

Geodesics and the Cosmological model of the Liouville Solution in General Relativity with a Scalar Field

D. Afanasev, Great Bay University, China

M. Katanaev*, Steklov Mathematical Institute, Moscow

arXiv:2410.03799;

Phys. Lett. B 864 (2025) 139439

JCAP (2025) accepted

* The work of M. Katanaev was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2025-303).

The Liouville metric (1849)

Notation:

(\mathbb{M}, g) - space-time = four dimensional manifold with Lorentzian signature metric.

$x = (x^\alpha)$, $\alpha = 0, 1, 2, 3$ - local coordinates

$$g_{\alpha\beta} := \Phi(x)\eta_{\alpha\beta}, \quad \eta_{\alpha\beta} := \text{diag}(+---)$$

$$\Phi := \phi_0(x^0) + \phi_1(x^1) + \phi_2(x^2) + \phi_3(x^3) \quad \text{- the conformal factor}$$

There is no Killing vector field for nontrivial functions $\phi_\alpha(x^\alpha)$

The advantage: the geodesic Hamilton-Jacobi equation admits complete separation of variables \rightarrow Geodesic equations are integrable for arbitrary $\phi_\alpha(x^\alpha)$.

The Stackel problem (1893): which metric admits complete separation of variables in the geodesic Hamilton-Jacobi equation?

It was solved for metrics of arbitrary signature in any number of dimensions:

Katanaev arXiv2305.02222,
PhysicaScripta (2023), TMΦ (2024)

The action

$$S := \int dx \sqrt{|g|} \left(R - 2\Lambda + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi) \right), \quad g := \det g_{\alpha\beta}$$

$\varphi(x)$ - scalar field

Λ - cosmological constant

$V(\varphi)$ - potential for a scalar field is specified later

There are no ghosts for $\Phi > 0$.

Field equations:

$$R_{\alpha\beta} = -\frac{1}{2} \partial_\alpha \varphi \partial_\beta \varphi + \frac{1}{2} \Phi \eta_{\alpha\beta} (V + 2\Lambda) \quad (1)$$

$$\eta^{\alpha\beta} \partial_{\alpha\beta}^2 \varphi + \frac{1}{\Phi} \Phi'^{\alpha} \partial_\alpha \varphi + \Phi V' = 0 \quad (2)$$

The additional assumption $\varphi(x) := \Psi(\Phi(x))$

Theorem. A general solution to the Euler-Lagrange equations for the action

$$S := \int dx \sqrt{|g|} \left(R - 2\Lambda + \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi) \right), \quad g := \det g_{\alpha\beta}$$

for the Liouville metric

$$g_{\alpha\beta} := \Phi(x) \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} := \text{diag}(+---)$$

$$\Phi := \phi_0(x^0) + \phi_1(x^1) + \phi_2(x^2) + \phi_3(x^3)$$

of signature $(+---) \Leftrightarrow \Phi > 0$ and scalar field $\varphi(x) := \Psi(\Phi(x))$

exists only for the potential

$$V(\varphi) = -2\Lambda + 12C^2 e^{\mp \frac{2}{\sqrt{3}} \varphi}$$

$C > 0$ - arbitrary constant.

It is $\Phi = s + c$ where $s := \eta_{\alpha\beta} x^\alpha x^\beta$ and $c \in \mathbb{R}$ is an integration constant.

$$\varphi = \pm \sqrt{3} \ln(C\Phi)$$

The solution is defined in the domain $\Phi > 0$.

