РФЯЦ-ВНИИЭФ
РОСАТОМ

Mathematical paradoxes of the Dirac equation representations

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Content

1. Dirac equation and fermion vacuum

Standard quantum electrodynamics (QED) employs the Dirac equation with a bispinor wave function. The Dirac equation for an electron with mass m and electric charge e < 0, interacting with an electromagnetic field $A^{\mu}(\mathbf{x},t)$*, can be written as*

$$p^0\psi_D(\mathbf{x},t)=H_D(\mathbf{x},t)\psi_D(\mathbf{x},t)=\Big(\boldsymbol{\alpha}\big(\mathbf{p}-e\mathbf{A}(\mathbf{x},t)\big)+\beta m+eA^0(\mathbf{x},t)\Big)\psi_D(\mathbf{x},t).$$

Here and below, the units $\hbar=c=1$ *are used.* $p^{\mu}=i\partial/\partial x_{\mu}$, $\mu=0,1,2,3;$

$H_D(\mathbf{x},t)$ *is the electromagnetic Hamiltonian;*

$A^{\mu}(\mathbf{x},t)$ *are the electromagnetic potentials;*

α^i,β *are the four-dimensional Dirac matrices (i =1,2,3); in the standard representation, the matrices* $\alpha^i,\beta,\Sigma^i,\gamma^5,\gamma^i$ *take the form*

$$\alpha^i=\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix},\ \beta=\gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\ \Sigma^i=\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix},\gamma^5=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\ \gamma^i=\gamma^0\alpha^i.$$

The bispinor $\psi_D(\mathbf{x},t)$ *can be expressed as* $\psi_D(\mathbf{x},t)=\begin{pmatrix}\varphi(\mathbf{x},t)\\ \chi(\mathbf{x},t)\end{pmatrix}.$

1. Dirac equation and fermion vacuum

In the free case (no interaction), the Dirac equation has normalized solutions with positive and negative energies ε

$$\left(\psi_D\right)_0^{(+)}\left(\mathbf{x},t\right)=\frac{1}{\left(2\pi\right)^{3/2}}\left(1+\frac{\mathbf{p}^2}{\left(\left|E\right|+m\right)^2}\right)^{-1/2}\begin{pmatrix}U_S\\ \dfrac{\boldsymbol{\sigma}\mathbf{p}}{\left|E\right|+m}U_S\end{pmatrix}e^{-i\left|E\right|t+i\mathbf{p}\mathbf{x}},\ \varepsilon=\left|E\right|>0$$

$$\left(\psi_D\right)_0^{(-)}\left(\mathbf{x},t\right)=\frac{1}{\left(2\pi\right)^{3/2}}\left(1+\frac{\mathbf{p}^2}{\left(\left|E\right|+m\right)^2}\right)^{-1/2}\begin{pmatrix}\dfrac{\boldsymbol{\sigma}\mathbf{p}}{\left|E\right|+m}U_S\\ U_S\end{pmatrix}e^{i\left|E\right|t-i\mathbf{p}\mathbf{x}},\ \varepsilon=-\left|E\right|<0.$$

Here, U_S ***is the normalized Pauli spinors*** $\left(\text{for } S_z=1/2, U_S=\begin{pmatrix}1\\0\end{pmatrix},\ \text{for } S_z=-1/2, U_S=\begin{pmatrix}0\\1\end{pmatrix}\right).$

1. Dirac equation and fermion vacuum

For stationary states in static electromagnetic fields, the Dirac equation also admits solutions with positive and negative energies.

Solutions of the Dirac equation with negative energies are physically unacceptable. This was well understood by the equation's author, P.A.M. Dirac [1]. *However, the complete set of solutions (with positive and negative energies) ensures mathematical completeness. Researchers in the first half of the twentieth century looked for a way out of the existing situation by interpreting solutions with negative energiesLet us consider two notable interpretations.*

[1]. P.A.M.Dirac, The Principles of Quantum Mechanics (Oxford University Press, 1930).

1. Dirac equation and fermion vacuum

1. *The physical vacuum of the Dirac equation is described as a "Dirac sea" of fully occupied negative-energy states. Holes in this sea are interpreted as antiparticles* [1].
2. *In the Stueckelberg-Feynman theory of positrons* [2] – [4], *positrons are electrons with negative energies moving backward in spacetime.*

The fermion vacuum in standard QED is non-empty, allowing for virtual particle-antiparticle creation and annihilation.

There exist versions of QED with an empty fermion vacuum, based on specific representations of the Dirac equation: the Foldy-Wouthuysen (FW representation) [5], *the Feynman-Gell-Mann (FG)* [6] *representation, and the representation with the Klein-Gordon (KG)-type fermion equations* [7] - [8].

For these representations, perturbative formalisms $(QED)_{FW}$ [9], [10], $(КЭД)_{FG}$ [11], *and* $(КЭД)_{KG}$ [12], [13] *have been developed, and certain physical effects have been calculated. The final physical results fully coincide with the results of standard QED with the Dirac equation.*

1. Уравнение Дирака и фермионный вакуум

[1]. P.A.M.Dirac, The Principles of Quantum Mechanics (Oxford University Press, 1930).

[2]. E. C. G. Stuekelberg, Helv. Phys. Acta. 14, L32, 588 (1941).

[3]. R. P. Feynman, Phys. Rev. 76, 749 (1949).

[4]. R. P. Feynman, Phys. Rev. 76, 769 (1949).

[5]. L. L. Foldy, S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

[6]. R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).

[7]. Y. B. Zeldovich and V. S. Popov, Sov. Phys. Usp. 14, 673 (1972).

[8]. V. P. Neznamov and I. I. Safronov, J. Exp. Theor. Phys. 128, 672 (2019), arxiv: 1907.03579.

[9]. V. P. Neznamov, Part. Nucl. 37, 86 (2006); arxiv: hep-th/0411050.

[10]. V. P. Neznamov, Part. Nucl. 43, 36 (2012); arxiv: 1107.0693.

[11]. L. M. Brown, Phys. Rev. 111, 957 (1958).

[12]. V. P. Neznamov and V. E. Shemarulin, Int. J. Mod. Phys. A 36, 2150086 (2021), arxiv: 2108.04664.

[13]. V. P. Neznamov, Int. J. Mod. Phys. A, 2150173 (2021), arxiv: 2110.03530.

