

# Some renormalization aspects of 2D nonlinear sigma model

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# Historical background

**Nonlinear 2D sigma-model**



**Perturbative methods**



**Divergences**



**Renormalizability**

**In dimensional regularization!  
What about other regularizations?**

## **Standard QFT:**

**Bogoliubov, Shirkov (1959)**  
**Faddeev, Slavnov (1978)**  
**Itzykson, Zuber (1980)**  
**Peskin, Schroeder (1995)**

## **Renormalization:**

**Faddeev, Slavnov (1978)**  
**Collins (1984)**  
**Zavialov (1990)**  
**Kazakov (2009)**

## **Regularization:**

**Brizola, Battistel,  
Sampaio, Nemes (1999)**  
**Cherchiglia and other (2021)**  
**Pauli, Villars (1949)**  
**Bollini, Giambiagi (1972)**  
**'t Hooft, Veltman (1972)**  
**Slavnov (1971)**  
**Bakeyev, Slavnov (1996)**  
**Stepanyantz (2020)**

## **Sigma-model:**

**Slavnov, Faddeev (1971)**  
**Polyakov (1975)**  
**Migdal (1976)**  
**Brezin (1976)**  
**Aref'eva (1979)**  
**Semenov-Tian-Shansky,  
Faddeev (1982)**  
**Friedan (1985)**  
**Faddeev, Reshetikhin (1986)**

## **Field of application:**

**Atiyah (1988)**  
**Segal (2004)**  
**Reshetikhin (2010)**  
**Stolz, Teichner (2011)**  
**Cattaneo, Mnev, Reshetikhin (2018)**  
**Kandel, Mnev, Wernli (2021)**

## **Cutoff regularization:**

**Oleszczuk (1994)**  
**Liao (1997)**  
**Bagaev (2008)**  
**Cynolter, Lendvai (2015)**  
**Ivanov, Kharuk,  
Akacevich (2019—H.B.)**

# Classical action

**Consider the group  $SU(n)$  and its algebra  $\mathfrak{su}(n)$ .**

**Let  $t^a$  are generators, such that  $[t^a, t^b] = f^{abc}t^c$ ,  $\text{tr}(t^a t^b) = -\frac{1}{2}\delta^{ab}$ .**

**Here  $f^{abc}$  are totally antisymmetric structure constants.**

**Consider  $g(\cdot) \in C^\infty(\mathbb{R}^2, SU(n))$  and define a set of functions  $C_\mu^a(x)$ , such that**

$$C_\mu^a(x)t^a = (\partial_{x^\mu}g(x))g^{-1}(x).$$

**Then the classical action is**

$$S[C] = \int_{\mathbb{R}^2} d^2x C_\mu^a(x)C_\mu^a(x).$$

# Quantum action

**BF method:**  
 DeWitt (1967)  
 't Hooft (1976)  
 Abbott (1972)  
 Aref'eva, Slavnov,  
 Faddeev (1974)

Let us use the background field method:  $g(x) = \exp(\gamma\phi(x))h(x)$ , where  $h(x)$  is a background field, and  $\phi(x)$  is a fluctuation.

We define one more background field  $B_\mu^a(x)$  as solution to  $B_\mu^a(x)t^a = (\partial_{x^\mu}h(x))h^{-1}(x)$ .

Then the symbolic form

$$W = -\ln\left(\int_{\mathcal{H}} \mathcal{D}g e^{-S[C]/4\gamma^2}\right)\Bigg|_{\text{reg}}$$

vertices with  $k$  external lines

can be rewritten as

$$W_{\text{reg}}[B, \Lambda] = \frac{S[B]}{4\gamma^2} - \frac{1}{2}(\ln \det(G_{\text{reg}}) - \varkappa_0) - \left[ \mathbb{H}_0^{sc} \exp\left(\sum_{k=3}^{+\infty} \frac{\gamma^{k-2}}{k!2} \Gamma_k\right) - \sum_{k=1}^{+\infty} \gamma^{2k} \varkappa_k \right].$$

regularised

connects the lines

takes only strongly connected part

# Regularization

Let us use a cutoff in the coordinate representation by averaging.

Define the operator  $O_{(\cdot)}^\Lambda$ , where  $\Lambda > 0$ , as

$$O_x^\Lambda f(x) = \int_{\mathbb{R}^2} d^2y \omega(|y|) f(x + y/\Lambda).$$

Here we need

$$\text{supp}(\omega(|\cdot|)) \subset \{y \in \mathbb{R}^2 : |y| \leq 1/2\}, \quad \int_{\mathbb{R}^2} d^2y \omega(|y|) = 1,$$

as well as

$$O_x^\Lambda O_x^\Lambda \ln(|x|) = \int_{\mathbb{R}^2} d^2z_1 \int_{\mathbb{R}^2} d^2z_2 \omega(|z_1|) \omega(|z_2|) \ln(|x + z_1/\Lambda + z_2/\Lambda|) \in C^2(\mathbb{R}^2, \mathbb{R}).$$

