Zoo of flows in 3d gauged supergravity with 2d spherical target space

based on a joint work with Lev Astrakhantsev and Misha Podoinitsyn (BLTP) `in progress`

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QFTHEP'270
July 3, MSU, Moscow

Outline

1. Motivation
2. $3d\ \mathcal{N}=2$ supergravity model
3. 2d autonomous dynamical systems, non-thermal flows $f=1$
4. 3d autonomous dynamical systems, thermal flows $f\neq 1$
5. Outlook

Motivation

Holography states that the on-shell value of the supergravity action in AdS$_{(d+1)}$ is equated with the generating functional of composite operators in CFT$_d$

$$\boxed{e^{-S_{AdS}(\phi)}|_{\lim_{z\to 0}(\phi(x,z)z^{\Delta-d})=\phi_0(x)} = \langle e^{\int d^dx\phi_{(0)}\mathcal{O}(x)}\rangle_{CFT},}$$

where $\phi_{(0)}$ is a d-dim. field is a boundary value of a $(d+1)$-dim. field ϕ, an operator $\mathcal{O}$ on the field theory side with the conformal dimension Δ.

`Maldacena'97,Witten'98, Gubser,Klebanov,Polyakov'98`

The source fields in d-dim. CFT and the partition function :

$$\mathcal{S} = S - \int d^dx\phi_{(0)}(x)\mathcal{O}(x), \quad Z(\phi_{(0)}) = e^{-W[\phi_{(0)}]} = \langle e^{\int d^dx\phi_{(0)}\mathcal{O}(x)}\rangle_{CFT}$$

- Holographic renormalization, RG flows, i.e. systematic removing the divergences and identifying the finite expressions, implies a careful analysis near the boundary.

`Akhmedov'98; de Boer et.al'98; Skenderis'99, de Haro et.al.'99`

`Papadimitriou&Skenderis'04`

The asymptotically AdS/dS metric (the domain wall) `Skenderis'99, de Haro et.al.'99`

$$ds^2 = e^{2\mathcal{A}(w)}\eta_{ij}dx^idx^j + dw^2, \quad \phi = \phi(w)$$

- Holographic QGP, holographic RG flows `Aref'eva'14, Aref'eva&Rannu'18`
- Irrelevant deformations, in particular , $T\bar{T}$-deformations `Chang,Ferko&Sethi'23`
- Thermal holography `Witten'98`
- Black hole interior `Hartnoll et.al.'20, Caceres et al.'23`
- de Sitter holography `Witten'01,Strominger'01, Maldacena'03`

$3d$ $\mathcal{N} = 2$ supergravity model

The supergravity model includes a graviton $e_\mu{}^a$, a gravitini ψ_μ, a gauge field A_μ and $\mathcal{N}=2$ multiplet (n scalar fields ϕ^α and n fermions λ^r)

`Deger, Kaya, Sezgin, Sundell (2000)`

The scalar field parametrizes the coset space

$$\frac{SU(2)}{U(1)} = \mathbb{S}^2$$

The supergravity Lagrangian for a complex scalar field Φ

$$\boxed{e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{e^{-1}}{16m\,a^4}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{|D_\mu\Phi|^2}{a^2(1+|\Phi|^2)} - V(\Phi) + \text{fermions},}$$

where $e = \det e_\mu{}^a$, $D_\mu\Phi = (\partial_\mu - iA_\mu)\Phi$, m is related to the cosmological constant, a is related to the curvature of the scalar manifold, the potential $V(\Phi)$ is given by

$$V(\Phi) = 2m^2C^2\left(2a^2|S|^2 - C^2\right) \qquad C = \frac{1-|\Phi|^2}{1+|\Phi|^2}, \quad S = \frac{2\Phi}{1+|\Phi|^2}.$$

Introducing the redefinition of the scalar field allows us to come to

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{e^{-1}}{a^4}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{4a^2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4a^2}|S|^2(\partial_\mu\theta + A_\mu)(\partial^\mu\theta + A^\mu) - V(\phi).$$

The truncated action ($\theta = 0$, $A^\mu = 0$) is given by

$$\boxed{S = \frac{1}{4}\int d^3x\sqrt{|g|}\left(R - \frac{1}{a^2}(\partial\phi)^2 - 4V(\phi)\right),}$$

with the potential of the scalar field ϕ given by

$$\boxed{V(\phi) = -2m^2\cos^2\phi\left((1+2a^2)\cos^2\phi - 2a^2\right).}$$

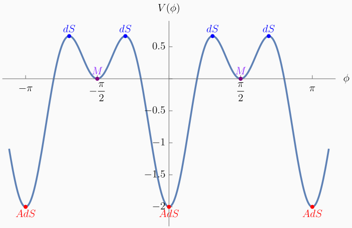

Figure 1: The behaviour of the potential for $a^2 = 1$: AdS , dS and Minkowski M extrema.