A general solution

$$g_{\alpha\beta} = (s + c)\eta_{\alpha\beta}, \quad s := \eta_{\gamma\delta} x^\gamma x^\delta$$

6 noncommuting Killing vector fields \longrightarrow Spontaneous symmetry emergence

Afanasev, Katanaev. Phys.Rev.D (2019), ibid. (2020)

$$\text{Curvature tensor } R_{\alpha\beta\gamma}{}^\delta = \frac{1}{\Phi} \left(3 - \frac{c}{\Phi} \right) \eta_{\alpha\gamma} \delta_{\beta}^\delta - \frac{3}{\Phi^2} (x_\alpha x_\gamma \delta_{\beta}^\delta - x_\alpha x^\delta \eta_{\beta\gamma}) - (\alpha \leftrightarrow \beta)$$

$$\text{Curvature invariants: } R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{12}{\Phi^4} \left(9 - \frac{6c}{\Phi} + \frac{5c^2}{\Phi^2} \right)$$

$$R^{\alpha\beta} R_{\alpha\beta} = \frac{36}{\Phi^4} \left(3 + \frac{c^2}{\Phi^2} \right)$$

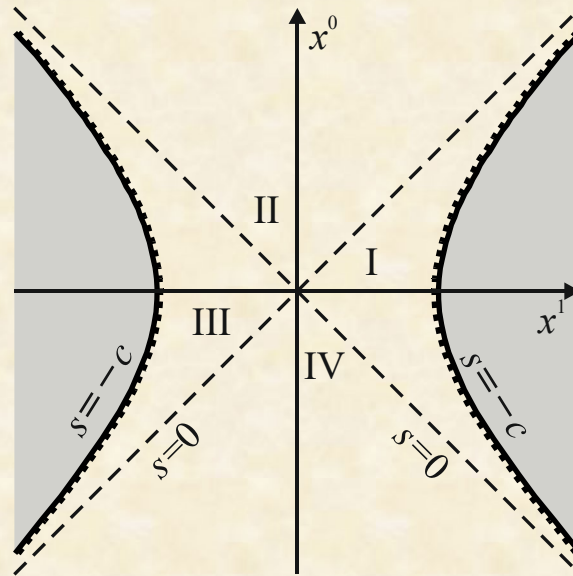
$$R = \frac{6}{\Phi^2} \left(3 + \frac{c}{\Phi} \right)$$

The only singularity is located at $\Phi = 0$

The spacetime is asymptotically flat at space and time infinities $|s| \rightarrow \infty$

Cosmological model

Naked singularity $\Phi > 0, c > 0$



The forbidden regions are shaded

The picture must be rotated in two extra space dimensions around x^0 axis

Naked singularity: test particles can live forever going through the throat or fall into the singularity at a finite proper time

The black hole solution

$$g_{\alpha\beta} = (s + c)\eta_{\alpha\beta}$$

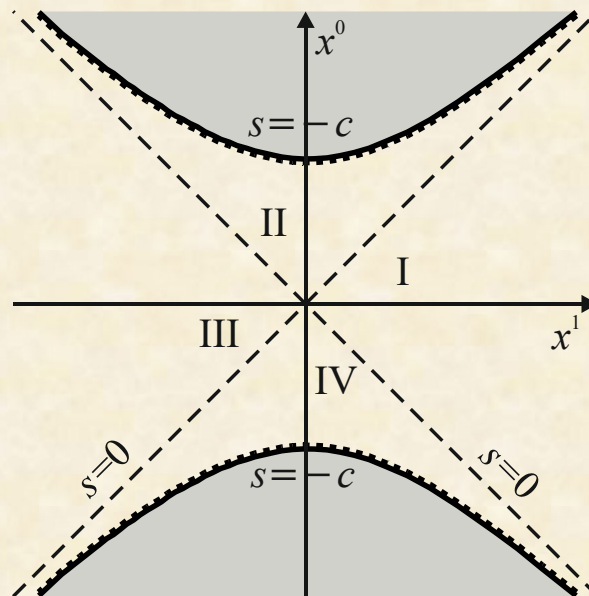
$$\Phi < 0 \Rightarrow s < -c$$

We must change the total sign of the action to avoid ghosts.