1. Dirac equation and fermion vacuum

Closed equations for fermions in the FW and FG representations have also been formulated for nonperturbative QED in strong electromagnetic fields **[1], [2].**

Using these representations ensures that only positive-energy solutions are required for computing physical effects, both for real and virtual intermediate fermion states. However, separate equations for fermions and antifermions are necessary, differing in the sign of the electric charge and mass terms.

The use of these representations in QED calculations has revealed contradictions between the physical premises of the theory and mathematical results. All contradictions arise from the inclusion of negative-energy fermion states in calculations.

[1]. V. P. Neznamov. The Foldy-Wouthuysen transformation with Dirac matrices in the chiral representation: closed expressions for energy operators of fermions moving in static electromagnetic fields. Int. J. Mod. Phys. A, DOI: 10.1142/S0217751X25500496.
[2]. V. P. Neznamov. Closed Foldy-Wouthuysen transformations for fermions moving in gauge-invariant time-dependent electromagnetic fields Int. J. Mod. Phys. A. DOI: 10.1142/S0217751X25500745.

2. Perturbation theory and Dirac equation representations

2.1 FW representation

In the FW representation, two conditions must be satisfied [1] *:*

1. The Hamiltonian or energy operator must be diagonal with respect to the upper and lower spinors of the transformed wave function $\psi_{FW}(\mathbf{x}),$ *that is, these operators do not mix the upper and lower components of* $\psi_{FW}(\mathbf{x})$.

2. The FW transformation must satisfy the wave function reduction condition. For time-independent Dirac Hamiltonians (static external fields), this condition can be written as:

$$\psi_D^{(+)}(\mathbf{x},t) = e^{-i\varepsilon t} A_{(+)} \begin{pmatrix} \varphi_c(\mathbf{x}) \\ \chi_c(\mathbf{x}) \end{pmatrix} \to \psi_{FW}^{(+)}(\mathbf{x},t) = U_{FW}^{(+)}(\mathbf{x}) \psi_D^{(+)}(\mathbf{x},t) = e^{-i\varepsilon t} \begin{pmatrix} \varphi_c(\mathbf{x}) \\ 0 \end{pmatrix}, \ \varepsilon > 0.$$

$$\psi_D^{(-)}(\mathbf{x},t) = e^{-i\varepsilon t} A_{(-)} \begin{pmatrix} \varphi_c(\mathbf{x}) \\ \chi_c(\mathbf{x}) \end{pmatrix} \to \psi_{FW}^{(-)}(\mathbf{x},t) = U_{FW}^{(-)}(\mathbf{x}) \psi_D^{(-)}(\mathbf{x},t) = e^{-i\varepsilon t} \begin{pmatrix} 0 \\ \chi_c(\mathbf{x}) \end{pmatrix}, \ \varepsilon < 0.$$

$A_{(+)}, A_{(-)}$ *are normalization operators.* $U_{FW}^{(+)}, U_{FW}^{(-)}$ *are operators of the FW transformation. These operators are not necessary identical for positive and negative energies.*

[1]. V. P. Neznamov and A. J. Silenko, J. Math. Phys. 50, 122302 (2009); arxiv: 0906.2069 (math-ph).

2. Perturbation theory and Dirac equation representations

In the FW representation, the Dirac equation for an electron interacting with an electromagnetic field $A^{\mu}\left(\mathbf{x},t\right)$ *can be expanded as a power series in the electromagnetic coupling constant by the unitary transformation* [1]

$$U_{FW}=\left(1+e\delta_1+e^2\delta_2+e^3\delta_3+...\right)U_{FW}^0,\; U_{FW}^{+}=U_{FW}^{-1}.$$

The resulting equation is

$$\varepsilon\psi_{FW}=H_{FW}\psi_{FW}=\left(\beta E_p+eK_1^{FW}\left(+m,A^{\mu}\right)+e^2K_2^{FW}\left(+m,A^{\mu},A^{\nu}\right)+e^3K_3^{FW}\left(+m,A^{\mu},A^{\nu},A^{\gamma}\right)+...\right)\psi_{FW}. \quad (1)$$

Here $E_p=\sqrt{m^2+\mathbf{p}^2},\; \psi_{FW}=U_{FW}\psi_D.$

The denotation $+m$ *in* K_n^{FW} *points to the use of the positive sign in front of* βm *in the Dirac equation. The terms with the negative sign in front of the mass m are absent in the equation (1). It follows from the structure of expressions* K_1^{FW}, K_2^{FW} ...

[1]. V. P. Neznamov, Part. Nucl. 37, 86 (2006); arxiv: hep-th/0411050

2. Perturbation theory and Dirac equation representations

In the free case $\varepsilon\left(\psi_{FW}\right)_0=\beta E_p\left(\psi_{FW}\right)_0$, *where, for the positive energy* $\varepsilon=|E|>0$

$$\left(\psi_{FW}\right)_0^{(+)}(\mathbf{x},t)=U_{FW}^0\left(\psi_D\right)_0^{(+)}=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}U_S\\0\end{pmatrix}e^{-i|E|t+i\mathbf{px}},$$

for the negative energy $\varepsilon=-|E|<0$

$$\left(\psi_{FW}\right)_0^{(-)}(\mathbf{x},t)=U_{FW}^0\left(\psi_D\right)_0^{(-)}=\frac{1}{(2\pi)^{3/2}}\begin{pmatrix}0\\U_S\end{pmatrix}e^{i|E|t-i\mathbf{px}}.$$

In FW representation, the equation (1) has non-covariant form and the Hamiltonian H_{FW} *is non-local. In this case, in the quantum field theory, it is difficult to use the standard methods of secondary quantization. However, the S-matrix approach and Feynman's propagator method* [1] – [3], [4] *can be employed. In this method, QED processes are described by the integral equations.*

[1]. E. C. G. Stuekelberg, Helv. Phys. Acta. 14, L32, 588 (1941).
[2]. R. P. Feynman, Phys. Rev. 76, 749 (1949).
[3]. R. P. Feynman, Phys. Rev. 76, 769 (1949).
[4]. J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics, McGraw-Hill Book Company (1964)

2. Perturbation theory and Dirac equation representations

The equation (1) for the four-dimensional x,y can be written as

$$\psi_{FW}\left(x\right)=\left(\psi_{FW}\right)_0^{(\pm)}\left(x\right)+\int d^4y\,S_{FW}\left(x-y\right)K^{FW}\left(y\right)\psi_{FW}\left(y\right), \qquad \text{(2)}$$

