# De Sitter entropy: on-shell versus off-shell

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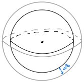
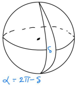
# Result

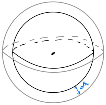
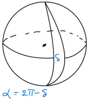
$$S_V^{\text{on shell}} =$$  $$= \int \frac{\delta Q}{T_H} = \int_0^{T_H} \frac{dT_H}{T_H} \int_{r_{ds}=\frac{1}{2\pi T_H}} d^{d-1}x\sqrt{g}\,\frac{\partial \rho}{\partial T_H}, \quad \rho = \frac{\Lambda(T_H)}{8\pi}$$

and

$$S_A^{\text{off shell}} =$$ 

$$= -(\alpha\partial_\alpha - 1)W[\alpha]\Big|_{\alpha=1} \sim A, \quad \text{where} \quad \alpha = \frac{\beta}{\beta_H}$$

- We show that entropy that computed on-shell precisely follows the area law in any dimension of space and in any theory of $f(R)$ gravity and coincide with Wald entropy and generalize it for the entanglenment entropy of minimally coupled scalar field:

 $$= S_V^{\text{on shell}} = S_A^{\text{off shell}} =$$ 


Show that:

$$-\left(\partial_\beta \log Z_{matter}^{ren}\right)\Big|_{\alpha=1} = \langle\hat{E}\rangle = \langle\hat{H}\rangle - \xi\langle\hat{Q}\rangle \quad \rightarrow \quad Z_{matter}^{ren} \neq Tr\left(e^{-\beta H}\right)$$

# Introduction

In Newtonian gravity, consider an ideal gas in a box:

$$Z_1 = \int d^3 x d^3 p e^{-\beta(p^2+mgz)} \sim AT^{5/2}(1-e^{-\frac{mgh}{T}})\frac{1}{mg} \tag{1}$$

$$Z_1 \sim \begin{cases} A, & g \to \infty \\ V, & g \to 0 \end{cases} . \tag{2}$$

In QFT, entropy is assumed to have the following expansion:

$$S_A = \frac{A}{4Gh} + \alpha_0 \log\left(\frac{A}{4Gh}\right) + \sum_n a_n \left(\frac{A}{4Gh}\right)^{-n}, \tag{3}$$

that follows from the analysis of the heat kernel:

$$\bar{K}_{M_\alpha}(s,x,x) = \frac{1}{(4\pi s)^{\frac{d}{2}}} \sum_n \bar{a}_n(x) s^n \tag{4}$$

and

$$\bar{a}_n(x) = a_n^{st}(x) + a_n^{\alpha}(x)(1-\alpha)\delta(\Sigma) \tag{5}$$

We will show that for de Sitter space time:

$$S_V^{\text{on shell}} = S_A^{\text{off shell}} \tag{6}$$

# Geometry of the de Sitter Space-Time

The de Sitter space is a vacuum solution to Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{7}$$

The $d$-dimensional de Sitter space can be visualized as a one-sheeted hyperboloid embedded in a $d+1$-dimensional ambient Minkowski space, described by:

$$dS_d = \{X \in \mathbb{R}^{d+1}, \quad X^\alpha X_\alpha = -X_0^2 + \sum_i X_i^2 = H^{-2}\} \tag{8}$$

The static coordinates of de Sitter space are given by:

$$X = \begin{cases} X^0 = H^{-1}\sqrt{1-r^2H^2}\sinh(tH) \\ X^i = rz_i, \qquad i = 1, \dots, d-1 \\ X^d = \pm H^{-1}\sqrt{1-r^2H^2}\cosh(tH) \end{cases}, \qquad t \in (-\infty, \infty), \; r \in (0, H^{-1}), \tag{9}$$

where $z_i$ are the coordinates on the $(d-2)$-dimensional sphere, and the $\pm$ in $X^d$ defines the right or left de Sitter wedges with the metric:

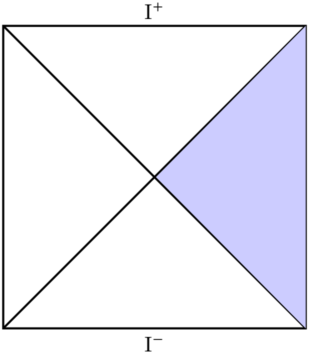
$$ds^2 = -(1 - r^2H^2)dt^2 + \frac{dr^2}{1 - r^2H^2} + r^2 d\Omega_{d-2}^2. \tag{10}$$

The static coordinates are bounded by a Killing horizon:

$$r_{\text{horizon}} = \frac{1}{H}, \tag{11}$$

where the metric degenerates.

