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Differential Renormalization in Curved Space-Time.

Renormalization.

Perturbation

$$\mathcal{S} = 1 + \sum_{n=1}^{\infty} \mathcal{S}_n, \qquad \mathcal{S}_n \sim \int \prod_{i=1}^{n} d^4 x_i : \phi^r : \left(\prod_{i=1}^{n} G(x_i) \right) : \phi^t :$$

The general idea of renormalization.

1-step: Momentum space representation

$$G(x-y) \Rightarrow \frac{1}{(2\pi)^4} \int d^4 k \exp(-ik(x-y)) G(k)$$

2-step: Regularization of Feynman diagrams.
Redefinition of the integral and making it finite.

$$D(k) = \frac{1}{k^2 - m^2} \Rightarrow D(k)_{reg} = \frac{1}{k^2 - m^2} + \sum_{i=1}^{N} \frac{c_i}{k^2 - M_i^2}$$

3-step: Substraction procedure

4-step: Renormalization

BPHZ-method

N.N.Bogolubov, O.S.Parasiuk , Acta. Math. (1957), v. 97, p.227
K.Hepp, Commun. Math. Phys. (1966), v.2, p. 301
W.Zimmermann,Commun. Math. Phys. (1969), v.15, p. 208

New ideas

F.A.Lunev, Phys.Rev. (1994), D50, p. 6589
F.A.Lunev, Phys.Rev. (1994), D50, p. 7735

$$G(x,m) = \frac{1}{(2\pi)^4}\int d^4p \frac{\exp(-ipx)}{p^2+m^2} \quad \Rightarrow \quad G(x,y) = \frac{1}{4\pi^3}\frac{1}{(x^2+y^2)^2}$$

So

$$\left(\Box_x + \Box_y\right)\hat{G}(x,y) = -\delta^{(4)}(x)\delta^{(2)}(y)$$

where

$$\Box \equiv \partial^2$$

and
$\hat{G}(x,y)$ is defined the renormalized Feynmann amplitude

Differential Renormalization

D.Z.Freedman, K.Jonson, J.I.Latorre, Nucl.Phys. (1992), B371, p. 353
Differential regularization and renormalization: a new method of calculation in quantum field theory

$$D^2(x) \sim \frac{1}{(x^2)^2} = -\frac{1}{4}\Box\frac{\ln\Lambda^2 x^2}{x^2}, \quad x \neq 0$$

This is the result

$$D(x) = \frac{1}{(2\pi)^4}\int d^4p \frac{\exp(-ipx)}{p^2 + i\varepsilon} \sim \frac{1}{x^2}$$

O.I.Zavialov, V.A.Smirnov, T. M. Phys. (1993), v.96, p. 287 (in russian)
V.A.Smirnov, Nucl. Phys. (1994), B427, p.325

$$\mathcal{R}\frac{1}{(x^2)^2} = -\frac{1}{4}\Box\frac{\ln\Lambda^2 x^2}{x^2}$$

and x is any
Improwed versions of DR: V.A.Smirnov, O.I.Zavialov, P.E.Haagensen, J.I.Latorr
and others (1995-2010)

Techniques for loop calculations in DR

J. Math. Phys (1997) v. 38, p. 738; Nucl. Phys, B537, p. 561 (1999); arXiv:hep-th/0605116v2 (2007); arXiv:hep-th/0706.1210v1(2007)

$$G(x) = \frac{1}{4\pi}\frac{1}{x^2}$$

$$\left[\frac{1}{(x^2)^2}\right]_R = -\frac{1}{4}\Box\left(\frac{\ln(x^2M^2)}{x^2}\right), \quad \left[\frac{1}{x^6}\right]_R = -\frac{1}{32}\Box\Box\frac{\ln(x^2M_1^2)}{x^2}$$

$$\frac{1}{(x^2)^2} = \frac{4}{x^2}\frac{d}{dx^2}\left((x^2)^2\frac{dG}{dx^2}\right)(*)$$