$K^{FW}\left(y\right)=\sum_{n=1}^{\infty}e^nK_n^{FW}\left(y\right)$ ***is an interaction Hamiltonian in the equation (1);***

$S_{FW}\left(x-y\right)$ ***is a Feynman propagator in the FW representation***

$$S_{FW}\left(x-y\right)=\frac{1}{\left(2\pi\right)^4}\int d^4p\,e^{-ip(x-y)}\frac{p^0+\beta E}{p^2-m^2+i\varepsilon}.$$

S-matrix elements can be written as

$$S_{fi}=\delta_{fi}-i\varepsilon_f\int d^4y\left[\left(\overline{\psi}_{FW}\right)_0^{(\pm)}\left(y\right)\right]_f K^{FW}\left(y\right)\left[\psi_{FW}\left(y\right)\right]_i.$$

Here, the bar over a function ψ_{FW} ***denotes the Hermitian conjucation,***

$\varepsilon_f=+1$ ***for the solution*** $\left[\left(\overline{\psi}_{FW}\right)_0^{(+)}\left(y\right)\right]_f$ ***and*** $\varepsilon_f=-1$ ***for the solution*** $\left[\left(\overline{\psi}_{FW}\right)_0^{(-)}\left(y\right)\right]_f$.

2. Perturbation theory and Dirac equation representations

So, we note:

1. The Hamiltonians H_{FW} and $K^{FW}(y)$ are diagonal with respect to the mixing of the upper and lower components of the bispinor ψ_{FW}. Each of equations (1) and (2) comprise two independent equations with spinor wave function $\sim U_S$. One equation includes states with positive energies, while the other includes states with negative energies.

The S-matrix elements can be calculated by considering only the positive-energy states. In this case, the negative-energy states are not used in calculations of physical QED processes. They are only required for mathematical completeness in the expansions of operators and wave functions.

2. ***In standard QED with the Dirac equation, positrons are electrons with negative energies moving backward in spacetime. In the Foldy-Wouthuysen representation, the situation changes. If in the equation for S-matrix elements we use*** $\left[\left(\bar{\psi}_{FW}\right)_0^{(+)}\right]_f$ ***on the left side of the equation and*** $-\left[\left(\psi_{FW}\right)_0^{(-)}\right]_i$ ***on the right side, then due to the structure of the bispinors to all orders of perturbation theory, we will obtain zero values for the corresponding elements of the S-matrix. Taking into account the bispinors, we can write***

$$\left\langle \frac{1}{(2\pi)^{3/2}} e^{i|E_f|t - i\mathbf{p_f x}} \begin{pmatrix} \bar{U}_S & 0 \end{pmatrix} \left| M_{FW} \right| \begin{pmatrix} 0 \\ U_S \end{pmatrix} \frac{1}{(2\pi)^{3/2}} e^{i|E_i|t - i\mathbf{p_i x}} \right\rangle = 0.$$

Here, by definition, M_{FW} ***is a diagonal operator.***

A similar result will be obtained if we use in the equation for S-matrix elements on the left side of the equation $\left[\left(\bar{\psi}_{FW}\right)_0^{(-)}\right]_f$***, and on the right side*** $-\left[\left(\psi_{FW}\right)_0^{(+)}\right]_i$.

Thus, in the FW representation, positrons cannot be described by electron states with negative energies. Positrons in the FW representation must be described by positive-energy states of a special equation for positrons.

2. Perturbation theory and Dirac equation representations

3. ***In the description of physical processes in (QED)FW involving real antiparticles, it has been found that in the original Dirac equations, the mass terms for particles and antiparticles must have opposite signs*** **[1], [2].** ***This is due to the impossibility of using states with negative energies in the theory.***

4. ***The Dirac equation for positrons takes the form*** $p^0\psi_D^C = \left(\boldsymbol{\alpha}\left(\mathbf{p}+e\mathbf{A}\right)-\beta m - eA^0\right)\psi_D^C.$

This equation is differ from the initial equation in the sign of the charge and term with βm.

The equation can be written in the FW representation

$$\varepsilon\psi_{FW}^C = \left(\beta E_p - eK_1^{FW}\left(-m, A^\mu\right) + e^2K_2^{FW}\left(-m, A^\mu, A^\nu\right) - e^3K_3^{FW}\left(-m, A^\mu, A^\nu, A^\gamma\right) + \ldots\right)\psi_{FW}^C.$$

The terms with the positive sign in front of the mass m are absent in the equation.

Thus, we arrive at the first paradox. After performing the unitary transformation of the Dirac equation into the FW representation, we have lost the interaction between positive- and negative-energy states. To restore the particle-antiparticle interaction, it becomes necessary to introduce a separate equation for positrons.

[1]. V. P. Neznamov, Voprosy Atomnoy Nauki i Tekhniki, Ser. Teoreticheskaya i Prikladnaya Fizika 1, 3 (1989) (in Russian).
[2]. V. P. Neznamov, Part. Nucl. 37, 86 (2006); arxiv: hep-th/0411050.

2. Perturbation theory and Dirac equation representations

2.2 Quantum electrodynamic (QED) with with Klein-Gordon-type fermion equations

Self-adjoint equations for electrons and positrons with spinor wave functions take the form **[1], [2]**

$$\left[\left(p^0 - eA^0\right)^2 - m^2 - \left(p^0 - eA^0 + m\right)^{1/2} \boldsymbol{\sigma}\left(\mathbf{p} - e\mathbf{A}\right) \frac{1}{p^0 - eA^0 + m} \boldsymbol{\sigma}\left(\mathbf{p} - e\mathbf{A}\right)\left(p^0 - eA^0 + m\right)^{1/2}\right]\Phi = 0,$$

$$\left[\left(p^0 + eA^0\right)^2 - m^2 - \left(p^0 + eA^0 - m\right)^{1/2} \boldsymbol{\sigma}\left(\mathbf{p} + e\mathbf{A}\right) \frac{1}{p^0 + eA^0 - m} \boldsymbol{\sigma}\left(\mathbf{p} + e\mathbf{A}\right)\left(p^0 + eA^0 - m\right)^{1/2}\right]\Phi^C = 0.$$

$\Phi = g_\varphi \varphi, \varphi\left(\mathbf{x}, t\right)$ ***is the upper spinor of the Dirac equation,*** $g_\varphi = \left(p^0 - eA^0 + m\right)^{-1/2}.$

The equation for positrons is differ from the equation for electrons in the signs of the charge e and the signs of the terms with masses m.

