General Lagrangian formulations for (ir)reducible mixed-antisymmetric higher integer spin fields in Minkowski spaces

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- I.L. Buchbinder, A.R, General Cubic Interacting Vertex for Massless Integer HS Fields, PLB (2021), [arXiv:2105.12030],
- I.L. Buchbinder, A.R, Covariant Cubic Interacting Vertices for Massless and Massive Integer Higher Spin Fields, Symmetry (2023) [arXiv:2212.07097] ,
- A.R., BRST–BV approach for interacting HS fields, TMPh (2023) [arXiv:2303.02870],
- I.L. Buchbinder, A.R, Consistent Lagrangians for irreducible interacting higher-spin fields with holonomic constraints [arXiv:2304.10358] PEPAN (2023)
- A.R, Towards Lagrangian dynamics for constrained mixed-symmetric interacting HS fields, [arXiv:2505.02190] PEPAN
- **I.L. Buchbinder, Yu.Bogdanova, A.R, General Lagrangian formulations for irreducible mixed-antisymmetric tensor fields on Minkowski spaces, to appear**

SINP and Faculty of Physics MSU, 30.06-5.07.2025

Motivations

Wigner-Bargmann (1939, 1948) classification (1939, 1948) of UIRs $ISO(1, d-1)$ is characterized by $[(d+1)/2]$ Casimirs; A. Isaev (2023-2024)

1. $P^2 = m^2, W^2 = -m^2 s(s+1)$ - massive Unitary irrep (UIR) with (half)integer spin;

2a. $P^2 = 0, W^2 = 0,\ W^\mu = \lambda P^\mu$ - massless helicity UIR;

2b. $P^2 = 0, W^2 = \mu^2$ - massless continuous spin UIR;

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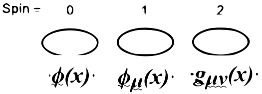
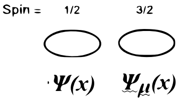
Lower Spin refers to consistent classical field theories ($s \leqslant 2$)

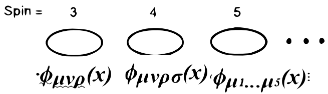


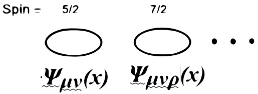


Higgs; (dark) photon, W, Z-bosons, gluons; graviton

leptons,quarks; gravitino (SYM, SUGRA)

Higher Spin (HS) stands for problematic construction ($s > 2$)





Fronsdal '78

Fang-Fronsdal '79

Interacting vertices (B,B,B), (B,B,B,B) and (F,F,B), (F,F,B, B), (F,F,F,F) in SM

Cubic vertices in SM for lower spins ($m = (\neq)0$): (1,1,1), (0,0,1), $\underline{(\frac{1}{2}, \frac{1}{2}, 0)}$, $\underline{(\frac{1}{2}, \frac{1}{2}, 1)}$

$$S_{\mathsf{SM}} = \int d^4x\, \mathcal{L}_{\mathsf{SM}}\ , \quad \mathcal{L}_{\mathsf{SM}} = \mathcal{L}_{\text{gauge fields}} + \mathcal{L}_{\text{leptons}} + \mathcal{L}_{\text{quarks}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{Higgs}}\ , \qquad (1)$$

$$\mathcal{L}_{\text{leptons}} = \sum_{k=1}^{3} \left[\bar{l}^k_L i\gamma^\mu \left(\partial_\mu - i\frac{g}{2} A^{\hat{a}}_\mu \tau_{\hat{a}} + i\frac{g'}{2} A_\mu \right) l^k_L + \bar{l}^k_R i\gamma^\mu \left(\partial_\mu + ig' A_\mu \right) l^k_R \right]\ ,$$

$$\mathcal{L}_{\text{quarks}} = \sum_{k=1}^{3} \left\{ \left[\begin{array}{c} \bar{u}_k \\ \bar{d}'_k \end{array} \right]_L i\gamma^\mu \left[\partial_\mu - i\frac{g_{\mathrm{s}}}{2} A^{\underline{\alpha}}_\mu \lambda_{\underline{\alpha}} - i\frac{g}{2} A^{\hat{a}}_\mu \tau_{\hat{a}} - i\frac{g'}{6} A_\mu \right] \left[\begin{array}{c} u_k \\ d'_k \end{array} \right]_L \right.$$

$$\left. + \bar{u}^k_R i\gamma^\mu \left[\partial_\mu - i\frac{g_{\mathrm{s}}}{2} A^{\underline{\alpha}}_\mu \lambda_{\underline{\alpha}} - i\frac{2g'}{3} A_\mu \right] u^k_R + \bar{d}'^k_R i\gamma^\mu \left[\partial_\mu - i\frac{g_{\mathrm{s}}}{2} A^{\underline{\alpha}}_\mu \lambda_{\underline{\alpha}} + i\frac{g'}{3} A_\mu \right] d'^k_R \right\}$$

$$d'^k = U^{kk'}_{\mathsf{CKM}} d^{k'}\ , \quad u^k = (u, c, t)\ , \quad d^k = (d, s, b)\ ,$$

The masses of particles are generated by the Yukawa interaction term

$$\mathcal{L}_{\text{Yukawa}} = -\frac{1}{\sqrt{2}} \sum_{k=1}^{3} \left\{ f^u_k \left[\begin{array}{c} \bar{u}^k \\ \bar{d}^k \end{array} \right]_L \varphi u^k_R + f^d_k \left[\begin{array}{c} \bar{u}^k \\ \bar{d}^k \end{array} \right]_L \varphi d^k_R + f^l_k \bar{l}^k_L \varphi l^k_R + \text{h.c.} \right\}\ ,$$

$$\mathcal{L}_{\text{Higgs}} = \frac{1}{2} \Big| \Big(i\partial_\mu + (g/2)\, A^{\hat{a}}_\mu \tau_{\hat{a}} + \left(g'/2\right) A_\mu \Big) \Big| \varphi^2 - \frac{\mu^2}{2} |\varphi|^2 - \frac{\lambda}{4} |\varphi|^4\ ,$$