# Penrose Diagram



# f(R) Gravity

De Sitter space serves as a solution in modified gravity theory:

$$W = \frac{1}{16\pi} \int d^d x \sqrt{g} \left( f(R) - 2\Lambda \right) + W_{\text{matter}}. \tag{12}$$

The variation of the action with respect to the metric yields:

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f'(R) + \Lambda g_{\mu\nu} = 8\pi T^{\text{matter}}_{\mu\nu}. \tag{13}$$

Thus, the de Sitter space, with Hubble constant equal to $H$, is a solution to the field equations in the absence of matter if the cosmological constant is given by:

$$\Lambda = \left( \frac{1}{2} f(R) - \frac{1}{d} f'(R) R \right) \Bigg|_{R=(d-1)dH^2}. \tag{14}$$

# Thermodynamics of de Sitter Space

An observer in the static patch of de Sitter space sees isotropic radiation with Gibbons-Hawking temperature:

$$T_H = \frac{H}{2\pi}. \tag{15}$$

In the semiclassical approximation, the entropy of de Sitter space obeys the area law:

$$S = \frac{A}{4}. \tag{16}$$

- The Gibbons-Hawking temperature is $10^{-30}K$, which is much lower than the temperature of the cosmic microwave background, $T = 2.73K$.
- This temperature implies the existence of entropy in de Sitter space, which is given by $2.6 \cdot 10^{122}$, vastly exceeding the entropy of all the matter and energy in our Universe, which is on the order of $10^{104}$.

# The de Sitter Temperature: $T_H = \frac{H}{2\pi}$

1. Imaginary Time Periodicity Trick

$$ds^2 = d\epsilon^2 + \epsilon^2 d(H\tau)^2 + H^{-2} d\Omega^2. \tag{17}$$

2. Tunneling Exponent

$$\Gamma \sim e^{-\frac{2\pi}{H}E}. \tag{18}$$

3. KMS Condition

$$G_{BD}(Z_{1,2}) = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta_H H}\phi(x_1)\phi(x_2)\right]. \tag{19}$$

4. Reduced Density Matrix

$$|BD\rangle = \sum_{n_\omega} e^{-\pi/H\omega n_\omega}|n_\omega, L\rangle \otimes |n_\omega, R\rangle \quad \rightarrow \quad \hat{\rho}_R = \mathrm{Tr}_L|BD\rangle\langle BD| = e^{-\beta_H H_R}. \tag{20}$$

5. Unruh-DeWitt Detector

$$\dot{P}(E_i \to E_j) = g^2|m_{ij}|^2 \int dt e^{-(E_j - E_i)t} G_{BD}(Z = \cosh(t)). \tag{21}$$

The probabilities satisfy the relation:

$$\dot{P}(E_i \to E_j) = \dot{P}(E_j \to E_i)e^{-\beta(E_j - E_i)}. \tag{22}$$

# Entropy (Replica Trick)

Let us consider a general curved spacetime $M$ with a Killing horizon, and impose periodic boundary conditions $\tau \sim \tau + \beta$, with a non-fixed inverse temperature $\beta = 2\pi\alpha/H$. In the limit $\alpha \to 1$, the Riemannian tensor contains delta-like surface contributions near the horizon:

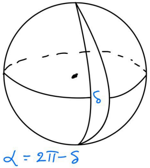
$$\lim_{\alpha\to 1} R^{M_\alpha}_{\mu\nu\rho\sigma} = R^{M}_{\mu\nu\rho\sigma} + 2\pi(1-\alpha)\left(n^1_\mu n^2_\rho - n^2_\mu n^1_\rho\right)\left(n^1_\nu n^2_\sigma - n^2_\nu n^1_\sigma\right)\delta\left(\Sigma\right) + O\left((1-\alpha)^2\right), \quad (23)$$

where $n^{1,2}$ are two orthonormal vectors orthogonal to the horizon surface $\Sigma$. The classical action in the limit $\alpha \to 1$ is given by:

$$W[\alpha] = \\ = -2\pi\alpha\frac{1}{16\pi}\int d^{d-1}x\sqrt{g}\left(f\left(R^M\right) - 2\Lambda\right) - \\ -\alpha(1-\alpha)\frac{1}{4}\int d^{d-1}x\sqrt{g}\delta\left(\Sigma\right)f'\left(R^M\right) + O\left((1-\alpha)^2\right). \quad (24)$$