3d $\mathcal{N} = 2$ gauged supregravity with $\mathbb{H}^2$: `Deger'02, AG&Usova'22, Arkhipova et al.'24, AG,Nikolaev&Podoinitsyn'24, AG,Gourgoulhon&Podoinitsyn'24`, Gutperle&Hultgreen-Mena'24 (Janus flows)

The superpotential

The potential is related to the superpotential W by the following relation

$$V = \frac{a^2}{4} W'^2 - \frac{1}{2} W^2,$$

the exact supersymmetric superpotential Deger'02 has the form

$$W_{\text{susy}} = -2m \cos^2 \phi.$$

$\mathcal{N}=2$ $D=5$ gauged supergravity `Behrndt'00`

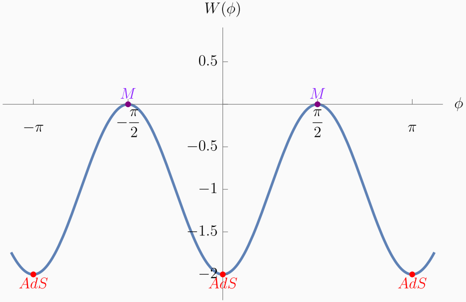

Figure 2: Plot of W with $m=1$. There are three AdS and two Minkowski M critical points on the period $\phi \in [-\pi, \pi]$.

BPS equations and half-supersymmetric solutions

Variations of gravitino and dilatino

$$\delta\psi_\mu = (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab})\epsilon + \frac{1}{2}W\gamma_\mu\epsilon, \quad \delta\lambda = \frac{1}{2}\left(-\gamma^\mu\partial_\mu\phi - \frac{2}{a}\frac{\partial W}{\partial\phi}\right)\epsilon$$

BPS equations

$$\boxed{\dot{A} = -W, \quad \dot{\phi} = a^2 W'.}$$

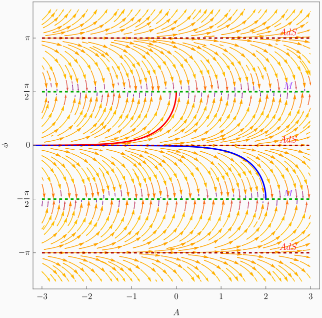

Figure 3: $m=1$, $a=1$, $n=0$ and different $\mathrm{c_A}$.

The ansatz for solutions and EOM

The ansatz for the metric and for the scalar field is given by

$$ds^2 = e^{2A(w)}(-f(w)dt^2 + dx^2) + \frac{dw^2}{f(w)}, \quad \phi = \phi(w), \quad f(w) \approx \begin{cases} 1, & \text{asympAdS} \\ 0, & \text{if } w \to w_h, \\ -1 & \text{asympdS} \end{cases}$$

The Hawking temperature is (dual to T of a dual field theory `Witten'98`)

$$T_H = \frac{e^{A(w_h)}}{4\pi} \left|\frac{df}{dw}\right| |_{w=w_h}.$$

The equations of motion

$$\ddot{A} + \frac{\dot{\phi}^2}{a^2} = 0,$$
$$\ddot{g} + \dot{g}^2 + 2\dot{g}\dot{A} = 0,$$
$$\dot{A}\dot{g} + 2\dot{A}^2 - \frac{\dot{\phi}^2}{a^2} + 4e^{-g}V = 0,$$
$$\ddot{\phi} + \dot{g}\dot{\phi} + 2\dot{A}\dot{\phi} - 2a^2e^{-g}V_\phi = 0,$$

where $g = \ln f$ and dot is a derivative with respect to w, V_ϕ is a derivative of the potential with respect to ϕ.