Cubic interacting vertex

Known results on cubic vertices

- metric formalism F. Berends, J. Van Reisen, NPB164 (1980), Berends, G. Burgers, H Van Dam, Nucl. Phys. B271 (1986); A. K. H. Bengtsson, I. Bengtsson, L. Brink, NPB (1983), E.S. Fradkin, M.A. Vasiliev, NPB 291 (1987), R. Manvelyan, K. Mkrtchyan, W. Ruhl, PLB 696 (2011), [arXiv:1009.1054 [hep-th]], E. Joung, M. Taronna, NPB 861 (2012) 145, arXiv:1110.5918[hep-th], I. Buchbinder, V. Krykhtin, M. Tsulaia, Cubic Vertices for $\mathcal{N} = 1$, NPB 967 (2021);

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- in BRST approach with (in)complete BRST operator for irreps $ISO(1,d-1)$ bosonic fields by I.Buchbinder, A.R. (2021-2023) ;
- in frame-like approach M. Vasiliev, Cubic Vertices for Symmetric higher spin Gauge Fields in (A)dS$_d$, NPB 862 (2012) 341 , arXiv:1108.5921[hep-th] arXiv:2208.02004, M. Khabarov, Yu. Zinoviev. JHEP 02 (2021);
- ***LF & (Non)Covariant Cubic vertex for irrep MAS integer HS fields*** $\Phi_{\mu^1[s_1],\dots,\mu^k[s_k]}$ ***not found*** *(in BRST approach with (in)complete $Q_{(c)}$)* FOR $k>2$, $k=1$: I.Buchbinder,V. Krykhtin (2009), Yu. Zinoviev;

  $k=2$ (X.Bekaert, Boulanger 2004, 2005), Yu. Zinoviev, 2016 , A.R. 2016

Contents

- BRST approach with complete BRST operator Q for irreducible free integer MAS HS fields on $R^{1,d-1}$;
    1. additive conversion of operator 2=nd class constraints: Verma and Fock modules;
    2. BRST complex with spin condition and Lagrangians;
- BRST approach with incomplete BRST operator Q_c for irreducible free integer and half-integer higher spins on $R^{1,d-1}$;
    1. Generating equations for superalgebra of incomplete operators: Q_c, spin σ_c and operators $\widehat{L}_{ij}, \widehat{T}_{rs}$;
    2. GI Lagrangian formulation with holonomic contraints;
- Deformation procedure with Q_c for interacting higher-spin fields;
- General solution of BRST equations for cubic vertices for constrained of helicities $(s[k_1], s[k_2], s[k_3])$ HS fields
    1. BRST-closed linear on oscillators operator $L^{(i)}$;
    2. BRST-closed cubic on oscillators operators $Z \equiv Z_{111}$;
- Summary

BRST approach with incomplete BRST operator Q_c

due to tensionless limit : $\Rightarrow$ for string BRST operator $\mathcal{Q}$ ($d = 26, 10$) for ($\alpha' \to \infty$): (G.Bonelli (2003), A. Sagnotti, M. Tsulaia, (2004)) now ($\forall d$).

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$$\boxed{\Longrightarrow \mathcal{Q} \stackrel{\alpha' \to \infty}{\longrightarrow} Q_c : \{\infty\}\texttt{many HS fields } \phi_\mu(x), .., \phi_{\mu(s)}(x) \texttt{ in string spectra}}$$

In BRST-BFV approach with incomplete Q_c (S. Ouvry, J. Stern, A. Bengtsson,, G. Barnich, M.Grigoriev, A.Semikhatov 2004, A.R. 2018)
instead of direct problem for generalized canonical quantization of Constrained DS by the aim inverse problem - is an construction of GI LF for HS fields with (m, s)

$$\boxed{\begin{array}{c}\texttt{irrep conditions}\\ \text{ISO(1,d-1), (SO(2,d-1))}\end{array}} \xrightarrow{\texttt{SFT}} \boxed{\begin{array}{c}\texttt{(super)algebra}\{o_I(x)\} = \{p^2 - m^2; a^+_{i\mu}p^\mu, a_{i\mu}p^\mu; \underline{o_a}, o^+_a\}\\ \{o_I(x)\} : \mathcal{H}, \ \ [o_I, o_J\} = f^K_{IJ}(o)o_k\end{array}}$$

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$$\xrightarrow[\texttt{I.B.,E.Fradkin, G.V., M.Henneaux}]{\texttt{BFV}} \boxed{\begin{array}{c}\texttt{BRST operator } \{o_I\}: Q_c(x)\\ Q_c = C^A o_A + \frac{1}{2}C^A C^B F^D_{AB}\mathcal{P}_D(-1)^{\varepsilon(o_A)+\varepsilon(o_D)}\end{array}}$$

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$\xrightarrow{\texttt{BFV}}$
`I.B.,E.Fradkin, G.V., M.Henneaux`

`BRST operator` $\{o_I\}$:$Q_c(x)$
$Q_c = C^A o_A + \frac{1}{2}C^A C^B F^D_{AB}\mathcal{P}_D(-1)^{\varepsilon(o_A)+\varepsilon(o_D)}$

$\xrightarrow{\texttt{LF}}$

$Q_c^2 = 0, \ [Q_c, \sigma_c\} = [Q_c, \mathcal{O}_a\} = 0, \ [\mathcal{O}_a, \sigma_c\} \sim \mathcal{O}_a$
`mass-shell` : $Q_c|\chi_c\rangle = 0, \mathrm{gh}(|\chi_c\rangle) = 0 \Rightarrow$ `action` : $S_c = \int d\eta_0 \langle\chi_c|Q_c|\chi_c\rangle$
`spin:` $(g_0 + \mathrm{more})(|\chi_c\rangle, |\Lambda_c\rangle, ...) = (s - d/2 + ...)(|\chi\rangle, |\Lambda\rangle, ...)$
`gauge symmetry:` $\delta|\chi_c\rangle = Q_c|\Lambda_c\rangle, \delta|\Lambda_c\rangle = Q_c|\Lambda^1_c\rangle, ...,$ `constr:` $\mathcal{O}_a(|\chi\rangle, |\Lambda\rangle, ...) = 0$