Applying the replica formula, we find that the only surface term contributes to the entropy:

$S_A^{\text{off shell}} =$




$$= -(\alpha\partial_\alpha - 1)W[\alpha]\Big|_{\alpha=1} = \frac{A}{4}f'(R). \quad (25)$$

# Local Thermodynamics

The first law of thermodynamics with non-zero pressure is given by:

$$T_H dS_V^{\text{on shell}} = \delta Q = dE + p dV. \tag{26}$$

Let us assume that:

- The temperature is given by the Gibbons-Hawking relation: $T_H = \frac{H}{2\pi}$.
- The energy density is given by the cosmological term:

$$\rho(2\pi T_H) = \frac{\Lambda(2\pi T_H)}{8\pi}, \quad \Lambda = \left(\frac{1}{2} f(R) - \frac{1}{d} f'(R) R\right) + 8\pi T_0^0 \tag{27}$$

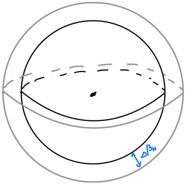
- The pressure is given by $p = -\rho$.

Thus, the total entropy of the de Sitter vacuum state is given by:

$$S_V^{\text{on shell}} = \int \frac{\delta Q}{T_H} = \int_0^{T_H} \frac{dT_H}{T_H} \int_{r_{ds} = \frac{1}{2\pi T_H}} d^{d-1}x \sqrt{g}\, \frac{\partial \rho}{\partial T_H}. \tag{28}$$

which leads to the conclusion that the entropy of the de Sitter vacuum state is given by:

$S_V^{\text{on shell}} =$



$$= \frac{A}{4} f'(R), \tag{29}$$

which exactly coincides with the Wald entropy of the Killing horizon.

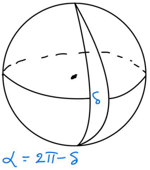
# Why does the bulk entropy of matter obeys the area law?

Let us expand the effective action of the matter field on a Euclidean space with a conical singularity near the Gibbons-Hawking temperature:

$$\log Z_\alpha = -W_0\left(T_H\right) - (1-\alpha)W_1\left(T_H\right) + O\left((1-\alpha)^2\right), \quad \text{where} \quad \alpha = \frac{\beta}{\beta_H}, \quad \beta_H = \frac{1}{T_H}. \tag{30}$$

Then the off-shell method to calculate the entropy gives:

$$S_A^{\text{off shell}} =$$




$$= -(\alpha\partial_\alpha - 1)\log Z_\alpha\Big|_{\alpha=1} = -W_0(T_H) - W_1(T_H). \tag{31}$$

The energy of the system is defined as follows:

$$\beta_H E = -\beta\partial_\beta \log Z_\alpha\Big|_{\beta=\beta_H} = -\partial_\alpha \log Z_\alpha\Big|_{\alpha=1} = -W_1(T_H). \tag{32}$$

The free energy obeys the standard relation:

$$F(T_H) = T_H W_0(T_H) = E(T_H) - T_H S_A^{\text{off shell}}(T_H). \tag{33}$$

At the same time, if we take the derivatives of the effective action with respect to the Gibbons-Hawking temperature, we obtain:

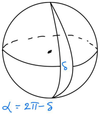
$$-\beta_H \frac{d}{d\beta_H}\log Z_{\alpha=1} = T_H \frac{d}{dT_H}\log Z_{\alpha=1} = H\frac{d}{dH}\log Z_{\alpha=1} = \tag{34}$$

$$= 2\int d^d x g^{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\log Z_{\alpha=1} = -\int d^d x\sqrt{g}\langle T^{\mu}_{\mu}\rangle^{\text{matter}} = -d\int d^d x\sqrt{g}\langle T^{0}_{0}\rangle^{\text{matter}}.$$

Hence free energy can be computed as follows:

$$\beta_H F(T_H) = -d\int\frac{dT_H}{T_H}\int_{r=\frac{1}{2\pi T_H}} d^d x\sqrt{g}\rho_{\text{matter}}(T_H) \tag{35}$$

