On natural volume forms on pseudo-Finslerian manifolds with *m*th root metrics

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Motivations

- There are attempts to solve problems of cosmology and quantum gravity with the help of Finslerian geometry.
- Let M^n be an n-dimensional oriented manifold. A *Finslerian* length element is $ds = F(x, dx) \geq 0$, where
$F(x, \lambda dx) = \lambda F(x, dx)$ and
$F(x, dx + dy) \leq F(x, dx) + F(x, dy)$ for any $x \in M^n$, $\lambda > 0$,
and $dx, dy \in T_x M^n$.
- In a *pseudo-Finslerian* manifold, it is possible that
$ds = F(x, dx) \ngeq 0$ and $F(x, dx + dy) \nleq F(x, dx) + F(x, dy)$
for some $dx, dy \in T_x M^n$. This case is more realistic in the relativistic physics.

Volume forms

- Field theories require integration on manifolds, i.e., *volume forms*.
- A differential n-form $\omega = \omega_{12\ldots n}(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ is called a volume form on M^n if $\omega_{12\ldots n}(x) > 0$.
- The key property is $\omega'_{12\ldots n}(x') = \omega_{12\ldots n}(x) \det \left[\frac{\partial x^j}{\partial x'^i}\right]$ under coordinate transformations $x^j = x^j(x'^1, \ldots, x'^n)$, i.e., the transformation law of a scalar density (here $\det \left[\frac{\partial x^j}{\partial x'^i}\right] > 0$).

Natural volume forms

- There are *natural* volume forms which are generated by additional geometric structures on a manifold, e.g., a metric.
- Let $\mathrm{Vol}(X)$ be the standard Euclidean volume of $X \subset \mathbb{R}^n$.
- The natural volume form on a Finslerian manifold is

$$\omega = \frac{\mathrm{Vol}(\text{the unit ball in } \mathbb{R}^n)}{\mathrm{Vol}(\{v \in T_x M^n \mid F(x,v) \leq 1\})} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

- In pseudo-Finslerian manifolds, $\mathrm{Vol}(\{v \in T_x M^n \mid F(x,v) \leq 1\}) = \infty$ for many cases. Thus, the above standard definition of the natural volume form *fails*!

The *m*th root metrics

- We consider n-dimensional oriented pseudo-Finslerian manifolds with the so-called "*m*th root metrics"

  $$ds^m = g_{i_1 i_2 \dots i_m}(x) dx^{i_1} dx^{i_2} \cdots dx^{i_m},$$

  where $m > 1$, $g_{i_1 i_2 \dots i_m}(x)$ are components of a symmetric covariant tensor field, and $i_1, i_2, \dots, i_m = 1, 2, \dots, n$.
- Our purpose is to define natural volume forms on these manifolds.

The main ideas

- We will construct scalar densities from $g_{i_1 i_2 \dots i_m}(x)$.
- $g'_{i_1 i_2 \dots i_m}(x') = \frac{\partial x^{j_1}}{\partial x'^{i_1}} \frac{\partial x^{j_2}}{\partial x'^{i_2}} \cdots \frac{\partial x^{j_m}}{\partial x'^{i_m}} g_{j_1 j_2 \dots j_m}(x)$
- We consider the contraction
  $G'_{i_1^1 i_2^1 \dots i_n^1} \equiv \varepsilon^{i_1^2 i_2^2 \dots i_n^2} \dots \varepsilon^{i_1^m i_2^m \dots i_n^m} g'_{i_1^1 i_1^2 \dots i_1^m} g'_{i_2^1 i_2^2 \dots i_2^m} \cdots g'_{i_n^1 i_n^2 \dots i_n^m}$
  (two-level indices run from 1 to n as well and $\varepsilon^{i_1^2 i_2^2 \dots i_n^2}$ is the Levi-Civita symbol with $\varepsilon^{12 \dots n} = 1$).
- It turns out that
  $G'_{i_1^1 \dots i_n^1} = \left(\det\left[\frac{\partial x^j}{\partial x'^i}\right]\right)^{m-1} \frac{\partial x^{j_1^1}}{\partial x'^{i_1^1}} \cdots \frac{\partial x^{j_n^1}}{\partial x'^{i_n^1}} G_{j_1^1 \dots j_n^1}$, where
  $G_{j_1^1 j_2^1 \dots j_n^1} \equiv \varepsilon^{j_1^2 j_2^2 \dots j_n^2} \dots \varepsilon^{j_1^m j_2^m \dots j_n^m} g_{j_1^1 j_1^2 \dots j_1^m} g_{j_2^1 j_2^2 \dots j_2^m} \cdots g_{j_n^1 j_n^2 \dots j_n^m}$,
  i.e., the transformation law of a tensor density of weight $m - 1$.
- $G'_{i_1^1 i_2^1 \dots i_n^1}$ and $G_{j_1^1 j_2^1 \dots j_n^1}$ are symmetric for odd $m > 1$ and antisymmetric for even $m > 0$.

