Landau method meets exact WKB

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WKB approximation

Consider 1D Schrödinger equation ($m = 1$)

$$\hbar^2 \psi''(x) + \underbrace{[2E - 2U(x)]}_{p^2(x)} \psi(x) = 0 \quad \text{at} \quad \hbar \to 0.$$

Solution is given by asymptotic series normalized at x_0 ($p(x_0) \neq 0$)

$$\psi(x) \sim a_+(\hbar)\xi^+_{x_0}(\hbar, x) + a_-(\hbar)\xi^-_{x_0}(\hbar, x),$$

$$\xi^{\pm}_{x_0}(\hbar, x) = \frac{\exp\left(\pm i \cdot \hbar^{-1} \int_{x_0}^{x} \mathcal{P}(\hbar, x')dx'\right)}{\sqrt{\mathcal{P}(\hbar, x)}}, \quad \mathcal{P}^2 = p^2 + \hbar^2 \sqrt{\mathcal{P}} \left(\frac{1}{\sqrt{\mathcal{P}}}\right)''.$$

$\xi^{\pm}_{x_0}(\hbar, x)$ are multivalued analytic.

Stokes lines and Stokes graphs

Stokes line

A curve L in $\mathbb{C}$ starting from turning point $x_* : \; p(x_*) = 0$

$$\operatorname{Im} i \int_{x_*}^{x} p(x')dx' = 0, \; x \in L.$$

Stokes graph

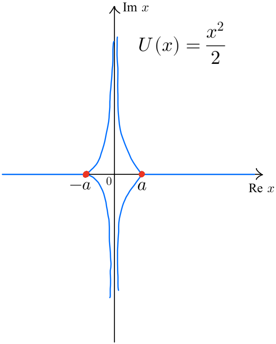
A collection of all Stokes lines for particular $p(x)$.

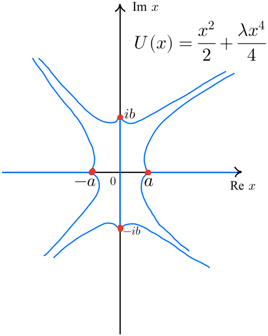
Stokes regions

Stokes lines divide $\mathbb{C}$ into unbounded regions $\mathcal{R}$.

Examples of Stokes lines

Classically forbidden region is an example of a Stokes line.





Examples of Stokes graphs for harmonic and anharmonic oscillators.

Boundary conditions

Consider L connecting x_* with ∞. For chosen determination of $p(x)$ one of $\xi^{\pm}$ will decay at ∞ (recessive), another will grow (dominant).

Condition $\psi \xrightarrow[x\to\infty \text{ on } L]{} 0$ unambiguously fixes asymptotic expansion of a solution.

$$\psi(x) \sim a_+(\hbar)\xi^+_{x_0}(\hbar, x),\ x \in L \quad \text{if e.g. } \xi^+ \text{ is recessive}$$

Example:
Condition that $\psi(x)$ decays in classically forbidden region of a potential well.

Normalization at infinity

For Landau method we chose another normalization condition than at x_0:

$$\xi^{\pm}_{x_*,\infty}(\hbar,x) = \frac{\exp\left(\pm i \cdot \hbar^{-1}\int_{x_*}^{x} p(x')dx' \pm i \cdot \hbar^{-1}\int_{\infty}^{x}[\mathcal{P}(\hbar,x') - p(x')]dx'\right)}{\sqrt{\mathcal{P}(\hbar,x)}},$$

$\mathcal{P}(\hbar,x) - p(x) = \mathcal{O}(\hbar^2)$ is integrable at ∞.

Relation between large x and $\hbar \ll 1$

If we choose contour to ∞ on Stokes line L and take $\psi(x) \sim a_{\pm}(\hbar)\xi^{\pm}_{x_*,\infty}(\hbar,x)$ recessive on L, then

$$\lim_{\hbar\to 0}\psi(x) = \lim_{x\to\infty}\psi(x).$$

Global exact WKB solution

Exact WKB deals with Borel summability of asymptotic series in $\hbar$ to the exact solution of the Schrödinger equation.

Borel summability

WKB solution $\psi(x) \sim a_+(\hbar)\xi^+_{x_*,\infty}(\hbar, x) + a_-(\hbar)\xi^-_{x_*,\infty}(\hbar, x)$ is Borel summable if

- x lies inside a Stokes region;
- $x \in L$ and $\psi(x)$ is recessive on L.

Connection problem

Starting from Borel summable expansion in a Stokes region one can connect exact solution to expansion in another Stokes region via suitable analytic continuation of $\xi^\pm$ on a Riemann surface $\mathbb{C}_2$ of $p(x)$.

Delabaere, Pham, '99

Original Landau method

We evaluate matrix elements $\langle n|f(x)|n'\rangle$; $n,\ n' \gg 1$; $n \geq n'$.

Preliminaries

- $\psi^*(x^*) = \psi(x)$;
- $f(x)$ is not exponentially large;
- LO $\psi(x) \sim \frac{C}{2}\left(\mathrm{e}^{-i\pi/4}\xi^- + \mathrm{e}^{i\pi/4}\xi^+\right)$ in classically allowed region.