Exact solutions of the model

Minkowski, AdS and dS solutions $f=1$

$$\ddot{A}+\frac{\dot{\phi}^2}{a^2}=0, \qquad A=\sqrt{-2V(\phi_*)}w+\mathrm{c}_2.$$

For Minkowski $V=0$, for AdS $V=-2$, for dS $V=\frac{2a^4}{2a^2+1}$, then we have

$$\boxed{\begin{array}{l} ds^2_M=-dt^2+dx^2+dw^2, \\ \\ ds^2_{AdS}=e^{2w}(-dt^2+dx^2)+dw^2, \quad ds^2_{dS}=e^{\frac{4a^2}{\sqrt{2a^2+1}}iw}(-dt^2+dx^2)+dw^2 \end{array}}$$

Double Wick rotation $t\to ir$, $-dt^2=dr^2$, $w\to i\tau$, $dw^2\to -d\tau^2$ gives de Sitter metric

$$ds^2=-d\tau^2+e^{-a^2\sqrt{\frac{1}{2a^2+1}}\tau}\left(dr^2+dx^2\right).$$

Half-supersymmetric exact solution (irrelevant deformation)`Deger'02`

$$\boxed{A(w)=-\frac{1}{4a^2}\ln[e^{-8ma^2w}+1]+\mathrm{c}_A\,, \qquad \phi=\pm\arctan[e^{4ma^2w}]\pm n\,\pi\,,}$$

where $\mathrm{c}_A\in\mathbb{R}$, $n\in\mathbb{Z}$ and $w\in(-\infty,\infty)$. AdS as $w\to-\infty$, Minkowski as $w\to\infty$.

BTZ black hole (non-rotating), $f\neq 1$

$$A(w)=\sqrt{-\frac{V(\phi_h)}{2}}\,w\,, \quad f(w)=e^{g(w)}=c-e^{-\sqrt{-2\,V(\phi_h)}(w-w_h)}, \quad \phi_h=\phi(w_h).$$

2d autonomous dynamical systems, non-thermal flows $f = 1$

New variable :

$$X=\frac{d\phi}{dA}=\frac{\dot{\phi}}{\dot{A}},\quad \phi\in[-\pi;\pi]$$

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Kiritsis et.al.'08'14-'19,Aref'eva,Policastro,AG'19

The equations of motion are brought to the dynamical system on $\mathbf{R}^2$

$$\frac{d\phi}{dA}=X,$$
$$\frac{dX}{dA}=\left(\frac{X^2}{a^2}-2\right)\left(X+\frac{a^2}{2}\frac{V'}{V}\right),$$

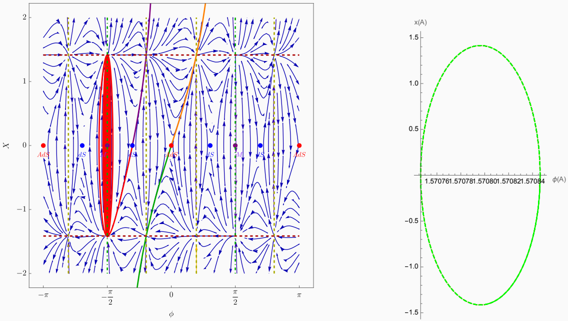

Figure 4: $(\phi,X)_{AdS}=(0,0)\quad(\phi,X)_{dS}=(\arccos(\pm\sqrt{\frac{a^2}{2a^2+1}}),0)$

de Sitter-Minkowski flow, the asymptotics near Minkowski

Leaving the leading term in the second equation of the system we are brought to

$$\frac{d\phi}{dA} = X,$$
$$\frac{dX}{dA} = \left(\frac{X^2}{a^2} - 2\right)\frac{a^2}{\phi - \frac{\pi}{2}}.$$

The system can be represented as a single equation for the scalar field

$$\left(\phi(A) - \frac{\pi}{2}\right)\phi''(A) - (\phi'(A))^2 + 2 = 0\,,$$

where we define $\phi'(A) := d\phi/dA$ and set $a^2 = 1$.