Q_c - for 1-st class constraints without holonomic ones with auxiliary fields on 2 stage

Approach with complete BRST operator Q for free MAS HS fields

A. Pashnev, M. Tsulaia (1998-2004), C. Burdik, I. Buchbinder, V. Rrykhtin, A.R., Takata, A. Isaev, S. Fedoruk (continuous spin in $d = 4$) In BRST-BFV approach with complete Q again instead of direct problem for generalized canonical quantization (BFV 1977-1983) of Constrained DS by the aim inverse problem - is an construction of GI LF for HS field with (m, s)

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$$\xrightarrow{\texttt{LF}} \left|\begin{array}{c}Q'=Q+(g^i_0+h^i+\text{more})C^i_g+\ldots:{Q'}^2=0\Rightarrow Q^2=2B^i(g^i_0+h^i+\text{more})\\ \texttt{mass-shell}:Q|\chi\rangle=0,\text{gh}(|\chi\rangle)=0\Rightarrow\texttt{action}:S=\int d\eta_0\langle\chi|KQ|\chi\rangle=\Phi_{\mu[s]}p^2\Phi^{\mu[s]}\\ \texttt{spin:}(g_0+\text{more})(|\chi\rangle,|\Lambda\rangle,\ldots)=-h(|\chi\rangle,|\Lambda\rangle,\ldots)\\ \texttt{gauge transfs:}\delta|\chi\rangle=Q|\Lambda\rangle,\delta|\Lambda\rangle=Q|\Lambda^1\rangle,\ldots\end{array}\right.$$

with auxiliary fields on 2,3 stages. It is the particular case of AKSZ model (1997).

Derivation of HS symmetry algebra $\mathcal{A}(Y[k], \mathbb{R}^{1,d-1})$

The m of g. spin $s = [\hat{s}_1, ..., \hat{s}_k] \equiv s[k]$ $ISO(1, d-1)$ group irrep with Young T. $Y[\hat{s}_1, ..., \hat{s}_k]$

$$\Phi_{\mu^1[s_1],...,\mu^k[s_k]} \longleftrightarrow \begin{array}{|c|c|c|c|c|}
\hline
\mu^1_1 & \mu^2_1 & \cdots & \mu^{k-1}_1 & \mu^k_1 \\
\hline
\mu^1_2 & \mu^2_2 & \cdots & \mu^{k-1}_2 & \mu^k_2 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\mu^1_{s_k} & \mu^2_{s_k} & \cdot & \mu^{k-1}_{s_k} & \mu^k_{s_k} \\
\hline
\mu^1_{s_k+1} & \mu^2_{s_k+1} & \cdot & \mu^{k-1}_{s_k+1} \\
\cline{1-4}
\cdot & \cdot & \cdot & \cdot \\
\cline{1-4}
\mu^1_{s_{k-1}} & \mu^2_{s_{k-1}} & \cdot & \mu^{k-1}_{s_{k-1}} \\
\cline{1-4}
\cdot & \cdot & \cdot \\
\cline{1-3}
\mu^1_{s_2+1} \\
\cline{1-1}
\cdots \\
\cline{1-1}
\mu^1_{s_1} \\
\cline{1-1}
\end{array} ,$$

$$(\partial^\mu \partial_\mu + m^2)\Phi_{\mu^1[s_1],...,\mu^k[s_k]} = 0,$$

$$(\partial_i \Phi)_{\mu^1[s_1-\delta_{i1}],...,\mu^k[s_k-\delta_{ik}]} \equiv \partial^{\mu^i_{l_i}} \Phi_{\mu^1[s_1],...,\mu^k[s_k]} = 0, \ 1 \leq l_i \leq s_i, \ i = 1, ..., k,$$

$$(\mathtt{Tr}^{ij} \Phi)_{\mu^1[s_1],...,\mu^k[s_k]} \equiv \eta^{\mu^i_{l_i} \mu^j_{l_j}} \Phi_{\mu^1[s_1],...,\mu^k[s_k]} = 0, \quad 1 \leq i < j \leq k,$$

$$(Y^{ij} \Phi)_{\mu^1[s_1],...,[\mu^i[s_i],...,\mu^j_{l_j}],\hat{\mu}^j[s_j-1],...,\mu^k[s_k]} = 0,$$

Aim to find LF for given HS field on $\mathcal{M}_{(c)}$:

$$\mathcal{S}_{(c)s[k]} : \mathcal{M}_{(c)} = \{(\Phi_{\mu^1[s_1],...,\mu^k[s_k]}, \Psi_{\mu^1[s_1-1],...,\mu^k[s_k]}, \ldots)\} \to \mathbb{R},$$

Primary constraints, All constraints, $sp(2k)$, $\mathcal{A}(Y[k],\mathbb{R}^{1,d-1})$

$$\texttt{SFT} \Longrightarrow \mathcal{H}: \{a^i_\mu, a^{j+}_\nu\} = -\eta_{\mu\nu}\delta^{ij}, \ \texttt{diag}\,\eta_{\mu\nu} = (+,-,..,-)$$