The natural volume form: even $m > 0$, integer $n > 1$

- We will use the notation $\mathrm{hdet}[g_{i_1 i_2 \dots i_m}(x)] \equiv G_{12\dots n}$. It is the simplest *hyperdeterminant* introduced by A. Cayley (in other terms and with respect to independent variables, not functions) in 1843. For $m = 2$, $\mathrm{hdet}[g_{i_1 i_2}] = \det[g_{i_1 i_2}]$.
- If $m > 0$ is even, then $\left|\mathrm{hdet}[g'_{i_1 i_2 \dots i_m}]\right|^{1/m} = \left|\mathrm{hdet}[g_{i_1 i_2 \dots i_m}]\right|^{1/m} \det\left[\frac{\partial x^j}{\partial x'^i}\right]$, i.e., the transformation law of a scalar density.
- We propose $\omega = \left|\mathrm{hdet}[g_{i_1 i_2 \dots i_m}(x)]\right|^{1/m} dx^1 \wedge \dots \wedge dx^n$ as the natural volume form on the n-dimensional oriented pseudo-Finslerian manifold (M^n, ds^m) for even $m > 0$ and integer $n > 1$.

The example: $m = 4$, $n = 2$

- Let us consider (M^2, ds^4).
- The metric is $ds^4 = g_{i_1 i_2 i_3 i_4}(x^1, x^2) dx^{i_1} dx^{i_2} dx^{i_3} dx^{i_4}$, where i_1, i_2, i_3, $i_4 = 1, 2$.
- The natural volume form on (M^2, ds^4) is $\omega = \left| g_{1111} g_{2222} - 4 g_{1112} g_{1222} + 3 (g_{1122})^2 \right|^{1/4} dx^1 \wedge dx^2$.

The natural volume form: odd $m > 1$, even $n > 0$

- Unfortunately, $\left|\mathrm{hdet}[g_{i_1 i_2 \ldots i_m}]\right|^{1/m}$ is not a scalar density for odd $m > 1$. However, we can iterate the above construction by replacing $g'_{i_1^1 i_1^2 \ldots i_1^m}$ with $G'_{i_1^1 i_1^2 \ldots i_1^n}$ everywhere.
- If $m > 1$ is odd and $n > 0$ is even, then $\left|\mathrm{hdet}[G'_{i_1 i_2 \ldots i_n}]\right|^{1/(mn)} = \left|\mathrm{hdet}[G_{i_1 i_2 \ldots i_n}]\right|^{1/(mn)} \det\left[\frac{\partial x^j}{\partial x'^i}\right]$, i.e., the transformation law of a scalar density.
- We propose $\omega = \left|\mathrm{hdet}[G_{i_1 i_2 \ldots i_n}(x)]\right|^{1/(mn)} dx^1 \wedge \cdots \wedge dx^n$ as the natural volume form on the n-dimensional oriented pseudo-Finslerian manifold (M^n, ds^m) for odd $m > 1$ and even $n > 0$.

The example: $m = 3$, $n = 2$

- Let us consider (M^2, ds^3).
- The metric is $ds^3 = g_{i_1 i_2 i_3}(x^1, x^2) dx^{i_1} dx^{i_2} dx^{i_3}$, where i_1, i_2, $i_3 = 1, 2$.
- The natural volume form on (M^2, ds^3) is

$$\omega = \Big| -(g_{111}g_{222})^2 + 6g_{111}g_{112}g_{122}g_{222} - 4g_{111}(g_{122})^3 \\ - 4(g_{112})^3 g_{222} + 3(g_{112}g_{122})^2 \Big|^{1/6} dx^1 \wedge dx^2.$$

Summary

- We have defined the natural volume forms ω on the n-dimensional oriented pseudo-Finslerian manifolds (M^n, ds^m) with the mth root metrics $ds^m = g_{i_1 i_2 \dots i_m}(x) dx^{i_1} dx^{i_2} \cdots dx^{i_m}$.
- $\omega = \left|\mathrm{hdet}[g_{i_1 i_2 \dots i_m}(x)]\right|^{1/m} dx^1 \wedge \cdots \wedge dx^n$ for even $m > 0$ and integer $n > 1$.
- $\omega = \left|\mathrm{hdet}[G_{i_1 i_2 \dots i_n}(x)]\right|^{1/(mn)} dx^1 \wedge \cdots \wedge dx^n$ for odd $m > 1$ and even $n > 0$.

Technical details and references can be found in Solov'yov A.V., Russ. J. Math. Phys., 2024, vol. 31, no. 2, pp. 317–324 (arXiv:2312.07518 [math.DG]).

Thank you for attention!