More accurate slow-roll approximations for inflation in scalar-tensor theories

The report is based on the paper
E.O. Pozdeeva, M.A. Skugoreva, A.V. Toporensky, S.Yu. Vernov. JCAP **05**, 081 (2025).

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Introduction

- There are the observational bounds on the inflationary parameters:
  the amplitude of scalar perturbations $A_s = (2.10 \pm 0.03)\times 10^{-9}$,
  the spectral index $n_s = 0.9654 \pm 0.0040$
  and the tensor-to-scalar ratio $r < 0.028.$
- The models with the minimally coupled scalar field and the quadratic and quartic potentials give rise too large value of ***r*** compared to the observed one.
- The models with the nonminimal coupling of the scalar field and the scalar curvature $F(\varphi)R$, where $F(\varphi) = 1/(8\pi G) + \xi\varphi^2$, and the same potentials are in good agreement with the observations. They are include the well-known Higgs-driven inflationary model.

Introduction

To calculate the inflationary parameters one usually uses **the slow-roll approximation**, which was applied initially for General Relativity (GR) models with minimally coupled scalar field. Then it was expanded for the nonminimal coupling (**D.I. Kaiser,** 1995).

However, it was shown in several papers, that **the standard slow-roll approximation** does not always work well for the models with the nonminimal coupling (**A.V. Toporensky, L. Järv**, 2022 – the nonminimal coupling with the scalar curvature,
M.A. Skugoreva et al., 2024, **E.O. Pozdeeva**, 2021 – the nonminimal coupling with the Gauss-Bonnet term).

Goal of the work

The goal of the present work was to obtain new more accurate slow-roll approximations, to compare them with the known ones, with the numerical calculations and the observational data, to find more suitable from them for the description of the inflation in the cosmological models with following Lagrangian

$$L=\sqrt{-g}\left[\frac{1}{2}F(\varphi)R-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-V(\varphi)\right],$$

where $F(\varphi)=1/(8\pi G)+\xi\varphi^2, \quad \xi>0,$

$$V(\varphi)=V_0\varphi^4, \quad V_0>0.$$

The FLRW metric $ds^2=-dt^2+a^2(t)\,dl^2$
and Planck units are used $c=\hbar=1$.

Methods of the investigation

Methods of the numerical integration,

algebraic methods

are applyed in this work.

Main equations

Equations of the gravitation and the scalar fields are derived by varying the action with the Lagrangian:

$$3H^2F(\varphi)=\frac{1}{2}\dot{\varphi}^2+V(\varphi)-3F_{\varphi}\dot{\varphi}H, \qquad (1)$$

$$2\dot{H}F(\varphi)=-\dot{\varphi}^2+F_{\varphi}\dot{\varphi}H-F_{\varphi\varphi}\dot{\varphi}^2-F_{\varphi}\ddot{\varphi}, \qquad (2)$$

$$\ddot{\varphi}+3H\dot{\varphi}+V_{\varphi}-3F_{\varphi}\left(\dot{H}+2H^2\right)=0, \qquad (3)$$

where $\frac{dV}{d\varphi}=V_{\varphi}, \quad \frac{dF}{d\varphi}=F_{\varphi}, \quad \frac{d^2F}{d\varphi^2}=F_{\varphi\varphi}.$

Main equations

The system (1)-(3) can be rewritten in Einstein form without the transformation to the Einstein frame:

$$3 M_{Pl}^2 Y^2 = \frac{A^2}{2}\dot{\varphi}^2 + V_{eff}, \qquad (4)$$

$$\dot{Y} = -\frac{A\sqrt{F}}{2 M_{Pl}^3}\dot{\varphi}^2, \qquad (5)$$

$$\ddot{\varphi} + 3\sqrt{\frac{F}{M_{Pl}^2}}\, Y\dot{\varphi} + \frac{A_\varphi}{2A}\dot{\varphi}^2 + \frac{V_{eff\,\varphi}}{A} = 0, \qquad (6)$$

where $M_{Pl}^2 = \frac{1}{8\pi G}$, $V_{eff} = M_{Pl}^4 \frac{V}{F^2}$,

$$A(\varphi) = \frac{M_{Pl}^4}{F^2}\left(1 + \frac{3F_\varphi^2}{2F}\right), \qquad Y = \frac{M_{Pl}}{\sqrt{F}}\left(H + \frac{F_\varphi \dot{\varphi}}{2F}\right).$$

The slow-roll and the inflation parameters.