An arbitrary "string-like"vector $|\Phi\rangle \in \mathcal{H}$

$$|\Phi\rangle = \sum_{s_1=0}^{[d/2]}\sum_{s_2=0}^{s_1}\ldots\sum_{s_k=0}^{s_{k-1}} \frac{\imath^{\sum_{p=1}^{k} s_p}}{s_1!\ldots s_k!}\Phi_{\mu^1[s_1],\ldots,\mu^k[s_k]}\prod_{i=1}^{k}\prod_{l_i=1}^{s_i}\hat{a}_i^{+\mu^i_{l_i}}|0\rangle,$$

$$\boxed{\left(l_0,\, l^i,\, l^{ij},\, t^{ij}\right)|\Phi\rangle = \left(\partial^\mu\partial_\mu + m^2,\, -i\hat{a}^i_\mu\partial^\mu,\, \tfrac{1}{2}\hat{a}^i_\mu\hat{a}^{j\mu},\, \hat{a}^{i+}_\mu\hat{a}^{j\mu}\right)|\Phi\rangle = \vec{0}} \Longleftrightarrow (1)-(4).$$

The set of $(k+1)^2$ even and k odd, l^i, primary constraints permits to realize $\Longleftrightarrow$ Eqs. as constraints on $|\Phi\rangle$ for each $s[k]$

Eqs. with number particles operators, g^i_0,

$$g^i_0|\Phi\rangle = (s_i - \frac{d}{2})|\Phi\rangle, \qquad g^i_0 = -\frac{1}{2}[\hat{a}^{i+}_\mu, \hat{a}^{\mu i}] \ = \ -\hat{a}^{i+}_\mu\hat{a}^{\mu i} - \frac{d}{2},$$

extract irrep with given $s[k]$

Construction is differed from LF with (in)complete BRST operator for MS HS field $\Phi_{\mu^1(s_1),\ldots,\mu^k(s_k)}$ (I.Buchbinder, A.R. NPB 2012)

All constraints, $sp(2k)$, $\mathcal{A}(Y[k],\mathbb{R}^{1,d-1})$

$$\{o_I\} = \{o_\alpha, o_\alpha^+;\ l_0, g_0^i\} \equiv \{o_a, o_a^+;\ l_0,\ l^i,\ l^{i+};\ g_0^i\}.$$

$$\langle\Psi|\Phi\rangle = \int d^dx \sum_{s_1=0}^{[d/2]}\sum_{s_2=0}^{s_1}\dots\sum_{s_k=0}^{s_{k-1}} \frac{(-1)^{\sum_p s_p}}{s_1!\dots s_k!}\Psi^\star_{\mu^1[s_1],\ldots,\mu^k[s_k]}(x)\Phi^{\mu^1[s_1],\ldots,\mu^k}$$

$$l_0 = \partial^2 + m^2,\ l^+_{ij} = \frac{1}{2}a^+_{j\mu}a^{+\mu}_i,\quad l^+_i = -\imath a_i^{+\mu}\partial_\mu,\ t^+_{ij} = a^{+\mu}_j a_{\mu i}\theta^{ji}, \theta^{ji} = 1(0), j > (<) i$$

which form together with $(l_i, l_{ij}, t_{ij}, g_0^i)$ integer HS symmetry superalgebra $\mathcal{A}(Y[k],\mathbb{R}^{1,d-1})$ w.r.t. [, }.

Subalgebra Lie of operators $S_{[k]} = \{l^{ij}, t^{ij}, g_0^i, l^+_{ij}, t^+_{ij}\} \overset{\texttt{Howe duality}}{\subset} sp(2k)$.
For $m=0$ the only o_I from upper and lower triangular subalgebras in $S_{[k]}$ compose an invertible matrix: $\|[o_{\mathbf{a}}, o_{\mathbf{b}}]\| = \|\Delta_{\mathbf{ab}}(g_0^i)\| + (o_I)$,
for $m \neq 0$ its number $2k(k-1)$ increases on $2k$ items l^i, l_i^+

additive conversion of operator 2=nd class constraints: Verma and Fock modules

To convert $\mathcal{A}(Y(k), \mathbb{R}^{1,d-1})$ with 2nd operator C.C. we have used the general procedure of additive conversion

$o_I \to O_I = o_I + o'_I : [o_I, o'_J] = 0,$ `so that` $[O_I, O_J] \sim O_K,\ o'_I : H' \to H';\ H' \cap \mathcal{H}^f =$

,
$\Rightarrow$ if $[o_I, o_J] = f^K_{IJ} o_K$, then $[o'_I, o'_J] = f^K_{IJ} o'_K$ & $[O_I, O_J] = f^K_{IJ} O_K$.
But, it's sufficient to convert only subalgebra $S_{[k]}$ for $\{o_a, o^+_a, g^i_0\}$.
So that the algebra of O_I is the same $\mathcal{A}_c(Y[k], \mathbb{R}^{1,d-1}) = \mathcal{A}(Y[k], \mathbb{R}^{1,d-1})$ as for o_I, but for o'_I - $S_{[k]}$.

Verma module for $S_{[k]}$

Cartan decomposition

$$S_{[k]} = \Big\{\overbrace{l'^{ij+}, t'^{+}_{rs}}\Big\} \oplus \Big\{\overbrace{g'^{i}_{0}}\Big\} \oplus \Big\{\overbrace{l'^{ij}, t'_{rs}}\Big\} \equiv \mathcal{E}^{-}_{k} \oplus H_{k} \oplus \mathcal{E}^{+}_{k}$$

Requirement: boundary conditions for o'_I from Cartan subalgebra:

$$g^{i}_{0} \to g'^{i}_{0}(h^{i}) = h^{i} + ...,$$

So that, following the result by C.Burdik 1985 we start with highest weight vector $|0\rangle_V$ & construct following Poincare–Birkhoff–Witt theorem

$$V(S_{[k]}) = U(\mathcal{E}^{-}_{k}) \otimes (|0\rangle_V) : \mathcal{E}^{+}_{k}|0\rangle_V = 0\,, g'^{i}_{0}|0\rangle_V = h^{i}|0\rangle_V\,,$$

to find $\{o'_I\} = \{o'_I(b_{ij}, b^{+}_{ij}, \overbrace{f_i, f^{+}_{i}}^{m\neq 0}, d_{ln}, d^{+}_{ln})\}$, $i,j,l,n = 1,...,k; i<j, l<n$:
$[b^{+}_{ij}, f_k, f^{+}_{k}, d_{ln}, d^{+}_{ln}] = [o_a, o^{+}_{a}]$: we use results
C. Burdik, O. Navratil, A. Pashnev, for $\mathcal{A}'_b(Y(1), AdS_d)$;
A. Kuleshov, A. R. arXiv:0905.2705 for $\mathcal{A}'(Y(1), AdS_d)$;
I.Buchbinder, A.R. NPB 2012 arXiv:1110.5044 for $sp(2k)$;
A.R. arXiv:1604.00620 for $\mathcal{A}'(Y[2], R^{1,d-1})$