Assuming the following initial conditions

$$\phi(0) = \frac{\pi}{2} + \phi_i\,, \qquad \phi'(0) = X_i\,,$$

where $\phi_i^2 \simeq 0$, $-\sqrt{2} < X_i < \sqrt{2}$. Near the Minkowski point the asymptotics is

$$\boxed{\phi(A) = \frac{\pi}{2} + \frac{\phi_i}{\sqrt{2 - X_i^2}}\left(\sqrt{2 - X_i^2}\cos\frac{A\sqrt{2 - X_i^2}}{\phi_i} + X_i \sin\frac{A\sqrt{2 - X_i^2}}{\phi_i}\right).}$$

The asymptotic solutions near AdS and dS

Near AdS fixed point (saddle)

$$\begin{bmatrix} \phi \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + k_1 e^{-\Delta_1 A} u_1 + k_2 e^{-\Delta_2 A} u_2$$

with the eigenvalues $\lambda_{1,2} = -1 \pm (1+2a^2)$ and the eigenvectors

$$u_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

$\Delta = 4$ for $a^2 = 1$, $\Delta = 1 + (1+2a^2)$ grows linearly, irrelevant.

Near dS fixed points (saddles)

$$\begin{bmatrix} \phi \\ X \end{bmatrix} = \begin{bmatrix} \pm\arccos(\sqrt{\frac{a^2}{2a^2+1}}) \\ 0 \end{bmatrix} + k_1 e^{-\Delta_1 A} u_1 + k_2 e^{-\Delta_2 A} u_2,$$

with $\Delta_{1,2} = -\lambda_{1,2}$, $\lambda_{1,2} = -1 \pm \sqrt{9+8a^2}$,

$$u_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

$\Delta = 1 + \sqrt{9+8a^2}$ grows linearly, irrelevant.

Global phase portrait of 2d system with $f=1$.

$$Z = 2\tan\left(\frac{\phi}{2}\right),$$

The Poincaré transformation

$$Z = \frac{z}{\sqrt{1-x^2-z^2}}, \qquad X = \frac{x}{\sqrt{1-x^2-z^2}},$$

with the constraint $x^2+z^2 \leq 1$. In the new coordinates the dynamical system takes the form

$$\dot{z} = \mathrm{p}(x,z),$$
$$\dot{x} = \mathrm{q}(x,z),$$

where the RHSs of equations are

$$\mathrm{p}(x,z) = \frac{x}{4}(z^2-1)\left(4x^2+3z^2-4\right) - xz\left(3x^2+2z^2-2\right)\cdot$$
$$\cdot\left(x + \frac{16z(1-x^2-z^2)\left(48(x^2-1)z^2+16(x^2-1)^2+33z^4\right)}{588(x^2-1)z^4+368(x^2-1)^2z^2+64(x^2-1)^3+285z^6}\right),$$

$$\mathrm{q}(x,z) = \frac{(x^2-1)x^2(z-4x)}{4} + (1-x^2-z^2)\Big(\frac{z}{380}(1408-1693x^2) + (x^2-1)x-$$
$$-\frac{4(7x^4-9x^2+2)z}{5(4x^2+5z^2-4)} - \frac{4(x^2-1)z(524x^4+x^2(591z^2-820)-306z^2+296)}{19(72(x^2-1)z^2+16(x^2-1)^2+57z^4)}\Big).$$

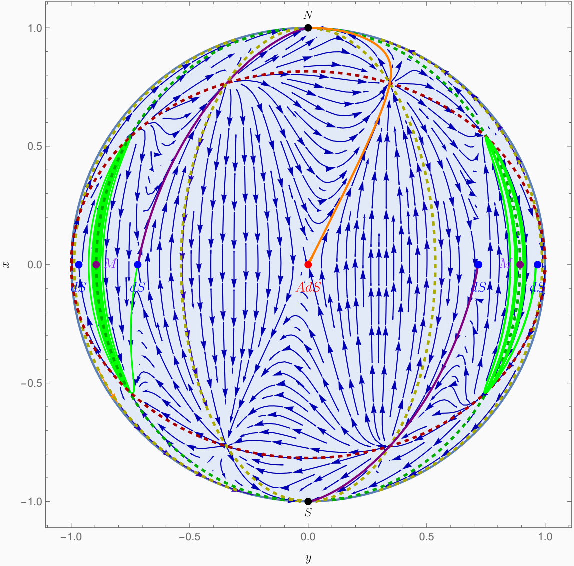

Figure 5: Global phase portrait of the dynamical system in the disk with numerical trajectories: AdS-Minkowski flow (orange), dS-Minkowski (spiraling) flows (green), dS-North Pole flow (violet)