$$S := \int dx \sqrt{|g|} \left(-R + 2\Lambda - \frac{1}{2} g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V(\varphi) \right)$$

$$V = 2\Lambda - 12C^2 e^{\mp \frac{2}{\sqrt{3}}\varphi}$$

The black hole solution corresponds to $c < 0$



Friedmann-like form of the Liouville metric

Pseudospherical
coordinates in quadrant II
(inside the future light cone):

$$x^0 := t \cosh \chi$$

$$x^1 := t \sinh \chi \sin \vartheta \cos \psi$$

$$x^2 := t \sinh \chi \sin \vartheta \sin \psi$$

$$x^3 := t \sinh \chi \cos \vartheta$$

$$\longrightarrow s = t^2 > 0$$

$$ds^2 = (t^2 + c) (dt^2 - d\Omega)$$

$$d\Omega := d\chi^2 + \sinh^2 \chi (d\vartheta^2 + \sin^2 \vartheta d\psi^2) \text{ - constant negative curvature metric}$$

Next coordinate transformation $t \rightarrow \tau(t)$: $\frac{d\tau}{dt} = \sqrt{t^2 + c}$

$$ds^2 = d\tau^2 - a^2(\tau) d\Omega$$

$$\frac{da}{d\tau} = \frac{2t^2 + c}{t^2 + c}$$

$$a(\tau) := t\sqrt{t^2 + c} \text{ - the scale factor}$$

$$\frac{d^2 a}{d\tau^2} = \frac{2ct}{(t^2 + c)^{5/2}} > 0 \text{ - accelerated expansion}$$

The expansion starts at the “horizon” $t = 0$ (future light cone), and there is no singularity in global coordinates. There are no Friedmann-like coordinates in quadrants I and III because surfaces $s = \text{const}$ are timelike.

Geodesics $x^\alpha(\tau)$ for the Liouville space-time (+---)

Lagrangian and Hamiltonian
for geodesics:
$$L := \frac{1}{2}(s+c)\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta,$$

$$H = \frac{1}{2(s+c)}\eta^{\alpha\beta}p_\alpha p_\beta, \quad p_\alpha = (s+c)\eta_{\alpha\beta}\dot{x}^\beta$$

Geodesic equations:
$$\ddot{x}^\alpha = -\Gamma_{\beta\gamma}^{\alpha}\dot{x}^\beta\dot{x}^\gamma = \frac{1}{s+c}(x^\alpha\dot{x}_\beta\dot{x}^\beta - 2x_\beta\dot{x}^\beta\dot{x}^\alpha)$$

The Hamilton-Jacobi equation
for the action function $W(x)$:
$$\frac{1}{s+c}\eta^{\alpha\beta}\partial_\alpha W\partial_\beta W = 2m^2 \quad m - \text{mass of a test particle}$$

Complete separation of variables:
$$W(x, d) = \sum_{\alpha=0}^3 W_\alpha(x^\alpha, d), \quad \det \frac{\partial^2 W}{\partial x^\alpha \partial d_\beta} \neq 0$$

$$W_\alpha'^2 = 2m^2 x_\alpha^2 + d_\alpha \quad d := (d_0, d_1, d_2, d_3)$$

$$d_0 - d_1 - d_2 - d_3 = 2m^2 c$$

Four independent parameters: d_1, d_2, d_3, m

We put $2m^2 = 1$ for timelike geodesics by stretching the canonical parameter τ

Lightlike (null) geodesics

The form of the lightlike geodesics is the same as in the Minkowskian spacetime. The only difference: the dependence on the canonical parameter.