Verma module for $S_{[k]}$

Explicit obtaining of the $V(S_{[k]})$ meet the technical obstacle because of not commuting of t'^{+}_{ln}, l'^{+}_{ij} with each other $\mathcal{E}^{-}_{k}$. The general $V(S_{[k]})$ vector

$$|\vec{n}_{ij}, \vec{p}_{rs}\rangle_V \;= |n_{12}, ..., n_{1k}, n_{23}, ..., n_{2k}, ..., n_{kk+1}; p_{12}, ..., p_{1k}, p_{23}, ..., p_{2k}, ..., p_{k-1k}\rangle_V \;,$$

$$|\vec{n}_{ij}, \vec{p}_{rs}\rangle_V \;\equiv |\vec{N}\rangle_V \equiv \textstyle\prod^{k}_{i<j} \big(l'^{+}_{ij}\big)^{n_{ij}} \prod^{k}_{r,r<s} \big(t'^{+}_{rs}\big)^{p_{rs}} |0\rangle_V ,$$

$$g'_{0i}|\vec{N}\rangle_V = \left(\sum_{l<i} n_{li} + \sum_{l>i} n_{il} - \sum_{s>i} p_{is} + \sum_{r<i} p_{ri} + h^{i}\right) \Big|\vec{N}\rangle_V \;\;,$$

$$t'^{+}_{r's'}|\vec{N}\rangle_V = \Big|\vec{N} + \delta_{r's',rs}\rangle_V \;-\sum_{k'=1}^{r'-1} p_{k'r'} \Big|\vec{N} - \delta_{k'r',rs} + \delta_{k's',rs}\rangle_V \;-\sum_{k'=1}^{r'-1} n_{k'r'} \times$$

$$\times \Big|\vec{N} - \delta_{k'r',ij} + \delta_{k's',ij}\rangle_V \;+ \sum_{k'=r'+1}^{s'-1} n_{r'k'} \Bigg|\vec{N} - \delta_{r'k',ij} + \delta_{k's',ij}\rangle_V - \sum_{k'=s'+1}^{k} n_{r'k'} \times$$

$$\times \Big|\vec{N} - \delta_{r'k',ij} + \delta_{s'k',ij}\rangle_V \;\;,$$

Explicit construction of $V(S_{[k]})$

$$l'^{+}_{i'j'}|\vec{N}\rangle_V \quad = \quad \Big|\vec{N} + \delta_{i'j',ij}\rangle_V \ , \qquad \texttt{for (-) root vectors} \in \mathcal{E}^{-}_{k}$$

$$\texttt{where}\ \ AB^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B^{n-k}\mathrm{ad}^k_B A\,, \mathrm{ad}^k_B A = [[...[A, \overbrace{B\}, ...], B}^{k\ \text{times}}],$$

To get the action of E^{α_i} on $|\vec{N}\rangle_V$ we get the recurrent relation

$$t'_{l'm'}|\vec{0}_{ij}, \vec{p}_{rs}\rangle_V = \Big|C^{l'm'}_{\vec{p}_{rs}}\rangle_V - \sum_{n'=1}^{l'-1} p_{n'm'} \Big|\vec{0}_{ij}, \vec{p}_{rs} - \delta_{n'm',rs} + \delta_{n'l',rs}\rangle_V$$

$$+ \sum_{k'=l'+1}^{m'-1} p_{l'k'}\Big[\prod_{r'<l',s'>r'} \prod_{r'=l',m'>s'>r'} (t'^{+}_{r's'})^{p_{r's'}-\delta_{l'k',r's'}}\Big] t'_{k'm'} \Big|\vec{0}_{ij}, \overset{\vec{p}_{rs}>}{\vec{p}_{q't'}}\rangle_V \ ,$$

Explicit construction of $V(S_{[k]})$

The solution of the above Eq. exists, so that the explicit form of $t'_{l'm'}$ action on the vector $|\vec{N}\rangle_V$ has the final form

$$
\begin{aligned}
t'_{l'm'} \; &= -\sum_{k'=1}^{l'-1} n_{k'm'} \left|\vec{N} - \delta_{k'm',ij} + \delta_{k'l',ij}\right\rangle_V \\
&+ \sum_{k'=l'+1}^{m'-1} n_{k'm'} \left|\vec{N} - \delta_{k'm',ij} + \delta_{l'k',ij}\right\rangle_V - \sum_{k'=m'+1}^{k} n_{m'k'} \left|\vec{N} - \delta_{m'k',ij} + \delta_{l'k',ij}\right\rangle_V \\
&+ \sum_{p=0}^{m'-l'-1} \Bigg\{ \sum_{k'_1=l'+1}^{m'-1} \ldots \sum_{k'_p=l'+p}^{m'-1} \prod_{j=1}^{p} p_{k'_{j-1}k'_j} \Bigg(\left|C^{k'_p m'}_{\vec{n}_{ij}, \vec{p}_{rs} - \sum_{j=1}^{p} \delta_{k'_{j-1}k'_j, rs}}\right\rangle_V \\
&- \sum_{n'=!k'_{p-1}}^{k'_p-1} p_{n'm'} \left|\vec{n}_{ij}, \vec{p}_{rs} - \sum_{j=1}^{p} \delta_{k'_{j-1}k'_j, rs} - \delta_{n'm',rs} + \delta_{n'k'_p,rs}\right\rangle_V \Bigg)\Bigg\}.
\end{aligned}
\tag{2}
$$