One can perform an expansion in powers of the charge e

$$\left[p_0^2 - \mathbf{p}^2 - m^2 \mp eV_1\left(\pm m, A^\mu\right) - e^2 V_2\left(\pm m, A^\mu, A^\nu\right) \mp e^3 V_3\left(\pm m, A^\mu, A^\nu, A^\gamma\right) - \ldots\right]\Phi\left(\pm m, \mathbf{x}, t\right) = 0.$$

Here, the upper signs in front of the charge and mass correspond to equation for the electron, while the lower signs correspond to equation for the positron, $\Phi\left(+m, \mathbf{x}, t\right) = \Phi,\ \Phi\left(-m, \mathbf{x}, t\right) = \Phi^C.$

[1]. V. P. Neznamov and I. I. Safronov, J. Exp. Theor. Phys. 128, 672 (2019), arxiv: 1907.03579.
[2]. V. P. Neznamov and V. E. Shemarulin, Int. J. Mod. Phys. A 36, 2150086 (2021), arxiv: 2108.04664.

2. Perturbation theory and Dirac equation representations

In the free case, the equations for electrons and positrons are reduced to Klein-Gordon equations with spinor wave functions

$$\left(p_0^2 - \mathbf{p}^2 - m^2\right)\Phi_0\left(\mathbf{x},t\right) = 0, \quad \left(p_0^2 - \mathbf{p}^2 - m^2\right)\Phi_0^C\left(\mathbf{x},t\right) = 0.$$

Orthonormal solutions of these equations takes the form

$$\Phi_0^{(\pm)}\left(\mathbf{x},t\right) = \frac{1}{\left(2\pi\right)^{3/2}}\frac{1}{\sqrt{2E}}e^{\mp i|E|t \pm i\mathbf{px}}U_S, \quad \left(\Phi_0^C\right)^{(\pm)}\left(\mathbf{x},t\right) = \frac{1}{\left(2\pi\right)^{3/2}}\frac{1}{\sqrt{2E}}e^{\mp i|E|t \pm i\mathbf{px}}U_S.$$

In works [1], [2], by analogy with $(QED)_{FW}$, the formalism of $(QED)_{KG}$ has been developed, and a number of physical effects have been calculated. Just as in the FW representation, the final results of the calculations coincide with those in standard QED.

In $(QED)_{KG}$, the calculations do not involve real or virtual states with negative energies. In the equations, the mass terms for particles and antiparticles have opposite signs.

[1]. V. P. Neznamov and V. E. Shemarulin, Int. J. Mod. Phys. A 36, 2150086 (2021), arxiv: 2108.04664.
[2]. V. P. Neznamov, Int. J. Mod. Phys. A, 2150173 (2021), arxiv: 2110.03530.

3. Nonperturbative QED and Dirac equation representations

3.1 The standard QED in the fields of hydrogen-like ions with a large charge number Z

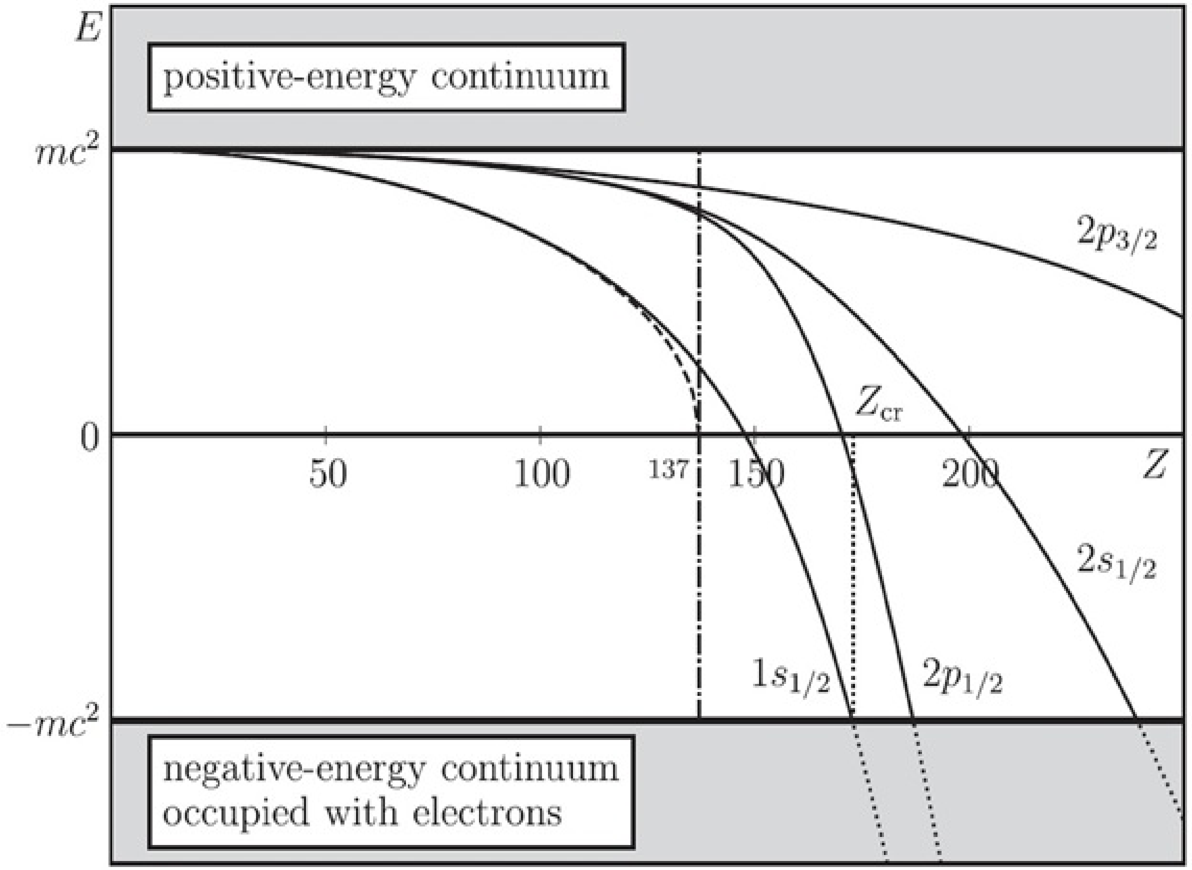


Fig. 1. The low-lying energy levels of the hydrogen-like ion as the function of the nuclear charge number Z.