Timelike geodesics for $\Phi > 0$

The Hamilton-Jacobi equation
$$g^{\alpha\beta} \partial_\alpha W \partial_\beta W = \sum_{\alpha=0}^3 g^{\alpha\alpha} W_\alpha'^2 = 1$$

The complete integral
$$W(x, d) = \sum_{\alpha=0}^3 W_\alpha(x^\alpha, d)$$

where
$$W_\alpha'^2 = (x^\alpha)^2 + d_\alpha > 0, \quad d_0 - d_1 - d_2 - d_3 = c$$

Four involutive quadratic conservation laws:
$$(p_\alpha)^2 = (x^\alpha)^2 + d_\alpha > 0$$

The metric belongs to class $[0, 4, 0]_2$ according to the classification of separable metrics in Katanaev arXiv2305.02222.

Four first integrals of the geodesic equations:
$$(s + c)^2 (\dot{x}^\alpha)^2 = (x^\alpha)^2 + d_\alpha$$

Timelike geodesics for $\Phi > 0$

The form of geodesics of general type $\left(\frac{dx^i}{dx^0}\right)^2 = \frac{(x^i)^2 + d_i}{(x^0)^2 + d_0}, \quad i = 1, 2, 3$

$$x^i = \left(\frac{C^i}{2} + \frac{d_i}{2C^i d_0}\right) x^0 + \left(\frac{C^i}{2} - \frac{d_i}{2C^i d_0}\right) \sqrt{(x^0)^2 + d_0} \quad C^i > 0$$

Three degenerate cases:

1) $d_0 = 0, d_i \neq 0$

$$x^i = \frac{(C^i x^0)^2 - d_i}{2C^i x^0}$$

3) $d_0 = d_i = 0$

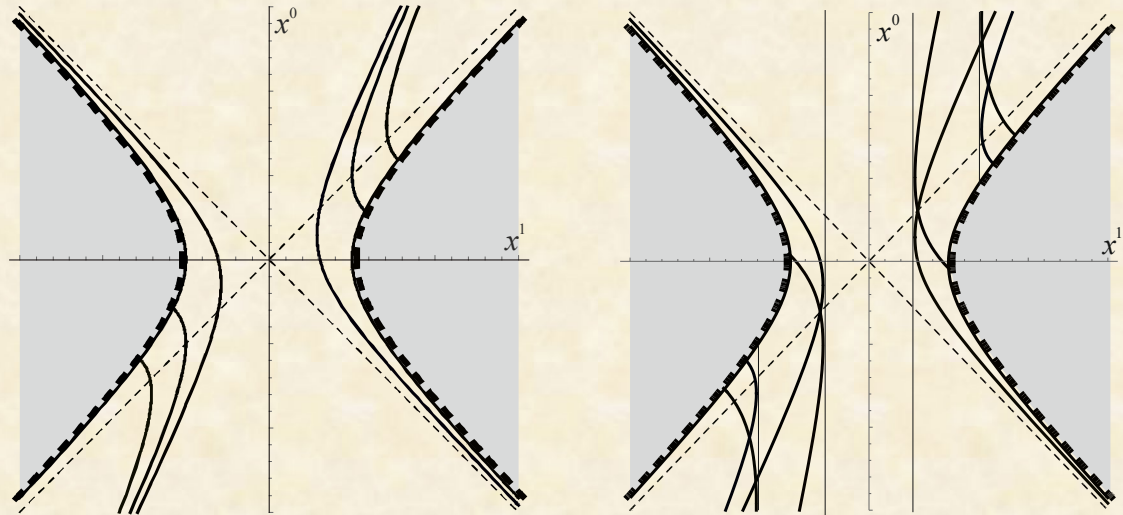
$$x^0 = C^i x^i, \quad x^0 = \frac{C^i}{x^i}$$