Analogously, the action of the rest E^{α_i}: $l'_{l'm'}$ on $|\vec{N}\rangle_V$ is determined with help of the "basic-block"vector $\left|C^{l'm'}_{\vec{p}_{rs}}\right\rangle_V$

$\Longrightarrow$ $V(S_{[k]})$ **is explicitly found!**

Making use of the mapping (C. Burdik, 1985)

$$|\vec{n}_{ij}, \vec{p}_{rs}\rangle_V \leftrightarrow |\vec{n}_{ij}, \vec{n}_s\rangle = \prod_{i,j\geq i}^{k} (b^+_{ij})^{n_{ij}} \prod_{r,s,s>r}^{k} (d^+_{rs})^{p_{rs}} |0\rangle \in \mathcal{H}' ,$$

$$\overset{m\neq 0}{\{f_i}\, , d^+_j\} = \delta_{ij} , \quad [b_{ij}, b^+_{lm}] = \delta_{il}\delta_{jm} , \ i<j, k<l , \qquad [d_{r_1 s_1} , d^+_{r_2 s_2}] = \delta_{r_1 r_2}\delta_{s_1 s_2} ,$$

Theorem

The polynomial oscillator realization for the $V(S_{[k]})$ *over Heisenberg-Weyl algebra* $A_{k\times k}$ *exists in the form*

$$C(b_{ij}, b^+_{lk}, d_{r_1 s_1} , d^+_{r_2 s_2}), \quad C \in \{t'_{l'm'}, t'^+_{l'm'}, l'_{i'j'}, l'^+_{i'j'} g'^i_0\}. \qquad (3)$$

explicit form of basic block $C^{lm}(d^+,d) \to \Big|C^{lm}_{\vec{p}_{rs}}\rangle_V$

$$C^{lm}(d^+,d) \equiv \Big(h^l - h^m - \sum_{n=m+1}^{k}(d^+_{ln}d_{ln} -! d^+_{mn}d_{mn}) -! \sum_{n=l+1}^{m-1} d^+_{nm}d_{nm} - d^+_{lm}d_{lm}\Big)d_{lm} + \sum_{n=m+1}^{k}\Big\{d^+_{mn} -! \sum_{n'=l+1}^{m-1} d^+_{n'n}d_{n'm}\Big\}d_{ln}.$$

so that, f.i. for t'_{lm}:

$$t'_{lm} = \sum_{p=0}^{m-l-1}\Big[\sum_{k_1=l+1}^{m-1}\cdots\sum_{k_p=l+p}^{m-1}\Big\{C^{k_p m}(d^+,d) - \sum_{n'=k'_{p-1}}^{k_p-1} d^+_{n'k_p}d_{n'm}\Big\}\prod_{j=1}^{p} d_{k_{j-1}k_j}\Big] - \sum_{n=1}^{l-1} b^+_{nl}b_{nm} + \sum_{n=l+1}^{m-1} b^+_{ln}b_{nm} - \sum_{n=m+1}^{k} b^+_{ln}b_{mn}\,, \quad k_0 \equiv l,$$

. Thus, the additive conversion of o_I into the 1st class O_I is realized! (It completely applicable for massive HS fields as well)

Complete BRST operator for Lie algebra $\mathcal{A}(Y[k], \mathbb{R}^{1,d-1})$

The BRST operator Q' for Lie algebra $\mathcal{A}_c(Y[k], \mathbb{R}^{1,d-1})$ by the standard rules of BFV-method .

$$Q' = O_I\mathcal{C}^I + \frac{1}{2}\mathcal{C}^I\mathcal{C}^J f^K_{JI}\mathcal{P}_K(-1)^{\varepsilon(o_I)+\varepsilon(o_K)}, \ {Q'}^2 = 0 \ \texttt{where} \ (\varepsilon, gh)Q' = (1,1), \quad (4)$$

$\mathcal{C}^I = (q; \vartheta, \eta; q^+, \vartheta^+, \eta^+), \mathcal{P}_K$ - ghost coordinates and momenta with opposite Grassmann parity to O_I with following non-vanishing C.R.

$$\begin{aligned} &\{\vartheta_{rs}, \lambda^+_{tu}\} = \{\lambda_{tu}, \vartheta^+_{rs}\} = \delta_{rt}\delta_{su}, && [q_i, p^+_j] = [p_j, q^+_i] = \delta_{ij} \ , \\ &\{\eta_{lm}, \mathcal{P}^+_{ij}\} = \{\mathcal{P}_{ij}, \eta^+_{lm}\} = \delta_{li}\delta_{jm} \ , && \{\eta_0, \mathcal{P}_0\} = \imath, \ \{\eta^i_{\mathcal{G}}, \mathcal{P}^j_{\mathcal{G}}\} = \imath\delta^{ij} \ ; \end{aligned} \quad (5)$$

and $gh(\mathcal{C}^I) = -gh(\mathcal{P}_I) = 1$.

Explicit form of Q'

$$Q' \quad = \quad Q_c + \sum_{i<j}\Big(\eta_{ij}\mathcal{L}^{+}_{ij} + \mathcal{L}_{ij}\eta^{+}_{ij} + \mathcal{T}_{ij}\vartheta^{+}_{ij} + \vartheta_{ij}\mathcal{T}^{+}_{ij}\Big) + \sum_{i}[\eta^{G}_{i}\sigma^{i}(G) + \imath\mathcal{B}^{i}\mathcal{P}^{G}_{i}], \qquad (6)$$

with definite operators $\mathcal{B}^i$ and with spin $\sigma^i(G)$, complete traceless $\mathcal{L}_{ij}$ and Young constraints (and theirs h,c. ones)