3d autonomous dynamical systems, thermal flows $f \neq 1$

3d autonomous dynamical systems, thermal flows $f \neq 1$

The new variable

$$Y = \frac{dg}{dA} = \frac{\dot{f}}{f\dot{A}}, \quad Y \in (-\infty, 0].$$

Then we have the following dynamical system:

$$\begin{aligned}
\frac{d\phi}{dA} &= X, \\
\frac{dX}{dA} &= \left(\frac{X^2}{a^2} - Y - 2\right)\left(X + \frac{a^2}{2}\frac{V_\phi}{V}\right), \\
\frac{dY}{dA} &= Y\left(\frac{X^2}{a^2} - Y - 2\right).
\end{aligned}$$

The dynamical system in the cylinder, (SUGRA with $\mathbb{H}^2$)

The initial conditions

$$z = [z_1 - \delta, z_1 + \delta] \quad x = 0, \quad y = 1 - \varepsilon,$$

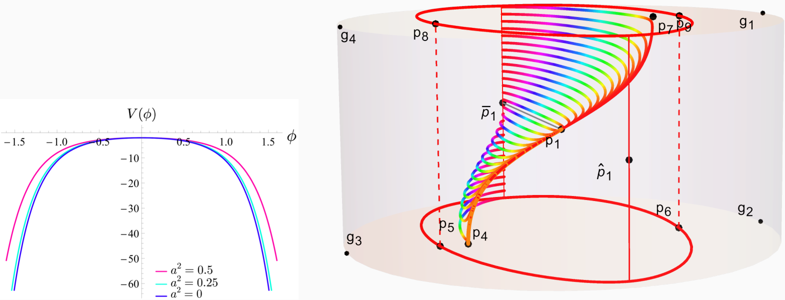

Figure 6: AG,Nikolaev&Podoinitsyn (2024), AG,Gougoulhon&Podoinitsyn (2024)

$$X=\frac{x}{\sqrt{1-x^2-y^2}},\quad Y=\frac{y}{\sqrt{1-x^2-y^2}}.$$

i.e. the system is mapped in the unit cylinder

$$\begin{aligned}\dot{\phi}&=\mathfrak{m}(x,y),\\ \dot{x}&=\mathfrak{p}(\phi,x,y),\\ \dot{y}&=\mathfrak{q}(\phi,x,y),\end{aligned}$$

where the RHSs are given by

$$\begin{aligned}&\mathfrak{m}(x,y)=x\sqrt{1-x^2-y^2};\\
&\mathfrak{p}(\phi,x,y)=2a^2\left(x^2-1\right)\tan\phi\Big[\frac{\left(x^2+y^2-1\right)\left(2\sqrt{1-x^2-y^2}+y\right)}{(2a^2+1)\cos 2\phi-2a^2+1}+\\
&+\frac{x^2\sqrt{1-x^2-y^2}\left(\left(2a^2+1\right)\cos 2\phi+1\right)}{(2a^2+1)\cos 2\phi-2a^2+1}\Big]+\\
&+\frac{x\left(x^2+y^2-1\right)\left(a^2\left(y\sqrt{1-x^2-y^2}+2(1-x^2-y^2)\right)-x^2\right)}{a^2};\\
&\mathfrak{q}(\phi,x,y)=y\Big[2x\tan\phi\Big(\frac{a^2(x^2+y^2-1)(2\sqrt{1-x^2-y^2}+y)}{(2a^2+1)\cos 2\phi-2a^2+1}+\\
&+\frac{x^2\sqrt{1-x^2-y^2}((2a^2+1)\cos 2\phi+1)}{(2a^2+1)\cos 2\phi-2a^2+1}\Big)+\\
&+\frac{(x^2+y^2-1)\left(a^2(y\sqrt{1-x^2-y^2}+2(1-x^2-y^2))-x^2\right)}{a^2}\Big].\end{aligned}$$

The dynamical system in the cylinder, (SUGRA with $\mathbb{S}^2$)

