Tachyonic and parametric resonances for massive particle production in an intense plane wave background.

arXiv:2504.08439 [hep-ph]

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Introduction

- Particle production on a plane wave background $\Phi_0 \cos(\omega t - kx)$
  - Massless or light plane wave
  - Parametric resonance

  A.Arza. PRD 105 (2022) 3, 036004 (2009.03870)
- Particle produuction in condensate $\Phi_0 \cos(m_\phi t)$
  - Preheating stage after inflation
  - Enhance of parametric resonance
  - It is well studied in the context of the Mathieu equation

  Kofman, Linde, Starobinskiy 94, 97 (hep-th/9405187,hep-ph/9704452)
  Khlebnikov, Tkachev 96 (hep-ph/9608458)
  Duffaeux et al 06 (hep-ph/0602144)
- Comparison of two approaches: a low-mass plane wave transforms into a condensate using a boost
- Parametric resonance for scalar electrons
  Heinzl, Ilderton, King PRD 94, 065039 (2016).

The model $g\phi\chi^2$

- Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}m_\chi^2\chi^2 - g\phi\chi^2,$$

- Classic wave ϕ in the initial state, $\phi(\vec{x}, t) = \Phi_0 \cos(px - \omega t)$
- Production through perturbation theory only for $m_\phi > 2m_\chi$
- Non-perturbative production χ particles even if $m_\phi < 2m_\chi$ for large amplitude Φ_0
- The exact solution of the Heisenberg equation for χ amplitude

Heisenberg equation. Plane wave

Equations of motion,

$$(\Box + m_\phi^2)\phi = -g\chi^2, \leftarrow \text{neglected in early times}$$
$$(\Box + m_\chi^2)\chi = -2g\phi\chi.$$

The Fourier transform for χ_k

$$\chi = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\Omega_{\vec{k}}}}\left(\chi_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + \chi_{\vec{k}}(t)^\dagger e^{-i\vec{k}\cdot\vec{x}}\right),$$

where $\Omega_{\vec{k}} = \sqrt{k^2 + m_\chi^2}$ and $[\chi_{\vec{k}}, \chi_{\vec{k}'}] = 0,\ [\chi_{\vec{k}}, \chi\dagger_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k} - \vec{k}')$.
Bogolyubov transformations $A_{\vec{k}} = \chi_{\vec{k}} + \chi^\dagger_{-\vec{k}}$, equation in the terms A_k:

$$(\partial_t^2 + \Omega_{\vec{k}}^2)A_{\vec{k}} = -\omega_{\vec{p}}^2\alpha\left(\sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}-\vec{p}}}}A_{\vec{k}-\vec{p}}e^{-i\omega_{\vec{p}}t} + \sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}+\vec{p}}}}A_{\vec{k}+\vec{p}}e^{i\omega_{\vec{p}}t}\right),$$

where $\alpha \equiv \frac{g\Phi_0}{\omega_{\vec{p}}^2} = \frac{g\sqrt{2\rho_\phi}}{\omega_p^3}$.

Plane wave. Low-mass approximation

$\chi_k \equiv a_k(t)e^{-i\Omega_k t}$ Looking for resonance in a_k

$$e^{-i\Omega_{\vec{k}}t}(\ddot{a}_{\vec{k}} - 2i\Omega_{\vec{k}}\dot{a}_{\vec{k}}) = \sigma_{\vec{p}-\vec{k}}a^\dagger_{\vec{p}-\vec{k}}e^{i(\Omega_{\vec{p}-\vec{k}}-\omega_{\vec{p}})t},$$

RWA $\ddot{a}_k \to 0$

$$-2i\Omega_{\vec{k}}\dot{a}_{\vec{k}} = \sigma_k a^\dagger_{\vec{p}-\vec{k}}e^{i(\Omega_{\vec{k}}+\Omega_{\vec{p}-\vec{k}}-\omega_{\vec{p}})t},$$

where $\sigma_k = g\sqrt{\frac{\rho_\phi/2}{\omega^2\Omega_k\Omega_{p-k}}}$, $\sigma_{p-k} = -\omega^2\alpha\sqrt{\frac{\Omega_k}{\Omega_{p-k}}}$