$$\begin{aligned} \mathcal{L}_{ij} \quad &= \quad L_{ij} + \frac{1}{2}q_{[i}p_{j]} + \sum_{p<j}\vartheta^{+}_{ip}\mathcal{P}_{pj} + \sum_{j<p}\vartheta^{+}_{jp}\mathcal{P}_{ip} - \sum_{j<p}\vartheta^{+}_{ip}\mathcal{P}_{jp} && (7)\\ &\quad + \frac{1}{4}\Big\{\sum_{p>j}\big[\eta_{ip}\lambda^{+}_{jp} - \eta_{jp}\lambda^{+}_{ip}\big] + \sum_{p<j}\eta_{pj}\lambda^{+}_{ip}\Big\}, \\ \mathcal{T}_{ij} \quad &= \quad T_{ij} + (q_j p^{+}_{i} + q^{+}_{i}p_j) + \sum_{p>j}\vartheta^{+}_{jp}\lambda_{ip} - \sum_{p<j}[\vartheta_{ip}\lambda_{pj} - \vartheta_{pj}\lambda_{ip}] && (8)\\ &\quad + \sum_{p>j}\eta_{jp}\mathcal{P}^{+}_{ip} - \sum_{i<p<j}\eta_{pj}\mathcal{P}^{+}_{ip} + \sum_{p<i}\eta_{pj}\mathcal{P}^{+}_{pi}. \end{aligned}$$

$Q'^{+}K = KQ'$, `in` $\mathcal{H}_{tot} = \mathcal{H}\otimes\mathcal{H}'\otimes\mathcal{H}_{gh}$ due to $V(S_{[k]})$ `osc.realization in` $\mathcal{H}'$

Unconstrained Lagrangian formulation

The obtaining of resulting LF takes standard character
As usual, we extract the spin operator from the Q':

$$\Longrightarrow Q' = Q + \eta^i_G(\sigma^i + h^i) + \mathcal{B}^i\mathcal{P}^i_G,$$
$$[Q, \sigma_i] = 0\,, .$$

The same applies to a scalar physical and gauge vectors

$$|\chi\rangle \;=\; \sum_{\{n\}_b=0}^{\infty} \sum_{\{n\}_f=0}^{1} \eta_0^{n_{\eta_0}} \prod_{i<j,i,j=1}^{k} \eta_{ij}^{+n_{\eta_{ij}}} \mathcal{P}_{ij}^{+n_{P_{ij}}} \prod_{r<s,r,s=1}^{k} \vartheta_{rs}^{+n_{\vartheta_{rs}}} \lambda_{rs}^{+n_{\lambda_{rs}}} \prod_{i=1}^{k} (\eta_i^G)^{n_i}$$
$$\times \prod_{i<j,i,j=1}^{k} f_j^{+n_{f_j}} b_{ij}^{+n_{b_{ij}}} \prod_{r<s,r,s=1}^{k} d_{rs}^{+n_{d_{rs}}} \Big|\chi_{n_{\eta_0} n_{\eta_{ij}} n_{\vartheta_{rs}} n_{P_{ij}} n_{\lambda_{rs}} n_i n_{q_i} n_{p_i} n_{f_j} n_{b_{ij}} n_d}$$

$|\chi^0\rangle, |\chi^s\rangle \in \mathcal{H}_{tot}$ i.e. $\partial(|\chi^0\rangle)/\partial\eta^i_G = 0$: gh$(|\chi^0\rangle, |\chi^s\rangle) = (0, -s)$

$|\chi\rangle = |\Phi\rangle + |\Phi_A\rangle, |\Phi_A\rangle_{\{(b,b^+,d,d^+) = \mathcal{C} = \mathcal{P} = 0\}} = 0$ `with` $|\Phi\rangle -$ `basic HS f.`

and with the use of the BFV–BRST EQUATION $Q'|\chi^0\rangle = 0$ that determines the physical states and a sequence of reducible gauge transformations,

Unconstrained Lagrangian formulation

$$
\begin{aligned}
& Q|\chi\rangle = 0, && (\sigma^i + h_i)|\chi\rangle = 0, && (\varepsilon, gh_H)\,(|\chi\rangle) = (\varepsilon_\chi, 0), && (11)\\
& \delta|\chi\rangle = Q|\chi^1\rangle, && (\sigma^i + h_i)|\chi^1\rangle = 0, && (\varepsilon, gh_H)\,(|\chi^1\rangle) = (\varepsilon_\chi + 1, -1), && (12)\\
& \delta|\chi^1\rangle = Q|\chi^2\rangle, && (\sigma^i + h_i)|\chi^2\rangle = 0, && (\varepsilon, gh_H)\,(|\chi^2\rangle) = (\varepsilon_\chi, -2), && (13)\\
& \ldots\ldots && \ldots\ldots && \ldots\ldots && \\
& \delta|\chi^{n-1}\rangle = Q|\chi^n\rangle, && (\sigma^i + h_i)|\chi^n\rangle = 0, && (\varepsilon, gh_H)\,(|\chi^n\rangle) = (\varepsilon_\chi + n \mod 2, -n). && (14)
\end{aligned}
$$

the middle Eqs. determines the spectrum of spin values for $|\chi\rangle$ and gauge pars. $|\chi^i\rangle$, $i = 1, ..., \sum_i s_i + \frac{1}{2}k(k-1)$, the corresponding proper eigenvalue and eigenvectors,

$$
(\sigma^i + h_i)|\chi\rangle_{s[k]} = \left(h_i + s_i - \frac{d - 6 + \theta_{m0}}{2} - 2i\right)|\chi\rangle_{s[k]} = 0 \Longleftrightarrow
$$

$$
h_i^{s[k]} = -s^i + \frac{d - 6 + \theta_{m0} + 4i}{2}, \ (\text{for } h_i^{s[k]} = h_i(s_i)) \tag{15}
$$

s_i from basic $|\Phi\rangle$: .