For the standard QED with the fluctuating fermionic vacuum in Fig.1, the low-lying energy levels of hydrogen-like ions are presented as the function of the nuclear charge number Z.

The figure is from the monograph *W. Greiner, J. Reinhardt, Quantum Electrodynamics, Springer-Verlag Berlin Heidelberg NewYork (2002).*

3. Nonperturbative QED and Dirac equation representations

Let us consider the level $1s_{1/2}$. For the Coulomb field of the nucleus point charge $+Z|e|$, the electron level $1s_{1/2}$ disappears at $Z=137$. If we take into account the finite dimensions of atomic nuclei **[1] – [4]**, ***the energy of state $1s_{1/2}$ becomes negative at $Z > 146$. At $Z_{cr} \approx 171$, the level $1s_{1/2}$ dives to the negative-energy continuum. Similarly, the energy of level $2p_{1/2}$ becomes negative at $Z > 168$; at $Z_{cr} \approx 184$, the level $2p_{1/2}$ dives to the negative-energy continuum. According to the theoretical predictions of the standard QED, when a level dives to the negative-energy continuum, a neutral vacuum disintegrates emitting two electron-positron pairs*** **[2], [3].**

[1]. I. Pomeranchuk and J. Smorodinsky, J. Phys. USSR 9, 97 (1945).
[2]. S. S. Gershtein and Y. B. Zeldovich, Zh. Eksp. Teor. Fiz. 57, 654 (1969) [Sov. Phys. JETP 30, 358 (1970)].
[3]. W. Pieper and W. Greiner, Z. Phys. 218, 327 (1969).
[4]. Y. B. Zeldovich and V. S. Popov, Sov. Phys. Usp. 14, 673 (1972).

3. Nonperturbative QED and Dirac equation representations

3.2 Feynman-Gell-Mann and Foldy-Wouthuysen representations

Let us write the Dirac equation in an external electromagnetic field in covariant form

$$\left(\gamma^0\left(p^0-eA^0\right)-\boldsymbol{\gamma}\left(\mathbf{p}-e\mathbf{A}\right)-m\right)\psi_D(\mathbf{x},t)=0.$$

Multiply on the left by an operator with the reversed sign of the fermion mass

$$\left(\gamma^0\left(p^0-eA^0\right)-\boldsymbol{\gamma}\left(\mathbf{p}-e\mathbf{A}\right)+m\right)\left(\gamma^0\left(p^0-eA^0\right)-\boldsymbol{\gamma}\left(\mathbf{p}-e\mathbf{A}\right)-m\right)\psi_D(\mathbf{x},t)=0.$$

As a result, we obtain a second-order equation with solutions degenerate with respect to the sign of the mass m.

$$\left[\left(p^0-eA^0\right)^2-\left(\mathbf{p}-e\mathbf{A}\right)^2-m^2+e\boldsymbol{\Sigma}\mathbf{H}-ie\alpha\mathbf{E}\right]\psi_D(\mathbf{x},t)=0.$$

Here $\mathbf{H}=\operatorname{rot}\mathbf{A},\ \mathbf{E}=-\dfrac{\partial\mathbf{A}}{\partial t}-\nabla A^0$ *are magnetic and electric fields.*

Below, we will consider the case of static electromagnetic fields, when $p^0\psi_D=\varepsilon\psi_D.$

To transition to the FG representation, it is necessary to use Dirac matrices in the chiral representation. This is achieved through a unitary transformation

$$S=S^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix} I & I\\ I & -I\end{pmatrix},\ \psi_{FG}\left(\mathbf{x},t\right)=S\psi\left(\mathbf{x},t\right)=\begin{pmatrix}\varphi_{FG}\left(\mathbf{x}\right)\\ \chi_{FG}\left(\mathbf{x}\right)\end{pmatrix}e^{-i\varepsilon t},\ \alpha_c^i=S\alpha^i S^{-1}=\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix},\ \Sigma_c^i=S\Sigma^i S^{-1}=\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}.$$

3. Nonperturbative QED and Dirac equation representations

In the FG representation, second-order equation does not mix the upper and lower components of the bispinor $\psi_{FG}(\mathbf{x},t)$. *In the case of stationary states this equation is reduced to two separate equations for the spinors* $\varphi_{FG}(\mathbf{x}),\ \chi_{FG}(\mathbf{x})$.

$$\left[\left(\varepsilon - eA^0\right)^2 - (\mathbf{p} - e\mathbf{A})^2 - m^2 + e\boldsymbol{\sigma}\mathbf{H} - ie\boldsymbol{\sigma}\mathbf{E}\right]\varphi_{FG}(\mathbf{x}) = 0, \quad (3) \qquad \left[\left(\varepsilon - eA^0\right)^2 - (\mathbf{p} - e\mathbf{A})^2 - m^2 + e\boldsymbol{\sigma}\mathbf{H} + ie\boldsymbol{\sigma}\mathbf{E}\right]\chi_{FG}(\mathbf{x}) = 0. \quad (4)$$

Similar equations were previously considered by Feynman and Gell-Mann [1].

It is noteworthy these equations are related to the equations in the Foldy-Wouthuysen representation [2], [3]. *Equation (3) in the FW representation is obtained for positive energies* $\varepsilon = |E| > 0$. *In this case,* $\varphi_{FG}(\mathbf{x}) = A_{(+)}\varphi_c(\mathbf{x}),\ \varphi_c(\mathbf{x})$ *is the upper spinor in the FW representation* $\left(\psi_{FW}^{(+)}(\mathbf{x}) = \begin{pmatrix} \varphi_c(\mathbf{x}) \\ 0 \end{pmatrix}\right)$.

Equation (4) in the FW representation is obtained for negative energies $\varepsilon = -|E| < 0$. *In this case,*

$\chi_{FG}(\mathbf{x}) = A_{(-)}\chi_c(\mathbf{x})$, *where* $\chi_c(\mathbf{x})$ *is the upper spinor in the FW representation* $\left(\psi_{FW}^{(-)}(\mathbf{x}) = \begin{pmatrix} 0 \\ \chi_c(\mathbf{x}) \end{pmatrix}\right)$.