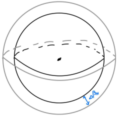
and as a result:




$$= S_A^{\text{off shell}}(\text{matter}) = \beta_H E(T_H) - \beta_H F(T_H) = \tag{36}$$

$$= \int dT_H \frac{\partial}{\partial T_H}\int_{r=\frac{1}{2\pi T_H}} d^d x\sqrt{g}\rho_{\text{matter}}(T_H) + d\int\frac{dT_H}{T_H}\int_{r=\frac{1}{2\pi T_H}} d^d x\sqrt{g}\rho_{\text{matter}}(T_H) =$$

$$= \int\frac{dT_H}{T_H}\int_{r=\frac{1}{2\pi T_H}} d^{d-1}x\sqrt{g}\,\frac{\partial\rho_{\text{matter}}(T_H)}{\partial T_H} = S_V^{\text{on shell}}(\text{matter}) =$$



Hence, the equality between the two expressions for the energy becomes evident. Nevertheless, we use at an intermediate step that derivative of the Euclidean function integral with respect to inverce temperature gives the energy of the system.

This relaion holds if there is the equality between the Euclidean functional integral and the thermal partition function:

$$Z_\alpha = Z^E \stackrel{?}{=} Z^C = \mathrm{Tr}[e^{-\beta \hat{H}}], \tag{37}$$

and if the energy of the system coincides with the expectation value of the Hamiltonian:

$$\langle \hat{E} \rangle \stackrel{?}{=} \langle \hat{H} \rangle = -\partial_\beta \log Z^C. \tag{38}$$

Both equalities hold in the absence of Killing horizon but for space-times with Killing horizons.

The diffence between the energy defined by the stress-energy tensor and the canonical Hamiltonian is given by:

$$E = H - \xi Q, \tag{39}$$

where $(\xi Q)$ is a boundary term, which vanishes for minimally coupled field theory or in the abcense of Killing horizons,

# Off shell method

To compute etropy off shell we considere the Euclidean static de Sitter space-time with not fixed period $\beta$ in time. Then expanding renormalized effenctive action $W_\beta^{ren}$ in terms of $\beta - \beta_H$ we will complute the renormalized entanglenment etropy at Gibbons-Hawkong temperature:

$$S_A^{\text{off shell}} = (\alpha\partial_\alpha - 1)\, W_\alpha^{ren}\Big|_{\alpha=1}, \qquad \alpha = \frac{\beta}{2\pi/H} \tag{40}$$

Since $\beta$ is an arbituary the Euclidean manifold has conical singularities at the horizon surface $\Sigma$.

$$W^{ren} = W_{gr}^{bar} + W_{matter}^{div} + O((1-\alpha)^2). \tag{41}$$

$$W_{gr} = \tag{42}$$

$$= \alpha \int_M d^4x\sqrt{g}\left(-\frac{1}{16\pi G_B}\left(R^M + 2\Lambda_B\right) + c_1^B R^M R^M + c_2^B R^M_{\mu\nu} R^{M,\mu\nu} + c_3^B R^M_{\mu\nu\rho\sigma} R^{M,\mu\nu\rho\sigma}\right) +$$

$$+4\pi(1-\alpha)\int d\Sigma\left(-\frac{1}{16\pi G_B} + 2c_1^B R^M + c_2^B R^M_{\mu\nu} n_i^\mu n_i^\nu + 2c_3^B R^M_{\mu\nu\rho\sigma} n_i^\mu n_i^\rho n_j^\nu n_j^\sigma\right) +$$

$$+O\left((1-\alpha)^2\right).$$

The Euclidean path integral for the non-minimally coupled scalar field is:

$$Z^{div}_{matter} = \int D\phi\ e^{-\frac{1}{2}\int_{M_\alpha} d^4x\sqrt{g}\ \phi\left(-\Box + m^2 + \xi R^{M_\alpha}\right)\phi}. \tag{43}$$

Ricci scalar contains delta-like singularity on the horizon surface:

$$R^{M_\alpha} = R^M + 4\pi(1-\alpha)\delta(\Sigma) + O\big((1-\alpha)^2\big). \tag{44}$$

We can expand the terms containing the delta function up to the first order in $(1-\alpha)$:

$$\log Z^{div}_{matter} = \log \bar{Z}^{div}_{matter} - 2\pi\xi(1-\alpha)\int d\Sigma \langle\phi^2\rangle^{div}_{\bar{Z}}, \tag{45}$$

where $\bar{Z}_{matter}$ is the Euclidean functional integral on the conical manifold but without delta-like singularity contribution in the action:

$$\bar{Z}^{div}_{matter} = \int D\phi\ e^{-\frac{1}{2}\int_{M_\alpha} d^4x\sqrt{g}\ \phi\left(-\Box + m^2 + \xi R^M\right)\phi}. \tag{46}$$