In our case we use the quasi-local deformation  $\phi(x) \rightarrow O_x^\Lambda \phi(x)$ , so we get

$$-\frac{\ln(|x|\sigma)}{2\pi} \longrightarrow \frac{\mathbf{f}(\Lambda, |x|^2 \Lambda^2)}{4\pi} + \frac{1}{2\pi} \begin{cases} \ln(\Lambda/\sigma), & \mathbf{for} \ |x| \leq 1/\Lambda; \\ -\ln(|x|\sigma), & \mathbf{for} \ |x| > 1/\Lambda. \end{cases}$$

# Renormalization

Standard multiplicative renormalization with the dimensional regularization is

$$\frac{1}{\gamma^2} \longrightarrow \frac{1}{\gamma_0^2} = \frac{1}{\gamma^2} - \sum_{i=0}^{+\infty} a_i \gamma^{2i}.$$

But! With a cutoff we have non-logarithmic singularities, so we expand the anzats

$$W_{\text{ren}}[B, \Lambda] = \frac{S[B]}{4\gamma_0^2} - \frac{1}{2} (\ln \det(G_{\text{reg}}) - \varkappa_0) - \left[ \mathbb{H}_0^{\text{sc}} \exp \left( \sum_{k=3}^{+\infty} \frac{\gamma_0^{k-2}}{k!2} (\Gamma_k + \Gamma_{r,k-2}) \right) - \sum_{k=1}^{+\infty} \gamma_0^{2k} \hat{\varkappa}_k \right].$$

↑  
renormalized coupling constant

↑  
additional renormalized vertices

After resummation we get

$$W_{\text{ren}}[B, \Lambda] = \frac{S[B]}{4\gamma^2} + W_0[B, \Lambda] + \gamma^2 W_1[B, \Lambda] + \gamma^4 W_2[B, \Lambda] + \dots$$

# First two loops

**First loop:**

$$W_0[B, \Lambda] = -\frac{1}{2} (\ln \det(G_{\text{reg}}) - \hat{\chi}_0) - \frac{a_0}{4} S[B],$$

$$a_0 = -\frac{c_2 L_1}{4\pi}.$$

**Second loop:**

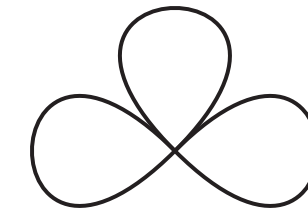
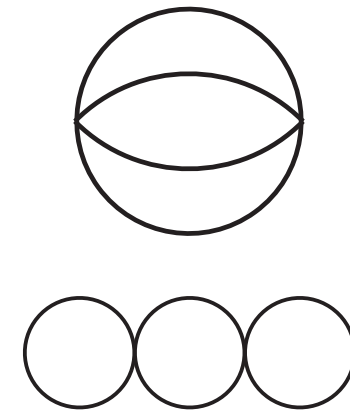
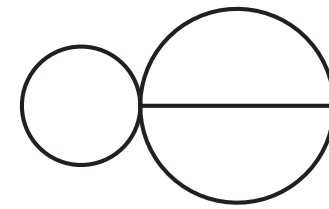
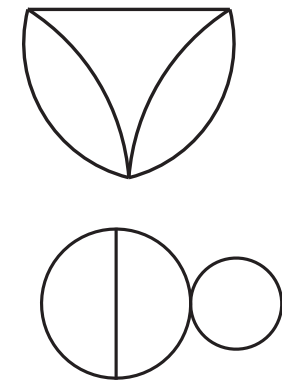
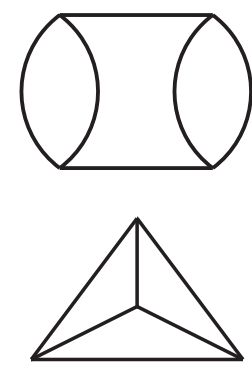
$$W_1[B, \Lambda] = -\frac{\mathbb{H}_0^{\text{sc}}(\Gamma_3^2)}{8(3!)^2} - \frac{\mathbb{H}_0^{\text{sc}}(\Gamma_4 + \Gamma_{r,2})}{2(4!)} - \frac{a_1}{4} S[B] + \hat{\chi}_1,$$

$$\Gamma_{r,2} = -\frac{\Lambda^2 c_2 \alpha_1(\mathbf{f})}{2\pi} \int_{\mathbb{R}^2} d^2x \phi^a(x) \phi^a(x), \quad a_1 = -\frac{L_1 c_2^2}{3(4\pi)^2} (3\theta_1 + 2\theta_2).$$

$$\frac{\theta_1}{2\pi} = \int_{\mathbb{R}^2} d^2x \left( A_0(x) G^{1,\mathbf{f}}(x) \right) \left( G^{1,\mathbf{f}}(x) - G^{1,\mathbf{f}}(0) \right)$$

$$\frac{\theta_2}{2\pi} = \int_{\mathbb{R}^4} d^2x d^2y \left( A_0(x) G^{1,\mathbf{f}}(x) \right) \left( A_0(y) G^{1,\mathbf{f}}(y) \right) \left( G^{1,\mathbf{f}}(x-y) - G^{1,\mathbf{f}}(x) \right)$$