[1]. R. P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958).
[2]. V. P. Neznamov. The Foldy-Wouthuysen transformation with Dirac matrices in the chiral representation: closed expressions for energy operators of fermions moving in static electromagnetic fields. Int. J. Mod. Phys. A, DOI: 10.1142/S0217751X25500496.
[3]. V. P. Neznamov. Closed Foldy-Wouthuysen transformations for fermions moving in gauge-invariant time-dependent electromagnetic fields. Int. J. Mod. Phys. A. DOI: 10.1142/S0217751X25500745.

3. Nonperturbative QED and Dirac equation representations

Let us write in according to* [1] *some of the equations for electrons and positrons in the FG and FW representations.

1. The equation for electrons with positive energies $\left(\varepsilon = |E| > 0,\ e = -|e| < 0\right)$

$$\left(\left(|E| + |e| A^0\right)^2 - \mathbf{p}^2 - m^2 - i|e|\boldsymbol{\sigma}\nabla A^0\right)\varphi_{FG}^{e} = 0.$$

2. The equation for electrons with negative energies $\left(\varepsilon = -|E| < 0,\ e = -|e| < 0\right)$

$$\left(\left(|E| - |e| A^0\right)^2 - \mathbf{p}^2 - m^2 + i|e|\boldsymbol{\sigma}\nabla A^0\right)\chi_{FG}^{e} = 0.$$

3. The equation for positrons with positive energies $\left(\varepsilon = |E| > 0,\ e = |e| > 0\right)$

$$\left(\left(|E| - |e| A^0\right)^2 - \mathbf{p}^2 - m^2 + i|e|\boldsymbol{\sigma}\nabla A^0\right)\varphi_{FG}^{p} = 0.$$

[1]. V. P. Neznamov. The Foldy-Wouthuysen transformation with Dirac matrices in the chiral representation: closed expressions for energy operators of fermions moving in static electromagnetic fields. Int. J. Mod. Phys. A, DOI: 10.1142/S0217751X25500496.

3. Nonperturbative QED and Dirac equation representations

In equation for electrons with positive energies, the spinor $\varphi_{FG}^{e}(\mathbf{x})$ *in the Feynman-Gell-Mann representation is proportional to the upper spinor* $\varphi_{c}^{e}(\mathbf{x})$ *in the FW representation*

$$\left(\varphi_{FG}^{e}(\mathbf{x}) = A_{(+)}^{e}\varphi_{c}^{e}(\mathbf{x});\ \left(\psi_{FW}^{(+)}(\mathbf{x})\right)^{e} = \begin{pmatrix} \varphi_{c}^{e}(\mathbf{x}) \\ 0 \end{pmatrix}\right).$$

In equation for electrons with negative energies, the spinor $\chi_{FG}^{e}(\mathbf{x})$ *in the Feynman-Gell-Mann representation is proportional to the lower spinor* $\chi_{c}^{e}(\mathbf{x})$ *in the FW representation*

$$\left(\chi_{FG}^{e}(\mathbf{x}) = A_{(-)}^{e}\chi_{c}^{e}(\mathbf{x});\ \left(\psi_{FW}^{(-)}(\mathbf{x})\right)^{e} = \begin{pmatrix} 0 \\ \chi_{c}^{e}(\mathbf{x}) \end{pmatrix}\right).$$

In equation for positrons with negative energies, the spinor $\varphi_{FG}^{p}(\mathbf{x})$ *in the Feynman-Gell-Mann representation is proportional to the upper spinor* $\varphi_{c}^{p}(\mathbf{x})$ *in the FW representation*

$$\left(\varphi_{FG}^{p}(\mathbf{x}) = A_{(+)}^{p}\varphi_{c}^{p}(\mathbf{x});\ \left(\psi_{FW}^{(+)}(\mathbf{x})\right)^{p} = \begin{pmatrix} \varphi_{c}^{p}(\mathbf{x}) \\ 0 \end{pmatrix}\right).$$

Above

$$A_{(+)}^{e} = \left(1 + \frac{m^2}{\left(|E| + \boldsymbol{\sigma}\mathbf{p} + |e|A^0\right)^2}\right)^{-1/2},\quad A_{(-)}^{e} = \left(1 + \frac{m^2}{\left(|E| + \boldsymbol{\sigma}\mathbf{p} - |e|A^0\right)^2}\right)^{-1/2},\quad A_{(+)}^{p} = \left(1 + \frac{m^2}{\left(|E| + \boldsymbol{\sigma}\mathbf{p} - |e|A^0\right)^2}\right)^{-1/2}.$$

3. Nonperturbative QED and Dirac equation representations

As a result, we see that equation for positrons with positive energies, $\varepsilon > 0$, coincides with equation for electrons with negative energies $\varepsilon < 0$.

In our case, the positrons are in the repulsive Coulomb field of ionized nuclei. For them, the upper continuum is available with the continuous energy spectrum $\varepsilon > m$. But in this case, there are no stationary bound states with $\varepsilon < m$.

The equality of these equations implies that the equivalent (accurate within the energy sign) continuous energy spectrum exists for electrons with negative energies and positrons with positive energies. But in the spectrum of equation for electrons with negative energies in the range of $Z_\Sigma = 147 \div 170$, there is the negative energy level $1s_{1/2}$ and in the range of $Z_\Sigma = 169 \div 183$ there is the negative energy level $2p_{1/2}$ (see Fig. 1).

3. Nonperturbative QED and Dirac equation representations

For equation for positrons with positive energies, the existence of such bound states is impossible for simple physical reasons. Hence, the existence of bound states with negative energies is a mathematical artifact.

Since the equations for electrons and positrons are obtained by unitary transformations of the Dirac equation, the conclusion about the absence of a physical (as opposed to mathematical) contribution of stationary bound states with negative energies to the calculated QED effects also holds for the original Dirac equation.

Let us note that equation for positrons with positive energies with the changed sign in front of $A^0(\mathbf{x})$ *(the motion of a positron in the attractive Coulomb field) coincides with equation for electrons. As it should be, discrete and continuous energy spectra of electrons and positrons, moving in the attractive Coulomb field, coincide with each other.*

3.3 QED with spinor Klein-Gordon-type equations

Let us write the selfconjugate equations for electrons and positrons with spinor wave functions in the presence of electrostatic fields.

