Cosmological dynamic of $f(R+\mu G)$ models in the FLRW background

Ekaterina Pozdeeva

SINP MSU

based on

N. Dimakis, A. Giacomini, G. Leon, A. Paliathanasis, E. Pozdeeva and S. Vernov, [arXiv:2505.02663 [gr-qc]]

The XXV International Workshop-School
High Energy Physics and Quantum Field Theory
QFTHEP'270

D.V. Skobeltsyn Institute of Nuclear Physics and Faculty of Physics of M.V. Lomonosov Moscow State University
Moscow, Russia, June 30 – July 5, 2025

The cosmological models with the Gauss-Bonnet term are motivated by string theory [1].

[1] [1] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. B 415, 497 (1994), arXiv:hep-th/9305025.
S. Kawai and J. Soda, Phys. Lett. B 460, 41 (1999), arXiv:gr-qc/9903017.
C. Cartier, J.-c. Hwang, and E. J. Copeland, Phys. Rev. D 64, 103504 (2001), arXiv:astro-ph/0106197.
J.-c. Hwang and H. Noh, Phys. Rev. D 71, 063536 (2005), arXiv:gr-qc/0412126.
M. Sami, A. Toporensky, P. V. Tretjakov, and S. Tsujikawa, Phys. Lett. B 619, 193 (2005), arXiv:hep-th/0504154.
G. Calcagni, S. Tsujikawa, and M. Sami, Class. Quant. Grav. 22, 3977 (2005), arXiv:hep-th/0505193.
S. Tsujikawa and M. Sami, JCAP 01, 006, arXiv:hep-th/0608178.
G. Cognola, E. Elizalde, S. Nojiri, S. Odintsov, and S. Zerbini, Phys. Rev. D 75, 086002 (2007), arXiv:hep-th/0611198.
S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rept. 692, 1 (2017), arXiv:1705.11098 [gr-qc].
S. V. Ketov, E. O. Pozdeeva, and S. Y. Vernov, JCAP 12, 032, arXiv:2211.01546 [gr-qc]

The Gauss-Bonnet term affects the dynamics in higher-dimensional gravitational models [2]

[2] D. Lovelock, J. Math. Phys. 12, 498 (1971).
G. Dotti, J. Oliva, and R. Troncoso, Phys. Rev. D 76, 064038 (2007), arXiv:0706.1830 [hep-th].
S. A. Pavluchenko and A. V. Toporensky, Mod. Phys. Lett. A 24, 513 (2009), arXiv:0811.0558 [gr-qc].
R. S. Bogadi, M. Govender, and S. Moyo, Eur. Phys. J. Plus 138, 426 (2023), [Erratum: Eur.Phys.J.Plus 138, 535 (2023)].
J. M. Z. Pretel, A. Banerjee, and A. Pradhan, Nucl. Phys. B 1008, 116702 (2024), arXiv:2107.03859 [gr-qc].
A. Giacomini, A. Paliathanasis, and A. Toporensky, (2025), arXiv:2503.07260 [gr-qc].

• We study the cosmological dynamics of Einstein-Gauss-Bonnet gravity models in a four-dimensional spatially flat FLRW metric. These models are described by $f(R, \mathcal{G}) = f(R + \mu\mathcal{G})$ theory of gravity. They are equivalent to models linear in the Ricci scalar R and in the Gauss-Bonnet scalar $\mathcal{G}$ with one nonminimally coupled scalar field without kinetic term.
• We analyze the stability of the de Sitter solutions and construct the phase space of the field equations to investigate the cosmological evolution.

A generic model with the Gauss-Bonnet term and a single scalar field can be described by the following Action Integral

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{\beta}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma) - F(\sigma)\mathcal{G}\right], \qquad (1)$$

where $U(\sigma)$, $V(\sigma)$, and $F(\sigma)$ are double differentiable functions of the scalar field σ. The Gauss-Bonnet term is given by

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \qquad (2)$$

At $\beta = 1$, we get standard scalar field models, whereas the case of $\beta = 0$ corresponds to $f(R, \mathcal{G})$ models.

• If the scalar field σ has no kinetic term, $\beta = 0$, then varying (1) by σ, we obtain

$$U_{,\sigma} R - F_{,\sigma}\mathcal{G} = V_{,\sigma}, \tag{3}$$

where the commas denote derivatives with respect to the scalar field σ, $A_{,\sigma} = \frac{dA}{d\sigma}$ for any function $A(\sigma)$. Using Eq. (3), one can eliminate σ from action (1) and get the corresponding $f(R,\mathcal{G})$ model.