To calculate $\bar{Z}^{div}_{matter}$ and $\langle\phi^2\rangle^{div}_{\bar{Z}}$ we can use the heat kernel expansions:

$$\log \bar{Z}^{div}_{matter} = \frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}e^{-sm^2}\int_{M_\alpha} d^4x\sqrt{g}\ \bar{K}_{M_\alpha}(s,x,x), \tag{47}$$

and

$$\langle\phi^2\rangle^{div}_{\bar{Z}} = \int_{\epsilon^2}^{\infty} ds e^{-sm^2}\bar{K}_{M_\alpha}(s,x,x). \tag{48}$$

The heat kernel on a manifold with a conical singularity is given by:

$$\bar{K}_{M_\alpha}(s,x,x) = \frac{1}{(4\pi s)^{\frac{d}{2}}}\sum_n \bar{a}_n(x)s^n, \tag{49}$$

with the coefficients expressed as:

$$\bar{a}_n(x) = a^{st}_n(x) + (1-\alpha)\,a^{\alpha}_n(x)\delta(\Sigma) + O\left((1-\alpha)^2\right), \tag{50}$$

where

$$a^{st}_0(x) = 1, \quad a^{st}_1(x) = \left(\frac{1}{6}-\xi\right)R^M, \tag{51}$$

$$a^{st}_2(x) = \frac{1}{180}R^M_{\mu\nu\rho\sigma}R^{M,\mu\nu\rho\sigma} - \frac{1}{180}R^M_{\mu\nu}R^{M,\mu\nu} - \frac{1}{6}\left(\frac{1}{5}-\xi\right)\Box R^M + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 R^M R^M.$$

Additional terms $a^{\alpha}_n(x)$ arise due to the presence of the conical singularity:

$$a^{\alpha}_0(x) = 0, \quad a^{\alpha}_1(x) = 4\pi, \tag{52}$$

$$a^{\alpha}_2(x) = \frac{2}{3}\pi\left(\frac{1}{6}-\xi\right)R^M - \frac{1}{45}\pi\left(R^M_{\mu\nu}n^{\mu}_i n^{\nu}_i - 2R^M_{\mu\nu\rho\sigma}n^{\mu}_i n^{\rho}_i n^{\nu}_j n^{\sigma}_j\right).$$

Thus, it can be seen that up to the zeroth and first orders in $(1-\alpha)$ all divergences are completely removed by the standard renormalization of the gravitational coupling constants:

$$W_{ren} = W_{gr} + W_{matter} = \tag{53}$$

$$= \alpha \int_M d^4x\sqrt{g}\left(-\frac{1}{16\pi G_{ren}}\left(R^M - 2\Lambda_{ren}\right) + c_1^{ren} R^M R^M + c_2^{ren} R^M_{\mu\nu} R^{M,\mu\nu} + c_3^{ren} R^M_{\mu\nu\rho\sigma} R^{M,\mu\nu\rho\sigma}\right) +$$

$$+4\pi(1-\alpha)\int d\Sigma\left(-\frac{1}{16\pi G_{ren}} + 2c_1^{ren} R^M + c_2^{ren} R^M_{\mu\nu} n_i^\mu n_i^\nu + 2c_3^{ren} R^M_{\mu\nu\rho\sigma} n_i^\mu n_i^\rho n_j^\nu n_j^\sigma\right) +$$

$$+O\left((1-\alpha)^2\right) + \text{finite terms.}$$

The renormalized constants are given by:

$$\frac{1}{G_{ren}} = \frac{1}{G_B} + \frac{(6\xi-1)}{12\pi\epsilon^2} + \frac{m^2(6\xi-1)}{6\pi}\log(\epsilon m), \tag{54}$$

$$c_1^{ren} = c_1^B - \frac{(1-6\xi)^2 \log(em)}{1152\pi^2},$$

$$c_2^{ren} = c_2^B + \frac{1}{2880\pi^2}\log(\epsilon m),$$

$$c_3^{ren} = c_3^B - \frac{1}{2880\pi^2}\log(\epsilon m),$$

$$\frac{\Lambda_{ren}}{G_{ren}} = \frac{\Lambda_B}{G_B} - \frac{1}{8e^4\pi} + \frac{m^2}{4e^2\pi} + \frac{m^4\log(em)}{4\pi}.$$