Resonance solution $a_{\vec{k}}(t) =$

$e^{i\epsilon_{\vec{k}}t/2}\Big(a_{\vec{k}}(0)(\cosh(s_{\vec{k}}t) - i\frac{\epsilon_{\vec{k}}}{2s_{\vec{k}}}\sinh(s_{\vec{k}}t)) + i\frac{\sigma_{\vec{p}-\vec{k}}}{2s_{\vec{k}}\Omega_{\vec{k}}}a^\dagger_{\vec{p}-\vec{k}}(0)\sinh(s_{\vec{k}}t)\Big)$,

where $s_{\vec{k}} = \frac{1}{2}\sqrt{\frac{\sigma^2_{\vec{p}-\vec{k}}}{\Omega^2_{\vec{k}}} - \epsilon^2_{\vec{k}}}$ и $\epsilon_{\vec{k}} = \epsilon_{\vec{p}-\vec{k}} = \Omega_{\vec{k}} + \Omega_{\vec{p}-\vec{k}} - \omega_{\vec{p}}$

$\ddot{a}_k \ll \Omega_k\dot{a}_k \quad \to \quad \alpha \ll 1, \mu \ll 1$

Bound: $\alpha > \mu^2/2$ - instability

Solution without approximation. $\mu \gtrsim 1$

We do not neglect $\ddot{a}_k$

The ansatz: $a_{\vec{k}}(t) = e^{i\epsilon_{\vec{p}-\vec{k}} t/2} \Big[a_{\vec{k}}(0) \left(\cosh(s_{\vec{p}-\vec{k}} t) - iC_1 \sinh(s_{\vec{p}-\vec{k}} t) \right) -$

$a^{\dagger}_{\vec{p}-\vec{k}}(0) \cdot iC_2 \sinh(s_{\vec{p}-\vec{k}} t) \Big]$

The commutation relation $\quad \rightarrow \quad C_1^2 - C_2^2 = -1$

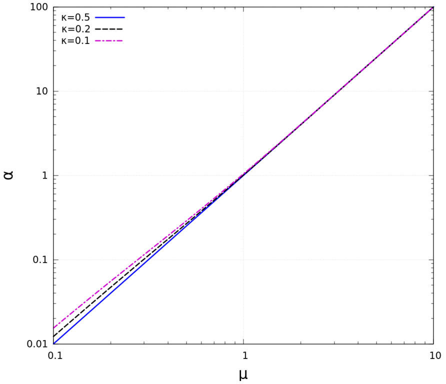
The solution

$$C_1 = \frac{\epsilon^2_{\vec{p}-\vec{k}}/4 - s^2_{\vec{p}-\vec{k}} - \Omega_{\vec{k}} \epsilon_{\vec{p}-\vec{k}}}{s_{\vec{p}-\vec{k}} (\epsilon_{\vec{p}-\vec{k}} - 2\Omega_{\vec{k}})}, \;\; C_2 = \frac{\sigma_{\vec{p}-\vec{k}}}{s_{\vec{p}-\vec{k}} (\epsilon_{\vec{p}-\vec{k}} - 2\Omega_{\vec{k}})},$$

$$s^2_{\vec{p}-\vec{k}} = -\frac{\epsilon^2_{\vec{p}-\vec{k}}}{4} - 2\Omega^2_{\vec{k}} + \epsilon_{\vec{p}-\vec{k}} \Omega_{\vec{k}} + \sqrt{\Omega^2_{\vec{k}} \epsilon^2_{\vec{p}-\vec{k}} + 4\Omega^4_{\vec{k}} + \sigma^2_{\vec{p}-\vec{k}} - 4\epsilon_{\vec{p}-\vec{k}} \Omega^3_{\vec{k}}}.$$

$\alpha = \mu^2/2 -$ the boundary of instability α, μ

The boundary of instability. Dependence $\alpha(\mu)$



$$\alpha_{\vec{\kappa}} = \frac{1}{4}\sqrt{\frac{\beta_{\vec{v}-\vec{\kappa}}}{\beta_{\vec{\kappa}}}(\beta_{\vec{\kappa}} + \beta_{\vec{v}-\vec{\kappa}} - 1)^2(-3\beta_{\vec{\kappa}} + \beta_{\vec{v}-\vec{\kappa}} - 1)^2},$$

for $k = p/2$ $\quad \alpha = \mu^2/2;\ \mu \gg 1 \quad \rightarrow \quad \alpha = \mu^2/2$ for any k