1. The equation for electrons with positive energies $\left(\varepsilon = |E| > 0,\ e = -|e| < 0\right)$

$$\left[\left(|E| + |e|A^0\right)^2 - m^2 - \left(|E| + |e|A^0 + m\right)^{1/2} \boldsymbol{\sigma}\mathbf{p} \frac{1}{|E| + |e|A^0 + m} \boldsymbol{\sigma}\mathbf{p} \left(|E| + |e|A^0 + m\right)^{1/2}\right] \Phi_e^{(+)} = 0.$$

2. The equation for electrons with negative energies $\left(\varepsilon = -|E| < 0,\ e = -|e| < 0\right)$

$$\left[\left(|E| - |e|A^0\right)^2 - m^2 - \left(|E| - |e|A^0 - m\right)^{1/2} \boldsymbol{\sigma}\mathbf{p} \frac{1}{|E| - |e|A^0 - m} \boldsymbol{\sigma}\mathbf{p} \left(|E| - |e|A^0 - m\right)^{1/2}\right] \Phi_e^{(-)} = 0.$$

3. The equation for positrons with positive energies $\left(\varepsilon = |E| > 0,\ e = |e| > 0\right)$

$$\left[\left(|E| - |e|A^0\right)^2 - m^2 - \left(|E| - |e|A^0 - m\right)^{1/2} \boldsymbol{\sigma}\mathbf{p} \frac{1}{|E| - |e|A^0 - m} \boldsymbol{\sigma}\mathbf{p} \left(|E| - |e|A^0 - m\right)^{1/2}\right] \Phi_p^{(+)} = 0.$$

As in Sec. 3.2, the analysis of these equations shows that the existence of bound states with negative energies in the spectrum of hydrogen-like ions is a mathematical artifact.

3.4 Dirac equation with non-relativistic Hamiltonian in the Foldy-Wouthuysen representation

The conclusions of the Sec. 3.2 and 3.3 are supported by the analysis of the non-relativistic Hamiltonian in the FW representation which was originally obtained in the first paper devoted to the FW transformation [1].

Let us consider the non-relativistic motion of electrons and positrons in an external electrostatic field $eA^0(\mathbf{x})$. *The non-relativistic Hamiltonian takes the form*

$$H_{FW}=\beta\left(m+\frac{\mathbf{p}^2}{2m}\right)+eA^0-\frac{ie}{8m^2}\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E}-\frac{e}{4m^2}\boldsymbol{\sigma}\left(\mathbf{E}\times\mathbf{p}\right)-\frac{e}{8m^2}\mathrm{div}\,\mathbf{E}.$$

Here, $\mathbf{E}=-\vec{\nabla}A^0$ *is an electric field.*

The Dirac equation with the Hamiltonian H_{FW} *is written in the form*

$$\left[\left(\varepsilon-eA^0\right)-\beta\left(m+\frac{\mathbf{p}^2}{2m}\right)+\frac{ie}{8m^2}\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E}+\frac{e}{4m^2}\boldsymbol{\sigma}\left(\mathbf{E}\times\mathbf{p}\right)+\frac{e}{8m^2}\mathrm{div}\,\mathbf{E}\right]\psi_{FW}\left(\mathbf{x},t\right)=0.$$

Here, for $\varepsilon>0$ $\quad\psi_{FW}^{(+)}\left(\mathbf{x},t\right)=\begin{pmatrix}\varphi(\mathbf{x})\\0\end{pmatrix}e^{-i\varepsilon t}$, *in this case* $\beta\psi_{FW}^{(+)}\left(\mathbf{x},t\right)=\psi_{FW}^{(+)}\left(\mathbf{x},t\right)$;

for $\varepsilon<0$ $\quad\psi_{FW}^{(-)}\left(\mathbf{x},t\right)=\begin{pmatrix}0\\\chi(\mathbf{x})\end{pmatrix}e^{-i\varepsilon t}$, *in this case* $\beta\psi_{FW}^{(-)}\left(\mathbf{x},t\right)=-\psi_{FW}^{(-)}\left(\mathbf{x},t\right)$.

[1]. L. L. Foldy, S. A. Wouthuysen, Phys. Rev. 78, 29 (1950).

3. Nonperturbative QED and Dirac equation representations

Let us consider the Dirac equation for electrons and positrons:

1. The equation for electrons with positive energies $\left(\varepsilon=|E|>0,\ e=-|e|<0\right)$

$$\left[\left(|E|+|e|A^0\right)-\left(m^2+\frac{\mathbf{p}^2}{2m}\right)-\frac{i|e|}{8m^2}\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E}-\frac{|\mathbf{e}|}{4m^2}\boldsymbol{\sigma}\left(\mathbf{E}\times\mathbf{p}\right)-\frac{|\mathbf{e}|}{8m^2}\mathrm{div}\,\mathbf{E}\right]\varphi^e\left(\mathbf{x}\right)=0.$$

2. The equation for electrons with negative energies $\left(\varepsilon=-|E|<0,\ e=-|e|<0\right)$

$$\left[\left(|E|-|e|A^0\right)-\left(m^2+\frac{\mathbf{p}^2}{2m}\right)+\frac{i|e|}{8m^2}\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E}+\frac{|\mathbf{e}|}{4m^2}\boldsymbol{\sigma}\left(\mathbf{E}\times\mathbf{p}\right)+\frac{|\mathbf{e}|}{8m^2}\mathrm{div}\,\mathbf{E}\right]\chi^e\left(\mathbf{x}\right)=0. \qquad (5)$$

3. The equation for positrons with positive energies $\left(\varepsilon=|E|>0,\ e=|e|>0\right)$

$$\left[\left(|E|-|e|A^0\right)-\left(m^2+\frac{\mathbf{p}^2}{2m}\right)+\frac{i|e|}{8m^2}\boldsymbol{\sigma}\,\mathrm{rot}\,\mathbf{E}+\frac{|\mathbf{e}|}{4m^2}\boldsymbol{\sigma}\left(\mathbf{E}\times\mathbf{p}\right)+\frac{|\mathbf{e}|}{8m^2}\mathrm{div}\,\mathbf{E}\right]\varphi^p\left(\mathbf{x}\right)=0. \qquad (6)$$

It can be seen that the equations (5) and (6) coincide with each other; that is, just as in the previous Sections dealing with the Coulomb field of atomic nuclei, the equation for positrons with $\varepsilon > 0$ *coincides with the equation for electrons with* $\varepsilon < 0$.