1. The slow-roll parameters:

$$\varepsilon_1 = -\frac{\dot{H}}{H^2}, \qquad \varepsilon_2 = \frac{d\ln|\varepsilon_1|}{dN}, \qquad \zeta_1 = \frac{\dot{F}}{FH}, \qquad \zeta_2 = \frac{d\ln|\zeta_1|}{dN} \quad .$$

Here $N = \ln a$.

2. The inflation parameters:

$$r = 8|2\varepsilon_1 + \zeta_1|, \qquad A_s = \frac{2H^2}{\pi^2 F r}, \qquad n_s = 1 - 2\varepsilon_1 - \zeta_1 - \frac{2\varepsilon_1\varepsilon_2 + \zeta_1\zeta_2}{2\varepsilon_1 + \zeta_1}.$$

9 The known slow-roll approximations. The simplest one.

In order to find **the slow-roll parameters** we solve the reduced system of the field equations

$$\begin{cases} H^2 \approx \dfrac{V}{3F}, \\ 3H\dot{\varphi} + V_{\varphi} - 6F_{\varphi}H^2 \approx 0. \end{cases}$$

$$\varepsilon_i \ll 1, \zeta_i \ll 1.$$

$$\zeta_1(\varphi) \approx \frac{F_{\varphi}(2F_{\varphi}V - FV_{\varphi})}{FV},$$

$$H^2(\varphi) \approx \frac{V}{3F}, \quad \dot{H} = \frac{\dot{\varphi}H}{2H^2}\frac{d(H^2)}{d\varphi} \Rightarrow \varepsilon_1 = -\frac{\dot{H}}{H^2} = -\frac{\dot{\varphi}H}{2H^4}\frac{d(H^2)}{d\varphi}$$

$$\varepsilon_1(\varphi) \approx \frac{FV_{\varphi} - 2F_{\varphi}V}{2V}\left(\frac{V_{\varphi}}{V} - \frac{F_{\varphi}}{F}\right).$$

10 The known slow-roll approximations. The more accurate one.

We do the transformation from the Jordan frame to the Einstein frame: $g_{\mu\nu}=\Omega^2\tilde{g}_{\mu\nu}$, $\Omega^2=\frac{M_{Pl}^2}{F(\varphi)}$. We find ε_1, ζ_1 and recalculate them for the Jordan frame.

$$\begin{cases}\tilde{H}^2\approx\frac{U(\Phi)}{3M_{Pl}^2},\\ \frac{d\Phi}{d\tilde{t}}\approx-\frac{1}{3\tilde{H}}\frac{dU(\Phi)}{d\Phi}.\end{cases}\quad\Rightarrow\quad\begin{cases}H(\varphi)^2\approx\frac{V}{3F},\\ 3H\dot{\varphi}+\frac{2(FV_\varphi-2F_\varphi V)}{2F+3F_\varphi^2}\approx0.\end{cases}$$

$$\varepsilon_i\ll1,\zeta_i\ll1.$$

$$\zeta_1(\varphi)\approx\frac{2F_\varphi(2F_\varphi V-FV_\varphi)}{V\left(2F+3F_\varphi^2\right)},$$

$$H^2(\varphi)\approx\frac{V}{3F},\quad \dot{H}=\frac{\dot{\varphi}H}{2H^2}\frac{d(H^2)}{d\varphi}\quad\Rightarrow\quad\varepsilon_1=-\frac{\dot{H}}{H^2}=-\frac{\dot{\varphi}H}{2H^4}\frac{d(H^2)}{d\varphi}$$

$$\varepsilon_1(\varphi)\approx-\frac{F(2F_\varphi V-FV_\varphi)}{V\left(2F+3F_\varphi^2\right)}\left(\frac{V_\varphi}{V}-\frac{F_\varphi}{F}\right).$$

New approximation I

In order to find **the slow-roll parameters** we solve the reduced system of the field equations

$$\varepsilon_i \ll 1, \zeta_i \ll 1.$$

$$\begin{cases} 3H^2F \approx V - 3F_{\varphi}\dot{\varphi}H, \\ 2\dot{H}F \approx -\dot{\varphi}^2 + F_{\varphi}\dot{\varphi}H, \\ 3H\dot{\varphi} + V_{\varphi} - 3F_{\varphi}(\dot{H} + 2H^2) \approx 0. \end{cases}$$

$$\zeta_1(\varphi) \approx \frac{2F_{\varphi}(2F_{\varphi}V - FV_{\varphi})}{2FV + 2FF_{\varphi}V_{\varphi} - F_{\varphi}^{\,2}V},$$