- If we choose

$$U(\sigma) = f_{,\sigma}, \quad F(\sigma) = 0, \quad V = \sigma f_{,\sigma} - f, \tag{4}$$

where $f(\sigma)$ is a nonlinear double differential function, then Eq. (3) gives $\sigma = R$, and we obtain the $f(R)$ gravity model with

$$S_{\mathcal{F}} = \int d^4x\sqrt{-g}\, f(R). \tag{5}$$

The phase-space analysis of this type of models has been presented in *L. JËarv and D. Kraiko, (2025), arXiv:2503.07544 [gr-qc].*

- The choice of

$$U(\sigma) = U_0, \quad F(\sigma) = \ - f_{,\sigma}, \quad V = \sigma f_{,\sigma} - f, \tag{6}$$

where U_0 is a constant, gives us $\sigma = \mathcal{G}$ as a solution of Eq. (3). So, action (1) transforms to

$$S_{\mathcal{G}} = \int d^4x\sqrt{-g}\left[U_0 R + f(\mathcal{G})\right]. \tag{7}$$

The phase-space analysis of such type of models has been made in *G. Papagiannopoulos, O. Luongo, G. Leon, and A. Paliathanasis, (2025), arXiv:2503.01620 [gr-qc].*

• We consider $f(R,\mathcal{G})$ models that correspond to nonconstant functions $U(\sigma)$ and $F(\sigma)$. Namely, we put

$$U(\sigma) = f_{,\sigma}, \quad F(\sigma) = -\mu f_{,\sigma}, \quad V = \sigma f_{,\sigma} - f, \tag{8}$$

where $\mu = \mu_0/M_{\rm Pl}^2$, μ_0 is a dimensionless constant. We thus, obtain $\sigma = R + \mu\mathcal{G}$, so action (1) takes the form

$$S_{R\,\mathcal{G}} = \int d^4x\sqrt{-g}\, f(R+\mu\mathcal{G}). \tag{9}$$

• Note that the action

$$S_{R\,\mathcal{G}} = \int d^4x\sqrt{-g}\, \left(U_0 R + f_1(R+\mu\mathcal{G})\right) \tag{10}$$

belongs to the class of models of (9), because $\sqrt{-g}\,\mathcal{G}$ is a total derivative, so we can choose

$$f(R+\mu\mathcal{G}) = U_0(R+\mu\mathcal{G}) + f_1(R+\mu\mathcal{G}) \tag{11}$$

and get action (10) from (9).
• All functions (5), (7), and (9) are particular cases of the action

$$S_{R\,\mathcal{G}} = \int d^4x\sqrt{-g}\, \left(U_0 R + f(c_1 R + c_2\mathcal{G})\right), \tag{12}$$

where c_1 and c_2 are constants.

- In the spatially flat FLRW geometry with the line element

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{13}$$

- the Ricci scalar R and the Gauss-Bonnet scalar $\mathcal{G}$ in terms of the scale factor $a(t)$ and of the lapse function $N(t)$ are as follows

$$R = 6\left(2H^2 + \frac{1}{N}\dot{H}\right), \tag{14}$$

$$\mathcal{G} = 24H^2\left(H^2 + \frac{1}{N}\dot{H}\right), \tag{15}$$

where the Hubble function is $H = \frac{\dot{a}}{N\,a}$ and the dots denote derivatives with respect to t.

• From action (1) with $\beta = 0$ in the spatially flat FLRW geometry with the line elementat $N = 1$, when t is the cosmic time, we derive the evolution equations:

$$6H^2U + 6HU_{,\sigma}\dot{\sigma} = V + 24H^3F_{,\sigma}\dot{\sigma}, \tag{16}$$

$$2\left(U - 4H\dot{F}\right)\dot{H} = -\ddot{U} + H\dot{U} + 4H^2\left(\ddot{F} - H\dot{F}\right), \tag{17}$$

$$V_{,\sigma} - 6\left(\dot{H} + 2H^2\right)U_{,\sigma} + 24H^2F_{,\sigma}\left(\dot{H} + H^2\right) = 0. \tag{18}$$

• Equations (17) and (18) can be presented as the following dynamical system [3]:

$\dot{\sigma} = \zeta,$

$\dot{\zeta} =$

$$\frac{(12H^2(U_{,\sigma\sigma}F_{,\sigma}+U_{,\sigma}F_{,\sigma\sigma})-48F_{,\sigma}F_{,\sigma\sigma}H^4-3U_{,\sigma\sigma}U_{,\sigma})\zeta^2+H(3B+4F_{,\sigma}V_{,\sigma}-6U_{,\sigma}^2)\zeta-\frac{V^2}{U}X}{3(4H^2F_{,\sigma}-U_{,\sigma})^2}$$

$\dot{H} = -\frac{V^2}{6(4H^2F_{,\sigma}-U_{,\sigma})U^2}X$

where $X = \frac{U^2}{V^2}\left[24H^4F_{,\sigma} - 12H^2U_{,\sigma} + V_{,\sigma}\right]$, $B = 3\left(4H^2F_{,\sigma} - U_{,\sigma}\right)^2$.