3. Nonperturbative QED and Dirac equation representations

However, in the non-relativistic case with $|E| - m \ll 1$, the equality of the equations for electrons and positrons does not lead to contradictions with physical reality. Indeed, in this case, the equation for electrons with negative energies lacks discrete negative energy levels (see Fig. 1).

Such levels mathematically appear only on the regime of strong electrostatic fields with Z > 146. It contradicts clear physical arguments about the absence of the discrete levels with negative energies.

Conclusions

Our analysis is identified two paradoxes.

<u>Paradox 1.</u>

After unitary transformation the Dirac equation in the external electromagnetic field and transition to the FW representation, S-matrix elements lose interaction between positive- and negative-energy states. To restore this interaction, it is necessary to introduce of additional equation for positrons with positive energies.

<u>Paradox 2.</u>

Contrary to clear physical premises, discrete energy levels with negative energies appear in the mathematical calculations of the energy spectrum of hydrogen-like ions with $Z_{\Sigma} = 147 \div 183$.

Conclusions

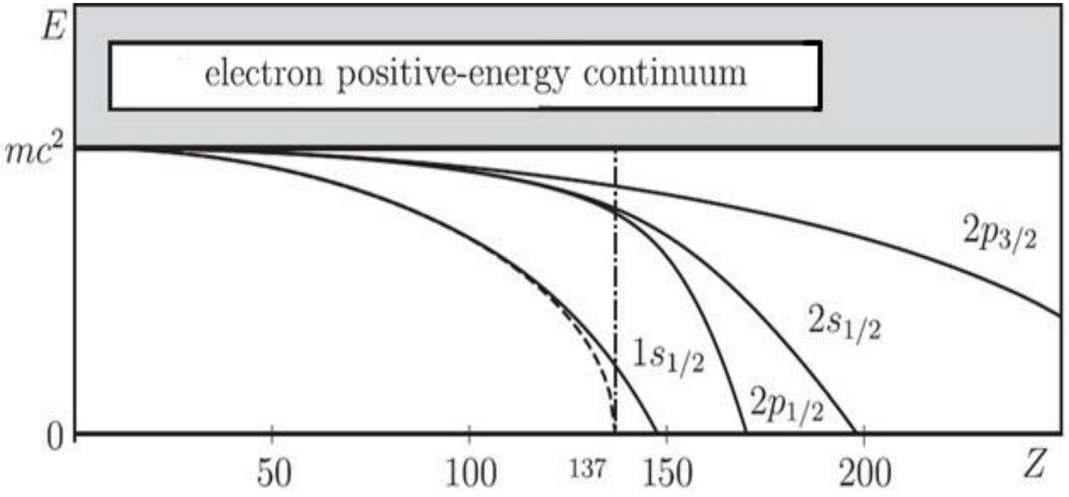
Both paradoxes vanish if only positive-energy states are used in QED calculations. Negative-energy states should only ensure mathematical completeness in expansions of the wave functions. Here, besides the equation for electrons with positive energies, an equation for positrons with positive energies is also introduced.

This approach applies to $(QED)_{FW}$, $(QED)_{KG}$ ***, and standard QED with the Dirac equation*** **[1]**. ***In the perturbative regime, physical results coincide with those in standard QED.***

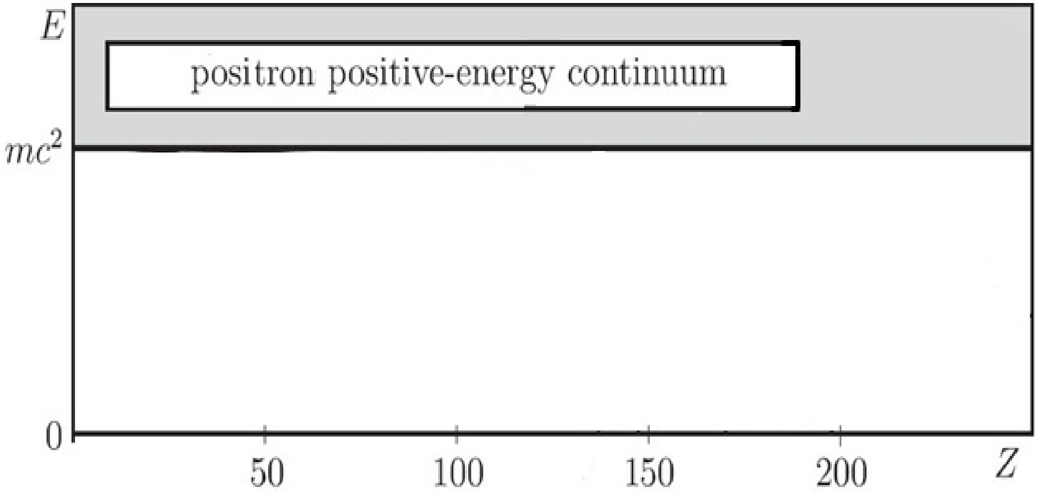
[1]. V. P. Neznamov, FIZMAT, V.2, №2, 94 (2024) (in Russian).

Conclusions

For nonperturbative QED, the spectrum of hydrogen-like ions with the charges Z|e| (Fig. 1) should be replaced by the spectrum of Fig. 2.



a)



b)

Fig. 2. The energy spectrum of: a) equation for electrons, b) equation for positrons

In the nonperturbative QED, the calculated spectrum Fig. 2a) (without discrete and continuous spectrum with negative energies) can be experimentally verified. In paper [1], to confirm this, a series of experiments using heavy ion collisions at colliders has been proposed.

[1]. V. P. Neznamov. The possibility to experimentally determine the structure of a fermionic vacuum in quantum electrodynamics. (2025). Int. J. Mod. Phys. A. DOI: 10.1142/S0217751X25501040.

Conclusions

Let us note that P.A.M. Dirac, dissatisfied with the presence of negative-energy states in his equation, turned toward the end of his course of life to the search for a relativistic wave equation with only positive-energy solutions **[1]*****. However, he failed to see his plan through to the logical end. In this work, we continue the pursuit of resolving the physical problems in quantum electrodynamics associated with the presence of unphysical fermionic states with negative energies.***

[1]. P.A.M.Dirac, Proc. Roy. Soc. A 322, 435 (1971); Proc. Roy. Soc. A 328, 1 (1972).

«Physical laws should have mathematical beauty»

P. A. M. Dirac

«What exists mathematically may, but does not necessarily, exist in reality»

Heino Falcke

Thank you for your attention