$$H^2(\varphi) \approx \frac{2FV - F_{\varphi}^{\,2}V + 2FF_{\varphi}V_{\varphi}}{3F(2F + 3F_{\varphi}^{\,2})},$$

$$\dot{H} = \frac{\dot{\varphi}H}{2H^2}\frac{d(H^2)}{d\varphi} \quad \Rightarrow \quad \varepsilon_1 = -\frac{\dot{H}}{H^2} = -\frac{\dot{\varphi}H}{2H^4}\frac{d(H^2)}{d\varphi}$$

$$\varepsilon_1(\varphi) \approx -\frac{F(2F_{\varphi}V - FV_{\varphi})}{2FV + 2FF_{\varphi}V_{\varphi} - F_{\varphi}^{\,2}V} \times$$
$$\times \left(\frac{2FV_{\varphi} + 2F_{\varphi}V + F_{\varphi}^{\,2}V_{\varphi} + 2FF_{\varphi\varphi}V_{\varphi} + 2FF_{\varphi}V_{\varphi\varphi} - 2F_{\varphi}F_{\varphi\varphi}V}{2FV + 2FF_{\varphi}V_{\varphi} - F_{\varphi}^{\,2}V} - \frac{F_{\varphi}}{F} - \frac{2F_{\varphi}(1 + 3F_{\varphi\varphi})}{2F + 3F_{\varphi}^{\,2}} \right).$$

New approximations II, III

In order to find **the slow-roll parameters** we solve the reduced system of the field equations

$$\varepsilon_i \ll 1, \zeta_i \ll 1.$$

$$\begin{cases} 3 M_{Pl}^2 Y^2 \approx V_{eff}, \\ 3\sqrt{\dfrac{F}{M_{Pl}^2}}\, Y\, \dot{\varphi} + \dfrac{V_{eff\,\varphi}}{A} \approx 0. \end{cases}$$

$$\zeta_1(\varphi) \approx \frac{2F_\varphi (2F_\varphi V - F V_\varphi)}{2FV + FF_\varphi V_\varphi + F_\varphi^2 V},$$

$$H^2(\varphi) = \frac{FY^2}{M_{Pl}^2\left(1+\frac{1}{2}\zeta_1\right)^2} \approx \frac{V}{3F\left(1+\frac{1}{2}\zeta_1\right)^2} \approx \frac{V}{3F(1+\zeta_1)}.$$

$$H^2(\varphi) \approx \frac{V}{3F(1+\zeta_1)} \approx \frac{V(2FV + FF_\varphi V_\varphi + F_\varphi^2 V)}{3F(2FV - FF_\varphi V_\varphi + 5F_\varphi^2 V)}, \qquad \textbf{II}$$

$$H^2(\varphi) \approx \frac{V}{3F\left(1+\frac{1}{2}\zeta_1\right)^2} \approx \frac{(2FV + FF_\varphi V_\varphi + F_\varphi^2 V)^2}{3FV(2F + 3F_\varphi^2)}. \qquad \textbf{III}$$

New approximations II, III

Differentiating the found expressions for $H^2(\varphi)$ we obtain the parameter $\varepsilon_1(\varphi)$.

$$\dot{H}=\frac{\dot{\varphi}H}{2H^2}\frac{d(H^2)}{d\varphi} \quad \Rightarrow \quad \varepsilon_1=-\frac{\dot{H}}{H^2}=-\frac{\dot{\varphi}H}{2H^4}\frac{d(H^2)}{d\varphi}$$