Lagrangian formulation with complete BRST Q operator

$\Longrightarrow$ The equations of motion and the sequence of reducible gauge transformations for the field with given $\mathbf{s} = s[k]$:

$$Q_{s[k]}|\chi^0\rangle_{[s]k} = 0, \delta|\chi^l\rangle_{[s]k} = Q_{[s]k}|\chi^{l+1}\rangle_{[s]k}, \ \delta|\chi^L\rangle_{[s]k} = 0,$$
$$l = 0, ..., L = \sum_i s_i + \frac{1}{2}k(k-1),$$

for $|\chi^0\rangle \equiv |\chi\rangle$, and can be obtained from the LAGRANGIAN ACTION

$$\mathcal{S}_{[s]k} = \int d\eta_0 \ {}_{[s]k}\langle\chi^0|K_{[s]k}Q_{[s]k}|\chi^0\rangle_{[s]k}, \ K_{[s]k} = K|_{h^i = -s^i + \frac{d-6+\theta(m0)+4i}{2}},$$

The corresponding LF of a bosonic field with a specific value of spin s subject to $Y(s_1, ..., s_k)$ is an UNCONSTRAINED REDUCIBLE GAUGE THEORY OF MAXIMALLY $L = k(k+1)$-TH STAGE OF REDUCIBILITY

Corollary: the result contains as a particular case LF for bosonic HS subject to $Y[s_1], Y[s_1, s_2]$ (Buchbinder, Krycktin, 2009, A.R., 2016)

Lagrangian formulation with incomplete BRST Q operator

Corresponding incomplete BRST operator $Q'_c = Q_c + \eta_i^G \sigma_c^i$ is easily derived from complete Q' operator:

$$
\begin{aligned}
Q'_c &= Q'\Big|_{\eta_{ij}^{(+)}=\mathcal{P}_{ij}^{(+)}=\vartheta_{rs}^{(+)}=\lambda_{rs}^{(+)}=b_{ij}^{(+)}=d_{rs}^{(+)}=h^i=0} && (16)\\
Q'_c &= \eta_0 l_0 + q_i^+ l_i + q_i l_i^+ + i q_i q_i^+ \mathcal{P}_0 + \eta_i^G \sigma_c^i(g) \ = \ Q_c + \eta_i^G \sigma_c^i(g), && (17)\\
\texttt{where} \qquad & \sigma_c^i(g) \ = \ g_0^i - q_i^+ p_i - q_i p_i^+ && (18)
\end{aligned}
$$

is incomplete spin operator. These operators as well as BRST-extended set of holonomic constraints $\widehat{L}_{ij}$, $\widehat{T}_{rs}$, (**??**) are given on the incomplete Hilbert space $\mathcal{H}_c$: $\mathcal{H}_c = \mathcal{H}^f \otimes H_{gh}^{o_A}$ The algebra of $(\sigma_c^i(g), \widehat{O}_a)$ is the same as one for $(g_0^i\,, o_a)$.:

$$
[\widehat{L}_{ij}, \sigma_c^i(g)\} \ = \ \widehat{L}_{ij}, \quad [\widehat{T}_{rs}, \sigma_c^i(g)\} \ = \ \widehat{T}_{rs}(\delta_{si} - \delta_{ri}). \qquad (19)
$$

the constrained gauge invariant LF of $(\sum_i s_i - 1)$ stage reducibility with the action $\mathcal{S}_{c|s[k]}$ for HS tensor field subject to Young tableaux $Y[s_1, \ldots, s_k]$ reads as,

$$
\mathcal{S}_{c|s[k]}(\chi_c) = \int d\eta_0 \ {}_{s[k]}\langle \chi_c | Q_c | \chi_c \rangle_{s[k]} \ , \qquad (20)
$$

$$
\Big(\delta;\ \widehat{L}_{ij},\ \widehat{T}_{rs}\Big)|\chi_c^l\rangle_{s[k]} = \Big(Q_c|\chi_c^{l+1}\rangle_{s[k]}\theta_{\sum_i s_i, l};\ 0,\ 0\Big),\ l = 0, 1, ..., \sum_i s_i. \qquad (21)
$$

For $\sum_i s_i = 0$ (which corresponds to the scalar field) the LF appears by non-gauge one.

Lagrangian formulation with incomplete BRST Q operator

Theorem: The set of solutions, $H_{(m,s[k])}$, for the equations (1)–(4) extracting the Poincare group massless ($m = 0$) irreducibile representation of spin $[s_1., ..., s_k]$ in terms of tensor field, $\Phi_{\mu^1[s_1],...,\mu^k[s_k]}$ is equivalent to the solutions of the Lagrangian equations of motion, for $l = -1$ in (21) subject to the reducible gauge transformations (21) for $l = 0, ..., \sum_i s_i$ and off-shell holonomic constraints:

$$
\begin{aligned}
H_{(0,s[k])} \; = \; & \left\{|\Phi\rangle| \Big(l_0,\, l_i,\, l_{ij},\, t_{rs},\, g_0^i - [s_i - d/2]\Big)|\Phi\rangle = 0\right\} \qquad (22) \\
= & \left\{|\chi^0_c\rangle \middle| \left[Q_c,\ \left\{\sigma^i_c - s_i + \frac{d-2}{2}\right\}\right]|\chi^0_c\rangle = 0, \right. \\
& \delta|\chi^l_c\rangle = Q_c|\chi^{l+1}_c\rangle,\ \delta|\chi_c^{\sum_i s_i}\rangle = 0 \\
& \left. \left(\widehat{L}_{ij}, \widehat{T}_{rs}, \left\{\sigma^i_c - s_i + \frac{d-2}{2}\right\}\right)|\chi^l_c\rangle = 0\right\}, \qquad (23)
\end{aligned}
$$

where, $l = 0, ..., \sum_i s_i$;

Conclusion

- Lagrangian formulations within BRST approachs with complete and incomplete BRST operators for irreducible free MAS HS field on Minkowski space $R^{1,d-1}$ subject to $Y[s_1, ..., s_k]$ is firstly constructed both for massless and massive bosonic field;
- it is shown the equivalence of respective Lagrangian dynamics within both approaches and its equivalence to solutions of Poincare group relations .
- the cubic interaction consruction now in progress

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Thank you very much