The general validity of the propagator equation

$$\Big[\Phi(x, x_1, ...x_n)(\Box_x - m^2)\Delta_m(x)\Big]_R = \Big[\Phi(x, x_1, ...x_n)(-\delta(x))\Big]_R,$$

$$\Delta_m(x) = \sqrt{\frac{m^2}{y^2}}K_1\left(\sqrt{m^2y^2}\right)$$

$$g_{\mu\nu}(x) \neq \eta_{\mu\nu} \equiv diag\{-1,1,1,1\}$$

$$\partial_\alpha \quad \rightarrow \quad \nabla_\alpha = \partial_\alpha + \Gamma_{\alpha ab} I^{ab}$$

Equations
for scalar field

$$\nabla_\nu \nabla^\nu \varphi + \left(m^2 + \xi R(x)\right)\varphi = 0$$

for spinor field

$$(\gamma^\alpha \nabla_\alpha + m)\psi = 0$$

for vector field

$$\nabla_\nu F^{\nu\mu} = 0, \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

We consider only scalar field theory with

$$L_{int} = \sqrt{-g}\frac{\lambda}{4!}\varphi^4$$

Simple results $\lambda\varphi^4$

$$\mathcal{L} = \frac{1}{2}\left[\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \left(Z_{(2)}m^2\right)\varphi^2\right] - \lambda Z_{(4)}\varphi^4$$

$$\Updownarrow$$

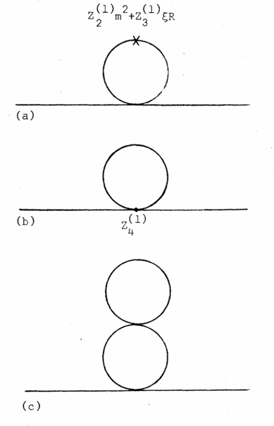
$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left[g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \left(Z_{(2)}m^2 + Z_{(3)}\xi R\right)\varphi^2\right] -$$

$$-\sqrt{-g}\lambda Z_{(4)}\varphi^4$$

Where

$$\varphi = Z_{(1)}\hat{\varphi}, \qquad Z_{(i)} = 1 + \sum_{r=1}^{\infty} Z_{(i)}^r \lambda^r$$

Loops



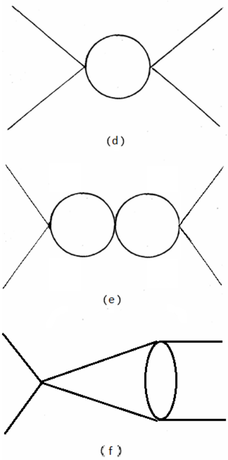


Figure: Caption for this figure with two images

Analitical Expressions

$$(a) \to -3\lambda^2 \int d^4x d^4x' \sqrt{g(x)g(x')} G(v,x)G(v',x)G^2(x,x')G(x,z)G(z'x')$$

$$(c) \to -9\lambda^2 \int d^4x d^4x' \sqrt{g(x)g(x')} G(x,x')G^2(x,x')G(x',x')G(x,z)$$

$$(d) \to 54\lambda^2 \int d^4x d^4x' \sqrt{g(x)g(x')} \left[Z_2^{(1)} m^2 + \right.$$

$$\left. + Z_3^{(1)} \xi R(x') \right] G(v,x)G^2(x,x')G(x,z)$$

and so on.

Ideas

Comellas J.,Haagensen E., Latorre J.I., Int.J. Mod.Phys A, v. 10, n. 19, p. 2819 (1995)

$$\frac{1}{(x^2)^2} = \frac{4}{x^2}\frac{d}{dx^2}\left((x^2)^2\frac{dG}{dx^2}\right)$$

$$\Downarrow$$

$$G^2(x) = -\frac{1}{16\pi^2}\left(\nabla_\nu\nabla^\nu - \alpha^2(g, R, ...)\right)\left(G(x)\ln\frac{g(x)}{M^2}\right)$$

$$\Downarrow$$

(a) Hyperboloid $H_n \subset E_{(n+1)}$
(b) Sphere $S_n \subset E_{(n+1)}$

Riemann Normal Coordinates

Bunch T.S, Parker L., Phys. Rev. D20, n. 10,(1979), pp.2499

$$\left(-\nabla_\nu \nabla^\nu + m^2 + \xi R(x)\right) G(x, x') = \frac{1}{\sqrt{-g}(x)} \delta(x - x') \quad (1)$$