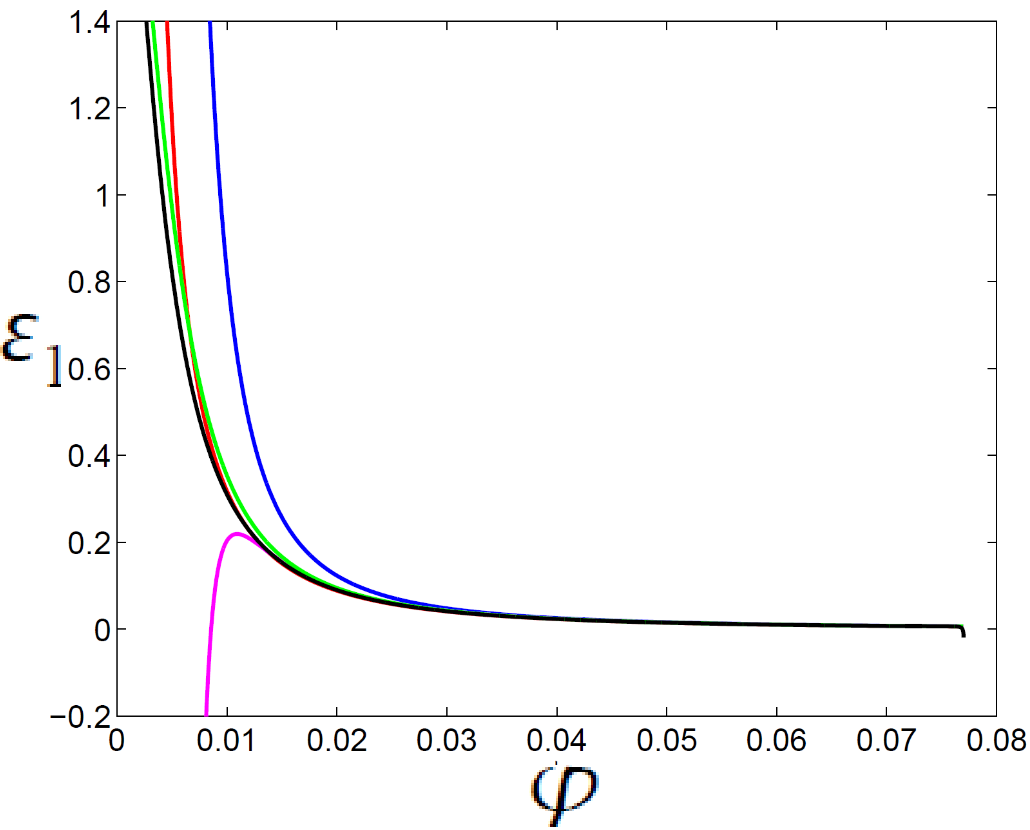
II

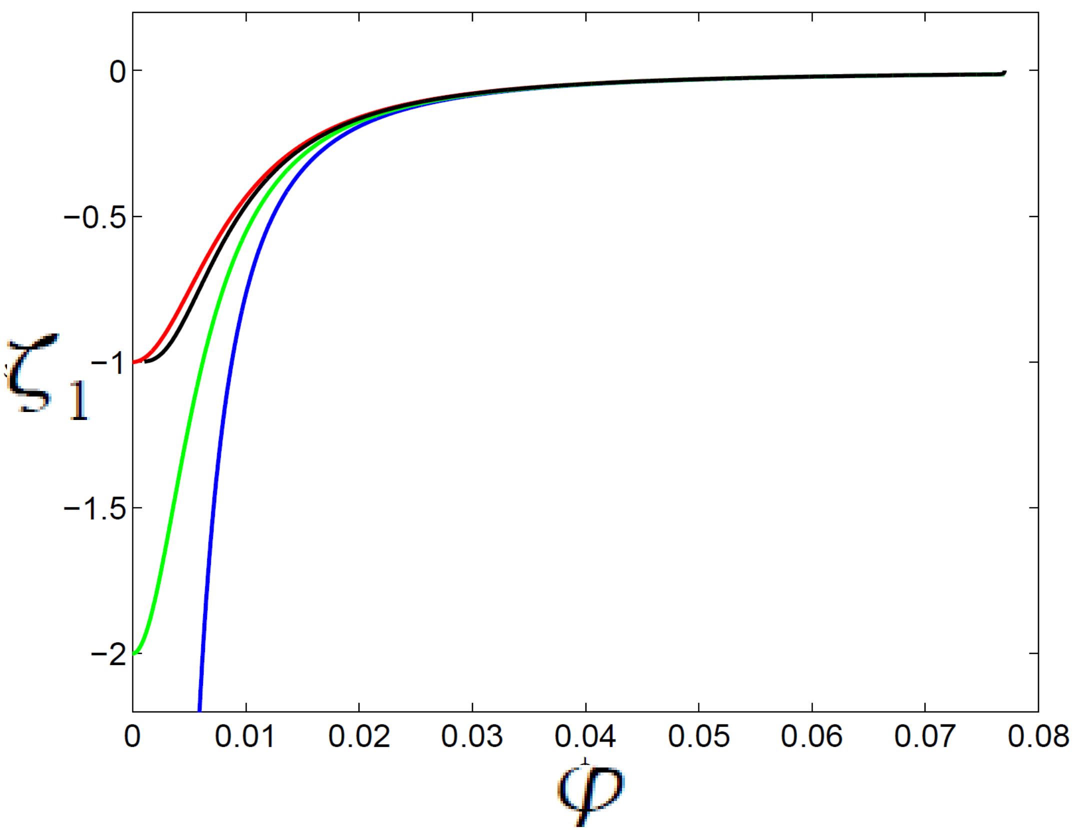
$$\varepsilon_1(\varphi)\approx-\frac{F(2F_\varphi V-FV_\varphi)}{2FV+FF_\varphi V_\varphi+F_\varphi{}^2V}\times$$
$$\times\left(\frac{V_\varphi}{V}+\frac{2FV_\varphi+2F_\varphi V+2F_\varphi{}^2V_\varphi+FF_{\varphi\varphi}V_\varphi+FF_\varphi V_{\varphi\varphi}+2F_\varphi F_{\varphi\varphi}V}{2FV+FF_\varphi V_\varphi+F_\varphi{}^2V}\right.$$
$$\left.-\frac{F_\varphi}{F}-\frac{2FV_\varphi+2F_\varphi V+4F_\varphi{}^2V_\varphi-FF_{\varphi\varphi}V_\varphi-FF_\varphi V_{\varphi\varphi}+10F_\varphi F_{\varphi\varphi}V}{2F+3F_\varphi{}^2}\right),$$