The condensate. Mathieu equation

Equation of motion

$$\ddot{\chi}_k + \left(k^2 + m_\chi^2 + 2g\Phi\cos(m_\phi t)\right)\chi_k = 0$$

Mathieu equation

$$\chi_k'' + (A_k + 2q\cos(2z))\,\chi_k = 0,$$

where

$$A_k = 4\frac{k^2 + m_\chi^2}{m_\phi^2}, \qquad q = 4\frac{g\Phi}{m_\phi^2}.$$

Narrow resonance $q \ll 1$
Broad resonance $q \gg 1$

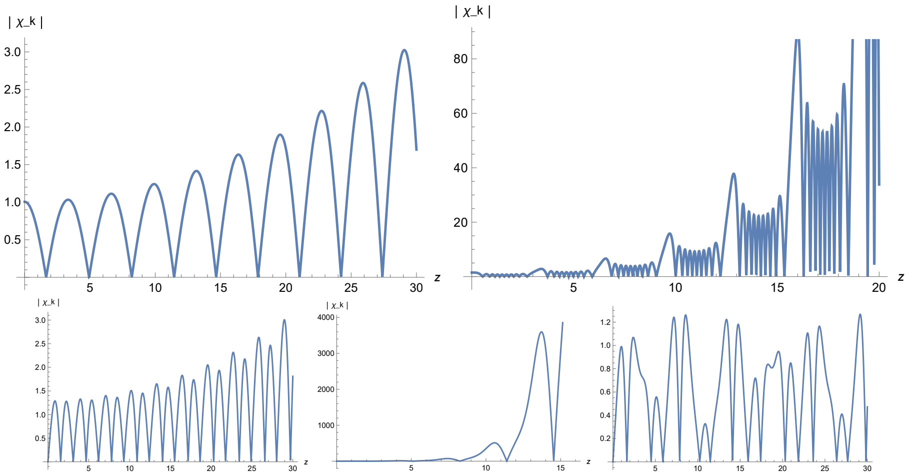
$$\frac{d\omega}{dt} \gtrsim \omega^2$$

The tachionic instability: broad resonance area + $A_k < 2q$

K. Losanov. Reheating after inflation. 2020

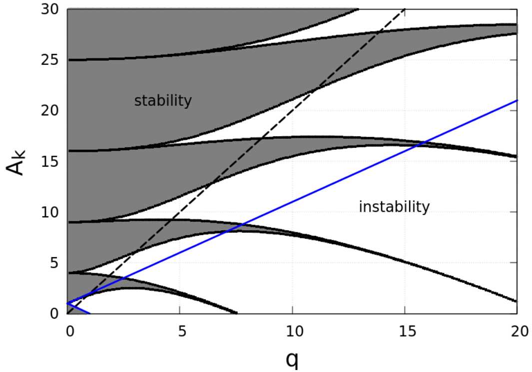
Kofman, Linde, Starobinskiy 97 (hep-ph/9704452)

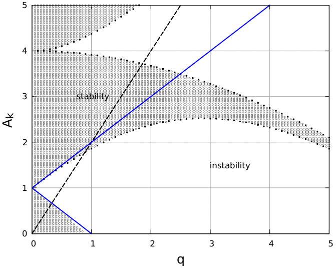
The resonance solutions



Upper line: Narrow resonance $A_k = 1$, $q = 0.1$. Broad resonance $A_k = 100$, $q = 50$.
Lower line: $N = 2$ resonance band, $A_k = 4$, $q = 1$. Tachionic resonance $A_k = 1$, $q = 1.5$.
Tachyonic unstable area, $A_k = 3$, $q = 2$

Mathieu instability diagram





Grey area - stability region. Blue solid line - instability bound for condensate, was obtained from Heisenberg equation. Black dashed line - $A_k = 2q$.

Comprasion of two approaches. Plane wave

We turn to the consideration of a low-mass plane wave ϕ instead of a massless one

The case $k = p/2 \implies \alpha = \mu^2/2$

In the terms of Mathieu equation

$$A_k = \frac{4(k^2 + m_\chi^2)}{\omega^2} = 4\beta_k^2, \qquad q = \frac{4g\Phi}{\omega^2} = 4\alpha,$$

$\omega^2 = p^2 + m_\phi^2$

Then we make a boost and move to a condensate with $p = 0$ and low mass $m_\phi < m_\chi$. Then

$$\alpha = \mu^2/2 \longrightarrow A_k = 2q$$

Interpretation through particles is

$N\phi \to 2\chi$, where N - the number of the peak

Our ansatz only for the first peak

Solution for condensate through Heisenberg equation. $p = 0$

$$e^{-i\Omega_k t}\left[\ddot{a}_k - 2i\Omega_k \dot{a}_k + 2\alpha\omega^2 \cos(m_\phi t) a_k\right] +$$
$$+e^{i\Omega_k t}\left[\ddot{a}^\dagger_{-k} + 2i\Omega_k \dot{a}^\dagger_{-k} + 2\alpha\omega^2 \cos(m_\phi t) a^\dagger_{-k}\right] = 0$$