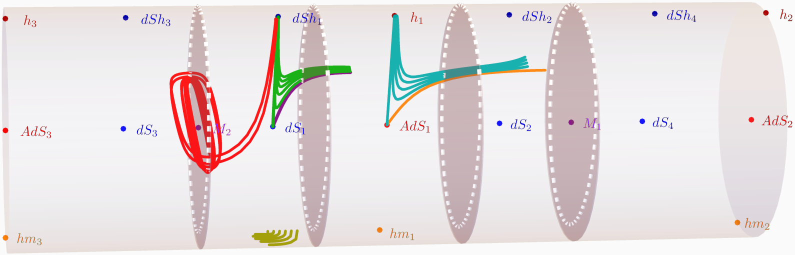

Figure 7: Initial conditions: green curves - $(\phi_0, x_0, y_0) = (\phi_{dS_1} - \delta, 0, 1 - \epsilon)$, $0 < \epsilon \ll 1$, $\delta \in (0, \frac{\pi}{2} - |\phi_{dS_1}|)$; cyan curves - $(\phi_0, x_0, y_0) = (0 + \delta, 0, 1 - \epsilon)$, $0 < \epsilon \ll 1$, $\delta \in (-|\phi_{n1}|, \phi_{n2})$; red curves - $(\phi_0, x_0, y_0) = (\phi_{dS_1} - \delta, 0, 1 - \epsilon)$, $0 < \epsilon \ll 1$, $\delta \in (0, \frac{\pi}{2} - |\phi_{dS_1}|)$.

The dynamical system with thermal flows $f \neq 1$

$$\frac{dZ}{dA} = X\left(1 + \frac{Z^2}{4}\right),$$
$$\frac{dX}{dA} = \left(\frac{X^2}{a^2} - Y - 2\right)\left(X + \frac{a^2}{2}\frac{V_\phi}{V}\right),$$
$$\frac{dY}{dA} = Y\left(\frac{X^2}{a^2} - Y - 2\right)$$

the Poincaré projection to place the dynamical system defined in $\mathbb{R}^3$ into the ball $\mathbb{B}^3$

$$Z = \frac{z}{\sqrt{1 - z^2 - x^2 - y^2}}, \quad X = \frac{x}{\sqrt{1 - z^2 - x^2 - y^2}}, \quad Y = \frac{y}{\sqrt{1 - z^2 - x^2 - y^2}},$$

where $(z, x, y) \in \mathbb{B}^3$ and are also related by the constraint

$$x^2 + y^2 + z^2 \leq 1.$$

Then we are brought to the 3d autonomous dynamical system in the ball

$$\dot{z} = \mathrm{m}(z, x, y),$$
$$\dot{x} = \mathrm{p}(z, x, y),$$
$$\dot{y} = \mathrm{q}(z, x, y),$$

The dynamical system in the unit ball, (SUGRA with $\mathbb{S}^2$)

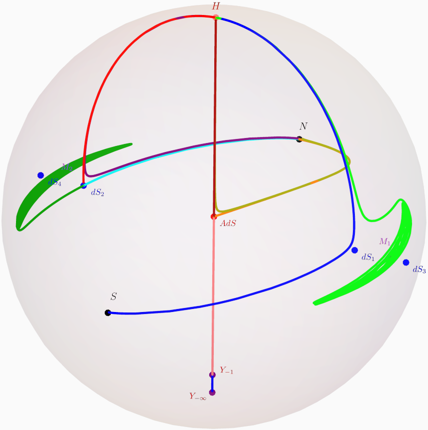

Figure 8: The flows: AdS-H (BTZ), AdS-N (AdS-Minkowski), H-dS (SdS), H-dS-S, H-AdS-N.

Outlook

Summary

- Approximate analytic form of the new de Sitter-asymptotically Minkowski flow with periodic asymptotics is found
- Deformations of BTZ and SdS solutions do not give asymptotically BTZ/SdS black holes
- New solutions with horizons and singularities (AdS(dS)-strings?)
- No flows between two dS or two AdS
- No flows between dS and AdS

To Do

- The stability analysis of the 3d dynamical system in the ball
- Prove analytically the absence of the flows between two dS/AdS
- Reconstruct analytically near-horizon metric of the observed flows with horizon and singularities
- Description of dual deformations for the flows which start from dS/AdS or its horizons

Thank you for attention!