III

$$\varepsilon_1(\varphi)\approx-\frac{F(2F_\varphi V-FV_\varphi)}{2FV+FF_\varphi V_\varphi+F_\varphi{}^2V}\times$$
$$\times\left(\frac{2(2FV_\varphi+2F_\varphi V+2F_\varphi{}^2V_\varphi+FF_{\varphi\varphi}V_\varphi+FF_\varphi V_{\varphi\varphi}+2F_\varphi F_{\varphi\varphi}V)}{2FV+FF_\varphi V_\varphi+F_\varphi{}^2V}-\frac{V_\varphi}{V}-\frac{F_\varphi}{F}-\frac{4F_\varphi(1+3F_{\varphi\varphi})}{2F+3F_\varphi{}^2}\right).$$

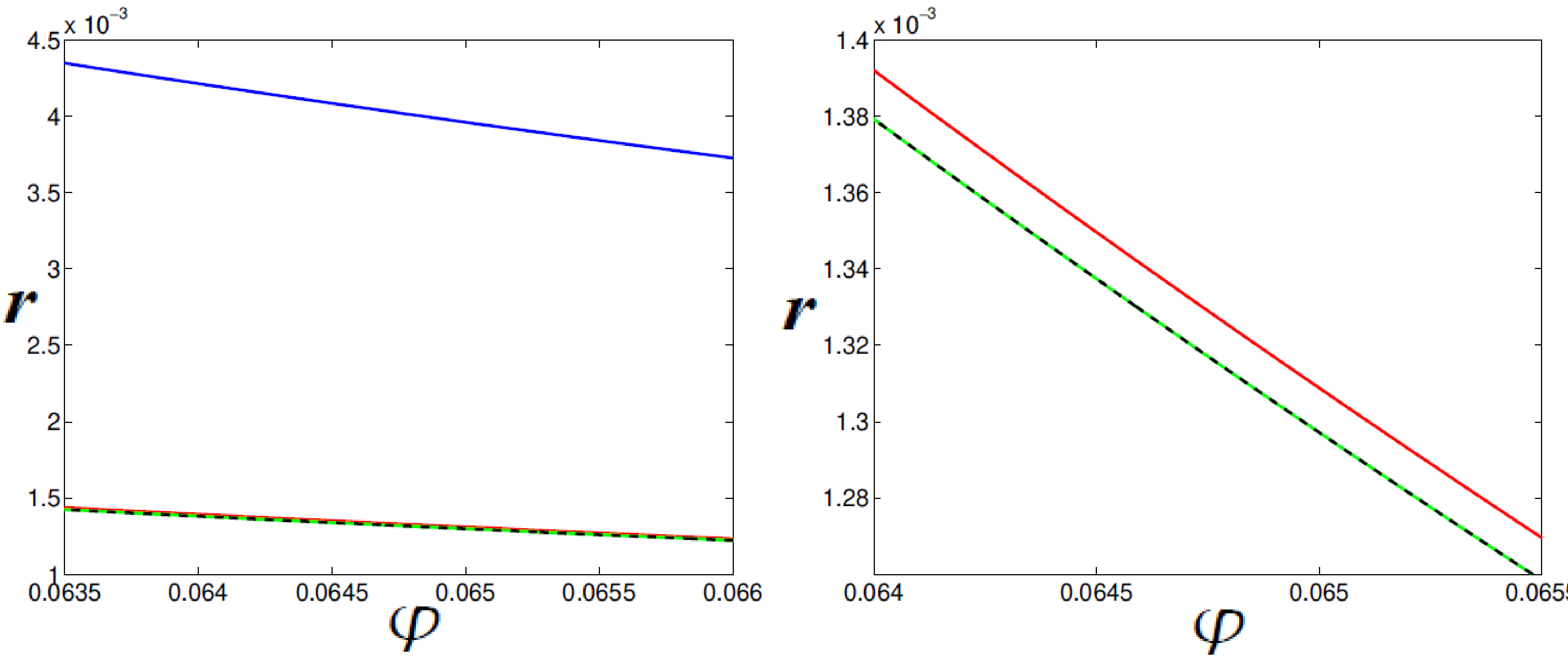
Fig. 1. The dependences ε_1, ζ_1 on the scalar field φ for the model $V(\varphi)=V_0\varphi^4$, $F(\varphi)=1/(8\pi G)+\xi\varphi^2$.