2) $d_0 \neq 0, d_i = 0$

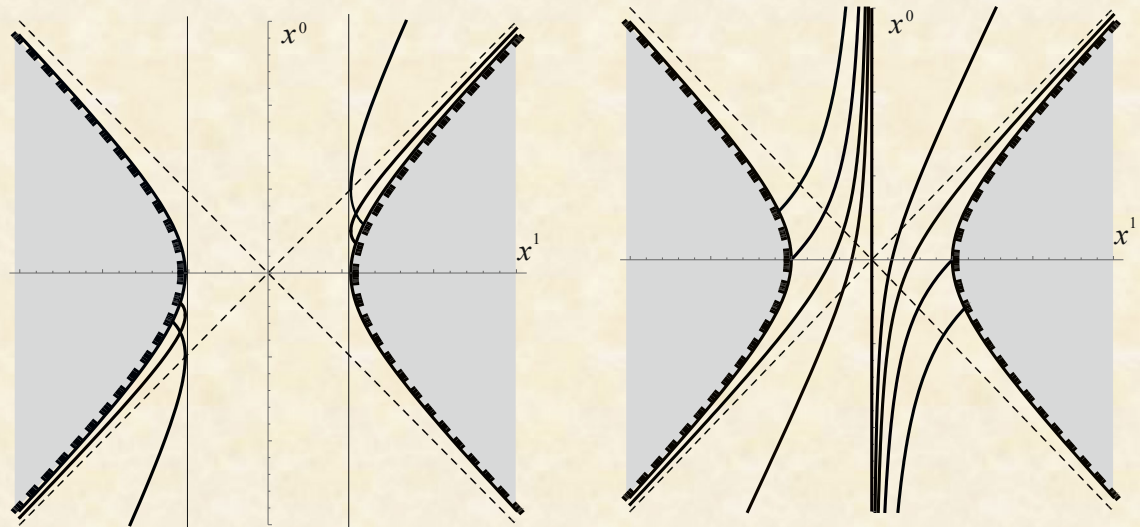
$$x^0 = \frac{(C^0 x^i)^2 - d_0}{2C^0 x^i} \quad C^0 \neq 0$$

~~Timelike geodesics for the naked singularity~~

Timelike geodesics
of general type:



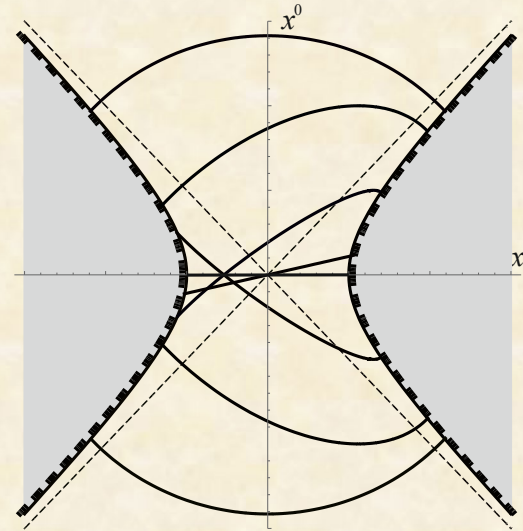
Timelike degenerate
geodesics:



The naked singularity located outside the Friedmann universe
attracts particles !

Spacelike geodesics for the naked singularity

Spacelike geodesics:



Conclusion

- 1) New global solution in General Relativity with a scalar field is found. The scalar field has exponential potential bounded from below, and there are no ghosts and tachyons.
- 2) The Liouville metric without a Killing vector is used as the initial ansatz. However the solution of Einstein's equations is invariant with respect to global Lorentz transformations and therefore has 6 noncommuting Killing vector fields (spontaneous symmetry emergence).
- 3) There are three types of global solutions with metric signature $(+ - - -)$ depending on the integration constant. Solutions with the naked singularity is of great interest in cosmology.
- 4) Metric for the naked singularity can be transformed into the Friedmann form only inside the light cone, where it describes accelerated expansion of the homogeneous and isotropic Universe without the Big Bang. The corresponding global solution is infinitely smooth and geodesically complete except the naked singularity.
- 5) Accelerated expansion is provided by the white hole which is located outside the Friedmann universe. There is no dark energy.
- 6) There are also three types of global solutions with metric signature $(- + + +)$ and exponential potential not bounded from below. It describes formation of the black hole by the scalar field.