Introducing normal coordinates $y^\mu = \left(x - x'\right)^\mu$ one has

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} \mathring{R}_{\mu\alpha\nu\beta} y^\alpha y^\beta - \frac{1}{6} \left(\nabla^\beta \nabla^\gamma \mathring{R}_{\mu\alpha\nu\beta}\right) y^\alpha y^\beta y^\gamma + \cdots \quad (2)$$

$$g(x) = \det |g_{\mu\nu}|, \quad \nabla_\mu = \partial_\mu + \mathring{\Gamma}_{\alpha\beta\mu} y^\alpha y^\beta + \cdots, \quad (3)$$

Where

$$\mathring{\Phi}_{\mu\alpha\nu\beta.....} \equiv \Phi_{\mu\alpha\nu\beta.....}(y = 0)$$

1. Momentum space representation of propagator in curved space-time

$$G(x,x') = \left(g(x)\right)^{-\frac{1}{4}} \hat{G}(x,x')$$

So

$$\hat{G}(x,x') \equiv \hat{G}(y) = \frac{1}{(2\pi)^4} \int d^4k \exp(-iky) \hat{G}(k)$$

Using (1) - (3) and representation

$$\frac{1}{(2\pi)^4} \int d^4k \exp(-iky) \frac{1}{k^2+m^2} = \sqrt{\frac{m^2}{y^2}} K_1\left(\sqrt{m^2y^2}\right)$$

where

$$\sqrt{\frac{m^2}{y^2}} K_1\left(\sqrt{m^2y^2}\right) = \frac{1}{y^2} + \frac{m^2}{4} \sum_{n=0}^{\infty} \frac{(m^2y^2/4)^n}{n!(n+1)!} \Big[\ln\Big(\frac{m^2y^2}{4}\Big) - \Psi(n+2) - \\ -\Psi(n+1) \Big]$$

2. Momentum space representation of propagator in curved space-time

$$\hat{G}(k) = \frac{1}{k^2+m^2} + \left(\frac{1}{6}-\xi\right)\mathring{R}\frac{1}{\left(k^2+m^2\right)^2} +$$

$$+i\left(\frac{1}{6}-\xi\right)\mathring{R}_{,\alpha}\frac{1}{k^2+m^2}\frac{\partial}{\partial k^\alpha}\frac{1}{k^2+m^2} +$$

$$+\left(\frac{1}{6}-\xi\right)^2\mathring{R}^2\frac{1}{\left(k^2+m^2\right)^3} + \mathring{A}^{\alpha\beta}\frac{1}{k^2+m^2}\frac{\partial}{\partial k^\alpha}\frac{\partial}{\partial k^\beta}\frac{1}{k^2+m^2} + \cdots$$

where

$$\mathring{A}_{\alpha\beta} = \frac{1}{2}(\xi-\frac{1}{6})\nabla_\alpha\nabla_\beta\mathring{R} + \frac{1}{120}\nabla_\alpha\nabla_\beta\mathring{R} - \frac{1}{40}\nabla_\nu\nabla^\nu\mathring{R}_{\alpha\beta} +$$

$$-\frac{1}{60}\mathring{R}_{\sigma\alpha\tau\beta}\mathring{R}^{\sigma\tau} - \frac{1}{60}\mathring{R}^{\sigma\mu\nu}{}_{\alpha}\mathring{R}_{\sigma\mu\nu\beta}$$