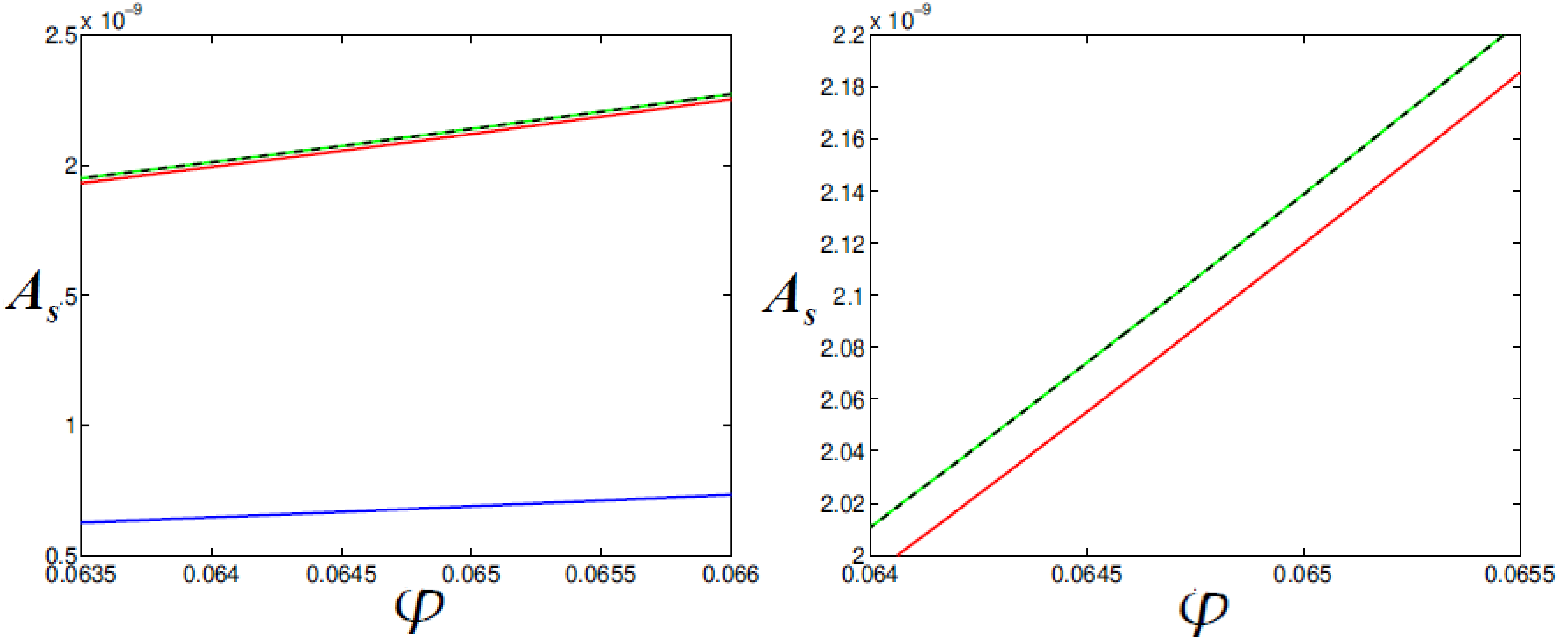
$$\xi=17367, \quad V_0=0.0125, \quad 8\pi G=1.$$

