanted!

Deformation of Starobinsky's model



Conclusion

- We have presented a procedure for calculating inflationary observables completely in the original (Jordan) frame.
- We have also provided sufficient conditions for the existence of inflationary attractors in $F(R)$ models.
- We have applied the described procedure to Starobinsky's model (as a "sanity check"), and to a new model, which can provide predictions for the CMB observables that fit the latest data.

Thank you!