# The third loop



diagrams with three loops

$$W_2[B, \Lambda] = -\frac{\mathbb{H}_0^{sc}(\Gamma_3^4)}{16(3!)^4 4!} - \frac{\mathbb{H}_0^{sc}(\Gamma_3^2 \Gamma_4)}{16(3!)^2 4!} - \frac{\mathbb{H}_0^{sc}(\Gamma_3 \Gamma_5)}{4(3! 5!)} - \frac{\mathbb{H}_0^{sc}(\Gamma_4^2)}{8(4!)^2} - \frac{\mathbb{H}_0^{sc}(\Gamma_6)}{2(6!)}$$

need to find

$$\frac{\mathbb{H}_0^{sc}(\Gamma_3^2 \Gamma_{r,2})}{16(3!)^2 4!} - \frac{\mathbb{H}_0^{sc}(\Gamma_4 \Gamma_{r,2})}{4(4!)^2} - \frac{\mathbb{H}_0^{sc}(\Gamma_{r,2}^2)}{8(4!)^2} - \frac{\mathbb{H}_0^{sc}(\Gamma_3 \Gamma_{r,3})}{4(3! 5!)} - \frac{\mathbb{H}_0^{sc}(\Gamma_{r,4})}{2(6!)}$$

counterterms from second loop

**Problem:**  
**Calculate asymptotics**  
 ↓  
**Find  $\Gamma_{r,3}$ ,  $\Gamma_{r,4}$ , and  $a_2$ .**

$$+ a_0 \left( W_1[B, \Lambda] + a_1 S[B]/4 \right) - \frac{a_2}{4} S[B] + \hat{\chi}_2.$$

counterterms from first loop

# Expectation

Omitting details we can split the deformed Green's function as follows

$$G_{\Lambda}^{ab}(x, y) = L_{\Lambda}^{ab}(x, y) + PS_{\Lambda}^{ab}(x, y).$$

↑  
singular and "local"

↑  
"non-local" and smooth enough  
depends on boundary conditions

$$W_2[B, \Lambda] = \overset{\sim PS^0}{W_{2,0}[B, \Lambda]} + \overset{\sim PS^1}{W_{2,1}[B, \Lambda]} + \overset{\sim PS^2}{W_{2,2}[B, \Lambda]} + \overset{\sim PS^3}{W_{2,3}[B, \Lambda]} + \dots$$

↑  
proportional  $S$   
gives  $a_2$

↑  
singularities do not appear  
due to renormalization of  
first two loops

↑  
not contain singularities

← ideal situation

can affect  $a_2$

But we have  $\Lambda^2$  and some "differences"!  
↓  
So we get  $\Gamma_{r,3}$  and  $\Gamma_{r,4}$ .

← real life

# Answers

$$\Gamma_{r,3} = -\frac{\ln(\Lambda/\sigma)}{3\pi} \int_{\mathbb{R}^2} d^2x f^{abc} \phi^b(x) f^{ced} \phi^e(x) \left( \partial_{x_\mu} B_\mu^{dg}(x) \right) f^{gha} \phi^h(x)$$

classical eq. of motion

does not appear in the final answer

$$\Gamma_{r,4} = \underbrace{\Lambda^2 \left( \frac{\alpha_1}{2\pi} - \frac{5\alpha_6}{2} \right) \tilde{\Gamma}_4}_{\sim \text{tr}(\phi^4)} - \frac{15\Lambda^2 c_2^2}{2\pi} R_0 \tau_5 \underbrace{\sim \text{tr}(\phi^2)} - \frac{5 \ln(\Lambda/\sigma)}{32\pi^2} \left( \underbrace{c_2^2 R_1 \tau_1 + c_2 R_2 \tau_2 + R_3 \tau_3}_{\sim \phi B^2 \phi} + \underbrace{c_2^2 R_4 \tau_4}_{\sim \phi B \partial \phi} \right)$$

Λ<sup>2</sup> part

logarithmic part  
non-zero for all cutoffs

$$a_2 = \underbrace{+ \frac{a_0 a_1}{2}}_{\text{pure part}} - \frac{c_2^3 \ln^2(\Lambda/\sigma)}{16(4!)^2 \pi^3} (2\tau_1 - 2\tau_2 + 2\tau_3 + \tau_4) + \ln(\Lambda/\sigma)t$$

↑  
pure part

**Many thanks!**