Thus, it is sufficient to utilize the standard renormalization procedure up to the second order in $(1-\alpha)^2$

We use the minimal DeWitt-Schwinger subtraction scheme to renormalize the effective action:

$$\log Z^{ren}_{matter} = \tag{55}$$

$$= \log \bar{Z}^{div}_{matter} - 2\pi\xi(1-\alpha)\int d\Sigma\langle\phi^2\rangle_{\bar{Z}} - \frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}\frac{e^{-sm^2}}{(4\pi s)^2}\left(a_0 + a_1 s + a_2 s^2\right).$$

And zeta function regularization instead of the heat kernel one, since the expansion of $\log \bar{Z}^{div}_{matter}$ in terms of $(1-\alpha)$ has been computed using the zeta function techniques in Fursaev:1993.
As a result:

$$\log Z^{ren}_{matter} = -W_0\left(T_H\right) - (1-\alpha)W_1\left(T_H\right), \tag{56}$$

The energy of the system is defined as the variation of the effective action (34):

$$\langle \hat{E} \rangle = \frac{H^2}{8\pi} \frac{d}{dH} \Big( \log Z^{ren}_{matter} \Big|_{\alpha=1} \Big). \tag{57}$$

Then we show that:

$$\boxed{-\Big( \partial_\beta \log Z^{ren}_{matter} \Big) \Big|_{\alpha=1} = \langle \hat{E} \rangle}. \tag{58}$$

This is the one of main relation since now it is clear that if the derivatives with respect to the inverse temperature of the renormalized effective action give exactly the energy of the system, then there should be the equality between the off-shell and on-shell methods.

At first glance, this relation seems obvious; nevertheless, due to the difference between the energy defined by the stress-energy tensor and the canonical Hamiltonian:

$$\langle \hat{E} \rangle = \langle \hat{H} \rangle - \xi \langle \hat{Q} \rangle, \tag{59}$$

it is evident that the statistical mechanics relation is not fulfilled:

$$-\Big( \partial_\beta \log Z^{ren}_{matter} \Big) \Big|_{\alpha=1} = \langle \hat{H} \rangle - \xi \langle \hat{Q} \rangle \neq \langle \hat{H} \rangle. \tag{60}$$

In our case, the expectation value of the boundary term $Q$ is given by:

$$\langle \hat{Q} \rangle = H \int_{r=\frac{1}{H}} d\Sigma \langle \phi^2 \rangle_{\bar{Z}}. \tag{61}$$

If we consider the renormalized effective action:

$$\log Z_{matter}^{ren} = \log \bar{Z}_{matter}^{ren} - 2\pi\xi(1-\alpha) \int d\Sigma \langle \phi^2 \rangle_{\bar{Z}}, \tag{62}$$

and take the derivatives with respect to the inverse temperature, we obtain the following relation:

$$-\left(\partial_\beta \log Z_{matter}^{ren}\right)\Big|_{\alpha=1} = -\left(\partial_\beta \log \bar{Z}_{matter}^{ren}\right)\Big|_{\alpha=1} - \xi \langle \hat{Q} \rangle. \tag{63}$$

As a result $\bar{Z}$ is equal to the thermal partition function since it satisfies the statistical mechanics relation:

$$-\left(\partial_\beta \log \bar{Z}_{matter}^{ren}\right)\Big|_{\alpha=1} = \langle \hat{H} \rangle = \frac{\mathrm{Tr}\left[e^{-\beta_H \hat{H}} \hat{H}\right]}{\mathrm{Tr}\left[e^{-\beta_H \hat{H}}\right]}. \tag{64}$$

At the same time, the free energies that arise from $Z_{\alpha=1}$ and $\bar{Z}_{\alpha=1}$ are equal to each other when $\beta = \beta_H$ or $\alpha = 1$. Then the entropy takes the form:

$$S_A^{\text{off shell}} = -\left(\beta\partial_\beta - 1\right) \log Z_{matter}^{ren}\Big|_{\beta=\beta_H} = \bar{S} - \xi \beta_H \langle \hat{Q} \rangle. \tag{65}$$

Here, $\bar{S}$ is the standard definition of the entropy: $\bar{S} = \beta_H \langle \hat{H} \rangle - \beta_H F$, which arises from the fact that the Euclidean functional integral $\bar{Z}$ is equal to the partition