3. Coordinate space representation of the Feynman propagator in curved space-time

So

$$\hat{G}(y) = \mathcal{K}_1(m,y) + \left(\frac{1}{6} - \xi\right)\mathring{R}\left(-\frac{\partial}{\partial m^2}\right)\mathcal{K}_1(m,y) +$$

$$+\left(\frac{1}{6} - \xi\right)\mathring{R}_{,\alpha}\left(\frac{\partial}{\partial m^2}\right)^2 \frac{\partial}{\partial y^\alpha}\mathcal{K}_1(m,y) +$$

$$+\left\{\left(\frac{1}{6} - \xi\right)^2 \mathring{R}^2 - 2\mathcal{A}^\alpha{}_\alpha\right\}\left(\frac{\partial}{\partial m^2}\right)^2 \mathcal{K}_1(m,y) +$$

$$+4\mathring{\mathcal{A}}^{\alpha\beta}\left(\frac{\partial}{\partial m^2}\right)^2 \frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}\mathcal{K}_1(m,y) + \cdot\cdot$$

where

$$\mathcal{K}_1(m,y) = \sqrt{\frac{m^2}{y^2}}K_1\left(\sqrt{m^2 y^2}\right)$$

Coordinate representation of the Feynman propagator in curved space-time

$$\hat{G}(y) = \sum_{\alpha_i=0}^{\infty} \sum_{n=0}^{\infty} \mathbb{A}^{(n)}_{\alpha_1....\alpha_i} \mathbb{D}^{\alpha_1....\alpha_i}_{(n)} \mathcal{K}_1(m, y)$$

$$\mathbb{D}^{\alpha_1....\alpha_i}_{(n)} = (-1)^n \prod_{j=1}^{I} \Big(-i \frac{\partial}{\partial y^{\alpha_j}} \Big) \Big(\frac{\partial}{\partial m^2} \Big)^n$$

where

$$\alpha_i = 0, n = 0 \rightarrow \mathbb{A}^0_0 = 1; \quad \alpha_i = 0, n = 1 \rightarrow \mathbb{A}^0_1 = \Big(\frac{1}{6} - \xi \Big) R$$

and so on.

Natural Renormalization

$$\Delta(y) = \mathcal{K}_1(m,y) = \frac{1}{y^2} - m^2 \int d^4x_1 \frac{1}{(y-y_1)^2}\frac{1}{y_1^2} +$$

$$+(-m^2)^2 \int\int dy_1^4 dy_2^4 \frac{1}{(y-y_1)^2}\frac{1}{(y_1-y_2)^2}\frac{1}{y_2^2} + \cdots$$

So

$$\Delta_{diff.ren}(y) = \frac{1}{y^2} + \frac{m^2}{4}\sum_{n=0}^{\infty}\frac{(m^2y^2/4)^n}{n!(n+1)!}\left[\ln\left(\frac{\Lambda_n^2 y^2}{4}\right) - 2C\right]$$

Denoting

$$\mathcal{K}_1^{(dr)}(y) \equiv \Delta_{diff.ren}(y)$$

$$\hat{G}_{(dr)}(y) = \sum_{\alpha_i=0}^{\infty}\sum_{n=0}^{\infty}\mathbb{A}^{(n)}_{\alpha_1....\alpha_i}\mathbb{D}^{\alpha_1....\alpha_i}_{(n)}\mathcal{K}_1^{(dr)}(m,y)$$

$$n - loops \sim \Big(G(y)\Big)^n \sim \int\int ....\int \sqrt{g(x_1)....g(x_l)}\Big(\hat{G}_{(dr)}(y)\Big)^n$$

$$\equiv \Big(\sum_{\alpha_i=0}^{\infty}\sum_{n=0}^{\infty}\mathbb{A}^{(n)}_{\alpha_1....\alpha_i}\mathbb{D}^{\alpha_1....\alpha_i}_{(n)}\mathcal{K}_1(m,y)\Big)^n$$

$$\Downarrow$$

$$\Big\{\frac{1}{y^2}+\frac{m^2}{4}\sum_{n=0}^{\infty}\frac{(m^2y^2/4)^n}{n!(n+1)!}\Big[\ln\Big(\frac{\Lambda_n^2y^2}{4}\Big)-2C\Big]\Big\}^n$$

$$\Downarrow$$

$$\equiv\sum_{s=0}^{\infty}a_{s,(n-s)}\big(\mathring{R},\mathring{R}_{,\alpha},\mathring{R}_{,\alpha\gamma..},\mathring{R}^l\big)\Big(\frac{1}{y^2}\Big)^s\Box^{(n-s)}\ln\Big(\frac{y^2}{\Lambda_s^2}\Big)$$

THANK YOU ON THE ATTENTION!