Fig. 2. The dependence *r* on the scalar field φ for the model $V(\varphi)=V_0\varphi^4,\ F(\varphi)=1/(8\pi G)+\xi\varphi^2$.



$$\xi=17367,\quad V_0=0.0125,\quad 8\pi G=1.$$

Fig. 3. The dependence A_s on the scalar field φ for the model $V(\varphi)=V_0\varphi^4$, $F(\varphi)=1/(8\pi G)+\xi\varphi^2$.



$$\xi=17367,\quad V_0=0.0125,\quad 8\pi G=1.$$

Fig. 4. The dependence n_s on the scalar field φ for the model $V(\varphi)=V_0\varphi^4$, $F(\varphi)=1/(8\pi G)+\xi\varphi^2$.

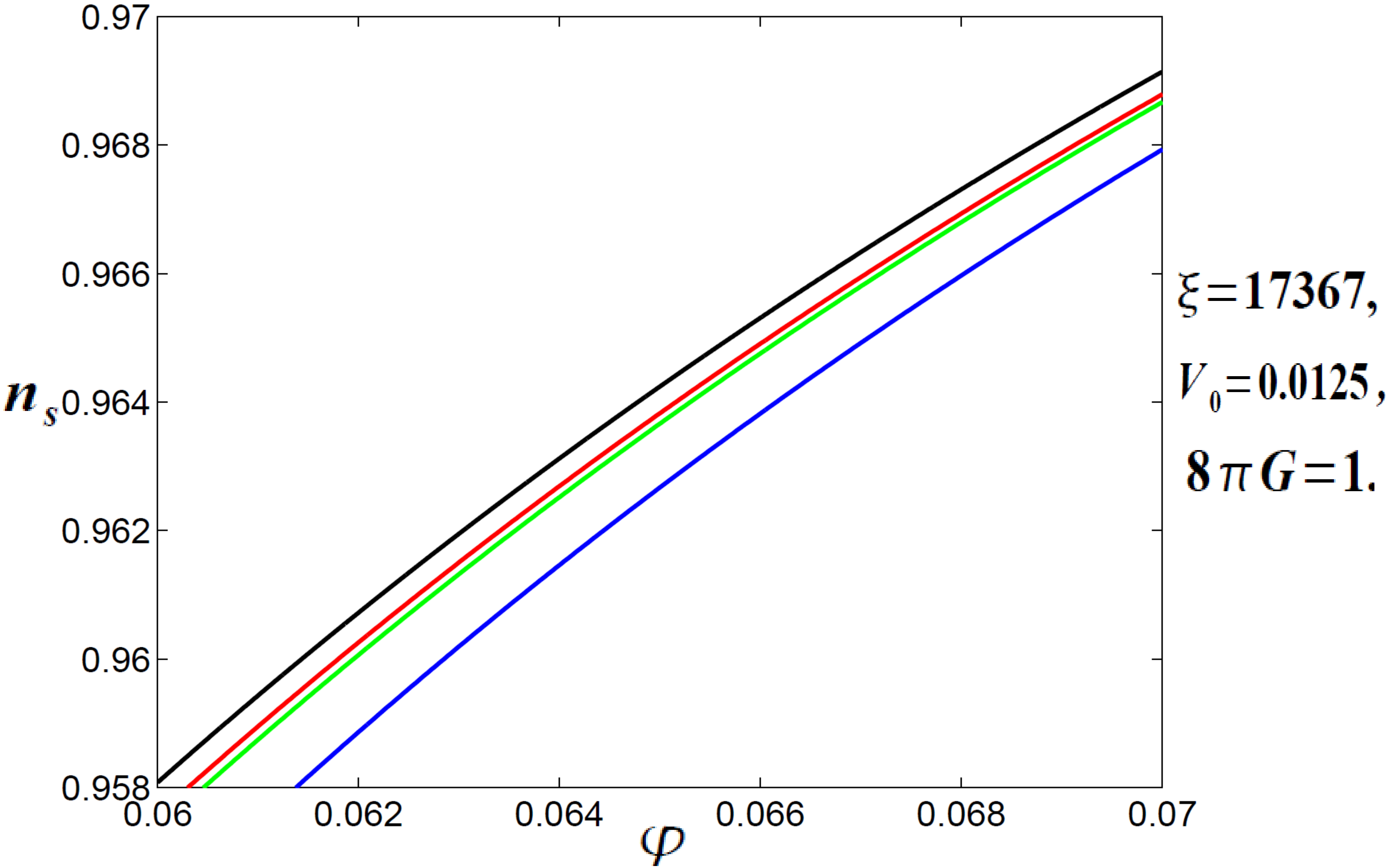
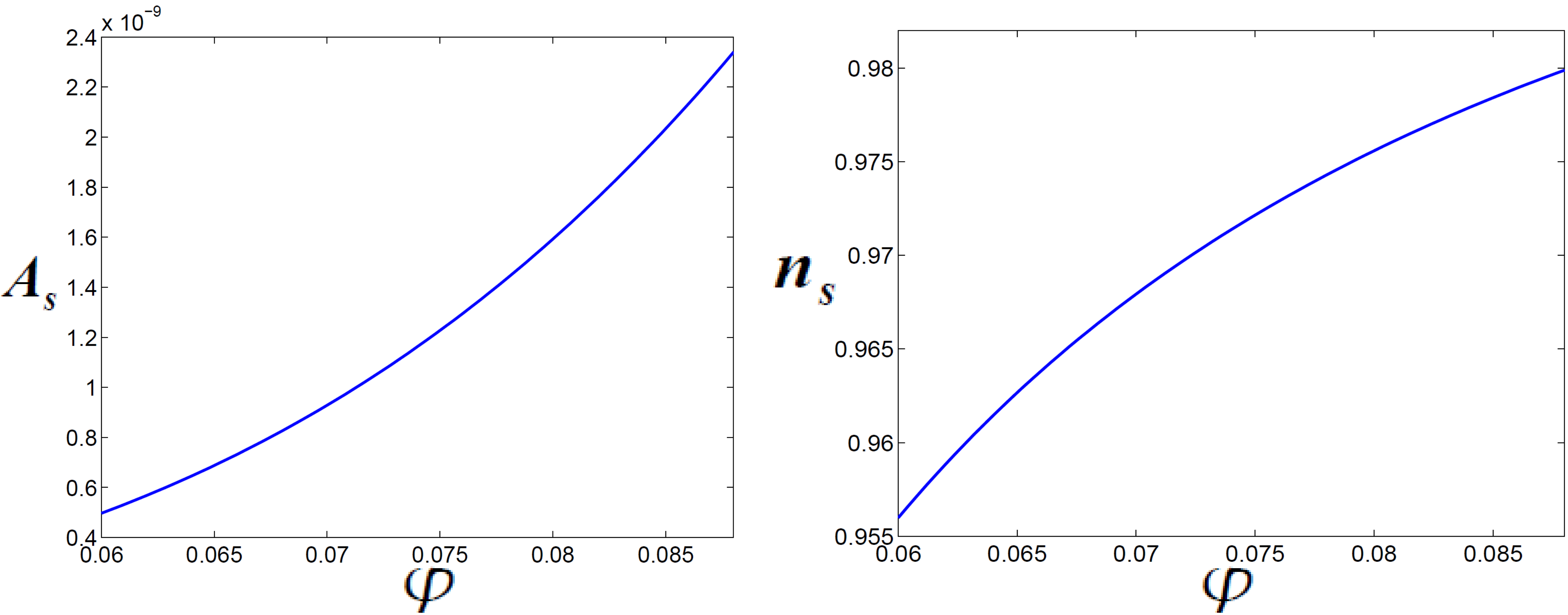


Fig. 5. The dependences A_s and n_s on the scalar field φ for the best known slow-roll approximation for the model $V(\varphi)=V_0\varphi^4, F(\varphi)=1/(8\pi G)+\xi\varphi^2$.



$$\xi=17367, \quad V_0=0.0125, \quad 8\pi G=1.$$

Conclusion

1. We have investigated the inationary models with the scalar field nonminimally coupled to gravity $F(\varphi)R$, where $F(\varphi)=1/(8\pi G)+\xi\varphi^2$, and the potential $V(\varphi)=V_0\varphi^4$.

2. We find **new slow-roll approximations** for these models. The inflationary dynamics have been investigated in the Jordan frame for three scenarios including the Higgs-driven inflation, which demonstrates that **new slow-roll approximations** are **more accurate** at the end of the inflation, but also give essentially more precise estimations for ***r*** и ***A*s**.

3. We have shown that **the best known slow-roll approximation are not suitable** for the calculations of the inflationary parameters in these models.

Thanks for attention!

Proposals of the work send to

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