Landau method

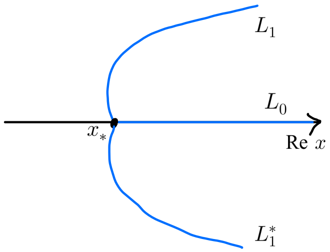
- Decompose $\psi_n(x)$ into sum of two complex conjugated solutions $\psi_n^\pm(x)$, $\psi_n^+(x) \propto \xi^+$ is decaying in the upper half plane;
- $\langle n|f(x)|n'\rangle \sim 2\mathrm{Re}\int_{-\infty}^{+\infty}\psi_n^+\, f\, \psi_{n'}dx \propto 2\mathrm{Re}\int_{-\infty}^{+\infty}\xi_n^+\, f\, \xi_{n'}^- dx$;
- Deform contour to singular point $\tilde{x}$ of $U(x)$. For polynomials $\tilde{x} = \infty$;
- $C^2/4\oint_\Gamma p^{-1}(x) \sim 1$, where Γ encycles classically allowed region.

Landau, Lifshitz, Vol. 3; Cornwall, Tiktopoulos, '93.

$\psi^{\pm}(x)$ in Exact WKB

Definition of $\psi^{\pm}$

- $\psi(x)$ is recessive on $L_0 \subset \mathbb{R}$, $\psi^{+}(x)$ is recessive on L_1 and $\psi^{-}(x)$ is recessive on L_1^*. They are all solutions of Schrödinger equation;
- $\psi(x) \sim \frac{C(\hbar)}{2} \mathrm{e}^{-i\pi/4} \xi^{-}_{x_*,\infty}(\hbar,\, x)$ on L_0, $C \in \mathbb{R}$, ξ^{-} is recessive on L_0;
- $\psi(x) = \psi^{+}(x) + \psi^{-}(x)$;
- $[\psi^{+}(x^*)]^* = \psi^{-}(x)$;
- $\langle n|f(x)|n'\rangle = 2\mathrm{Re} \int_{-\infty}^{+\infty} \psi_n^{+}(x) f(x) \psi_{n'}(x) dx$, $n \geq n'$



Up to exponentially small corrections

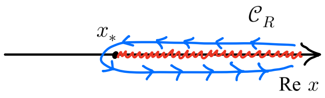
$$\psi_{n'}(x) \sim \frac{C_{n'}}{2}\xi^{-}_{x_*,\infty,n'}(\hbar,\, x), \quad \psi^{+}_{n}(x) \sim \frac{C_n}{2}\mathrm{e}^{-i\Delta S_R(n)}\xi^{+}_{x_*,\infty,n}(\hbar,\, x).$$

Matrix element

$$\langle n|f(\hat{x})|n'\rangle \sim \mathrm{Re}\frac{C_n C_{n'}}{2}\mathrm{e}^{-i\Delta S_R(n)}\int_{-\infty+i\varepsilon}^{+\infty+i\varepsilon}\xi^{+}_{x_*,\infty,n}(x)f(x)\xi^{-}_{x_*,\infty,n'}(x)dx.$$

C_n is obtained in the same fashion via $n' = n,\ f(x) = 1$ and $\langle n|n\rangle = 1$.

$$\Delta S_R(n) = \hbar^{-1}\int_{\mathcal{C}_R}[\mathcal{P}_n(x) - p_n(x)]dx.$$



λx^4 oscillator

We consider anharmonic oscillator with a potential

$$U(x) = \frac{x^2}{2} + \frac{\lambda x^4}{4}.$$

After coordinate change $x = z/\sqrt{\lambda}$, Schrödinger equation is in WKB-like form ($\hbar \to \lambda,\ E \to \lambda E$):

$$\lambda^2 \psi''(z) + (2\lambda E - z^2 - z^4/2)\psi(z) = 0.$$

At $\lambda \to 0$ work both WKB and perturbation theory.

Matrix elements $\langle n|\hat{x}|0\rangle$ are related to $\langle n|\hat{\phi}(0)|0\rangle$ in $\lambda\phi^4$ (see presentation of D. Levkov).

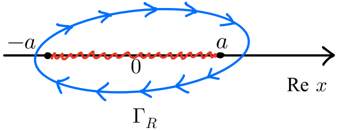
Normalization and quantization

Quantization condition (up to exponentially small corrections):

$$\oint_{\Gamma_R} \mathcal{P}(z)dz \sim 2\pi(\lambda n + \lambda/2).$$

Normalization condition (up to exponentially small corrections):

$$\frac{C^2\lambda^{1/2}}{4}\mathrm{e}^{-i\Delta S_R} \oint_{\Gamma_R} \mathcal{P}^{-1}(z)dz \sim 1.$$



Matrix element and concluding remark

Up to exponentially small corrections exact Landau method gives

$$\langle n|\hat{x}|0\rangle \sim \left((-1)^{n+1}+1\right) 2\pi i \operatorname*{Res}_{x=+\infty} x\psi_n^+(x)\psi_0(x),$$

where Res means coefficient $\propto x^{-1}$ in $x \to +\infty$ expansion.

Conclusions

- Landau method can become perturbatively exact and may be truly exact;
- Limits $x \to \infty$ and $\lambda \to 0$ in WKB are connected to each other;
- Exact WKB let us prove the formula for amplitudes from the talk of D. Levkov.

Remaining questions

- How expression for $\langle n|f(\hat{x})|0\rangle$ differs from $\langle n|\hat{x}|0\rangle$?
- Is the expression for $\langle n|\hat{x}|0\rangle$ exact?
- How to take exponentially small contributions into account?
- How situation changes for other potentials?

Thank you for your attention!