The solution

$$a_k(t) = e^{i\epsilon_k t/2}\Big(a_k(0)\left(\cosh(st) - i\frac{\epsilon^2/4 - s^2 - \Omega_k\epsilon_k}{s(2\Omega_k - \epsilon_k)}\sinh(st)\right) -$$
$$-ia^\dagger_{-k}\frac{\alpha\omega^2}{s(2\Omega_k - \epsilon_k)}\sinh(st)\Big),$$

где

$$s = \sqrt{\Omega_k^2(2\Omega_k - \epsilon_k)^2 + \alpha^2 m_\phi^4} - \Omega_k(2\Omega_k - \epsilon_k) - \frac{\epsilon_k^2}{4}.$$

In the terms of the Mathieu equation for $s^2 = 0$:

$$q = |A_k - 1|,$$

where $A_k = 4\frac{k^2 + m_\chi^2}{m_\phi^2}$, $\qquad q = 4\frac{g\Phi}{m_\phi^2}$.

Scalar QED and Mathieu equation (work in progress)

The Klein-Gordon equation for scalar QED

$$(\Box + 2ie\mathcal{A}^\mu\partial_\mu - e^2\mathcal{A}^2 + m^2)\psi = 0$$

$$\psi = e^{-ip\cdot x}F(\phi),\ \ F(\phi) = w(\zeta)exp[i(\frac{(p\cdot k)}{k^2})\phi],$$

where $k = (\omega, 0, 0, n\omega)$ and $p = (\sqrt{(\frac{n\omega}{2})^2 + m^2 + p_1^2 + p_2^2}, p_1, p_2\frac{n\omega}{2})$

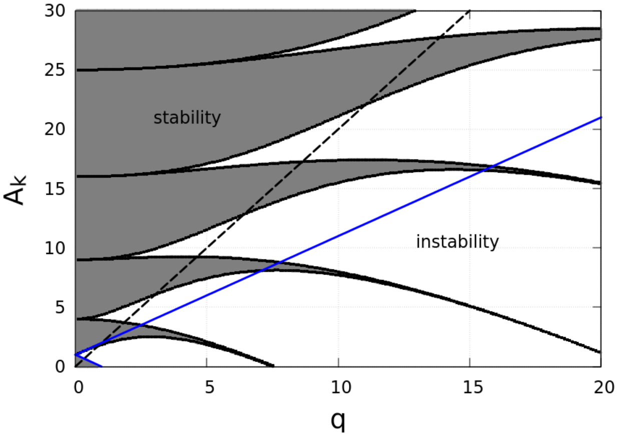
Mathieu equation $\frac{d^2w}{d\zeta^2} + (A_k - 2q\cos(2\zeta))w = 0,$

$$A_k = \frac{4}{\omega^2(1-n^2)}\left(\frac{\left(\sqrt{(\frac{n\omega}{2})^2 + m^2 + p_1^2 + p_2^2} - \frac{n^2\omega}{2}\right)^2}{1-n^2} - \frac{e^2E_0^2}{\omega^2}\right),$$

$$q = \left|\frac{4eE_0}{\omega^3(1-n^2)}\right|(p_1^2 + p_2^2)^{1/2},$$

where $e-$charge, E_0-amplitude of electromagnetic field. We are consider circularly polarized monochromatic wave. The medium with refractive index $n < 1$ (Millar et al. PRD 107, 055013(2023), Lapine et al. PRB 87, 024408 (2013))

Stability diagram for Mathieu equation



Boundary condition:

Narrow resonance $q = |A_k - 1|$, first peak, blue solid line

Broad resonance $A_k = 2q$, black dashed line

Conclusion

- In the case of a large mass, a high energy density is required. Its threshold value:
$$\rho_\phi \geq \frac{m_\chi^4 \omega^2}{2g^2}$$
- For a massless plane wave, for the case of $k = p/2$, after the boost, we see that the instability boundary for solving the Heisenberg equation and for solving the Mathieu equation coincide, with the exception of peaks.
- For a plane massless wave, there will be only 1 line of instability, since, based on the interpretation of particles ($N\phi \to 2\chi$), two massless waves will not interact with each other.
- When solving the Heisenberg equation for condensate, the parameters can be selected so that the solution is within the range of the broad resonance of the Mathieu equation.
- For scalar electrons we can find a resonantly growing solution if we reduce the Klein-Gordon equation to the Mathieu equation

Thank you for your attention!