[3] S. Vernov and E. Pozdeeva, Universe 7, 149 (2021), arXiv:2104.11111 [gr-qc].

A de Sitter solution corresponds to $\dot{\zeta}_{dS} = 0$, $\zeta_{dS} = 0$, and $\dot{H}_{dS} = 0$, so, $X(H_{dS}, \sigma_{dS}) = 0$. Using Eq. (16) and assuming $H_{dS} > 0$, we get

$$H_{dS} = \sqrt{\frac{V(\sigma_{dS})}{6U(\sigma_{dS})}}, \tag{19}$$

hence,

$$\begin{aligned} X(H_{dS}, \sigma_{dS}) &= \frac{2}{3}F_{,\sigma}(\sigma_{dS}) - \frac{2U(\sigma_{dS})U_{,\sigma}(\sigma_{dS})}{V(\sigma_{dS})} + \frac{U^2(\sigma_{dS})V_{,\sigma}(\sigma_{dS})}{V^2(\sigma_{dS})} \\ &= V_{eff\,,\sigma}(\sigma_{dS}) = 0, \end{aligned} \tag{20}$$

where V_{eff} is the effective potential

$$V_{eff} = \frac{2}{3}F - \frac{U^2}{V} = -\frac{2\mu}{3}\,f_{,\sigma} - \frac{f_{,\sigma}^2}{\sigma f_{,\sigma} - f}. \tag{21}$$

The same result can be obtained from Eqs. (3) and (19).

In the case of $U(\sigma_{dS}) > 0$, stable de Sitter solutions correspond to $V_{eff,\sigma\sigma}(\sigma_{dS}) > 0$, whereas unstable de Sitter solutions correspond to $V_{eff,\sigma\sigma}(\sigma_{dS}) < 0$, the proof is given in Ref. [4]. Note that the effective potential can be useful for the consideration of slow-roll inflation in models described by the action (1) with $\beta = 1$, see Refs. [5]

[4] S. Vernov and E. Pozdeeva, Universe 7, 149 (2021), arXiv:2104.11111 [gr-qc].

[5] E. O. Pozdeeva, M. R. Gangopadhyay, M. Sami, A. V. Toporensky, and S. Y. Vernov, Phys. Rev. D 102, 043525 (2020), arXiv:2006.08027 [gr-qc].
E. O. Pozdeeva, M. A. Skugoreva, A. V. Toporensky, and S. Y. Vernov, JCAP 09, 050, arXiv:2403.06147 [gr-qc].
E. O. Pozdeeva, M. A. Skugoreva, A. V. Toporensky, and S. Y. Vernov, (2025), arXiv:2502.13008 [gr-qc]

• Introducing the two Lagrange multipliers λ_1, λ_2 to $S=\int d^4x\sqrt{-g}f\left(R,\mathcal{G}\right)$ we get

$$S_{f(R,\mathcal{G})}=\int d^4x\sqrt{-g}\left[f\left(R,\mathcal{G}\right)-\lambda_1\left(R-6\left(2H^2+\frac{1}{N}\dot{H}\right)\right)\right.$$
$$\left.-\lambda_2\left(\mathcal{G}-24H^2\left(H^2+\frac{1}{N}\dot{H}\right)\right)\right]$$

Variation of the action with respect to the R, $\mathcal{G}$ gives $\lambda_1=f_{,R}$,$\lambda_2=f_{,\mathcal{G}}$.
• We introduce the $f_{,R}=\phi$, $f_{,\mathcal{G}}=\psi$, and the scalar field potential related to the function $f\left(R,\mathcal{G}\right)$ by the following expression,

$$V\left(\phi,\psi\right)=f_{,R}R+f_{,\mathcal{G}}\mathcal{G}-f\left(R,\mathcal{G}\right), \tag{22}$$

• Using $\dot{H}=-\frac{\dot{N}}{N^2}\frac{\dot{a}}{a}+\frac{1}{N}\frac{\ddot{a}}{a}-\frac{1}{N}\left(\frac{\dot{a}}{a}\right)^2$, and integration by parts we get

$$S_{f(R,\mathcal{G})}=\int d^4x\left(-\frac{6}{N}\phi a\dot{a}^2-\frac{6}{N}a^2\dot{a}\dot{\phi}-\frac{8}{N^3}\dot{\psi}\dot{a}^3-Na^3V\left(\phi,\psi\right)\right) \tag{23}$$

If we consider the $f(R,\mathcal{G}) = f(\mathcal{X})$ model, with $\mathcal{X} = R + \mu\mathcal{G}$, from the chain rule for the scalar fields it follows

$$\phi = f_{,R} = \frac{df}{d\mathcal{X}}\frac{d\mathcal{X}}{dR} = \frac{df}{d\mathcal{X}}, \tag{24}$$

$$\psi = f_{,\mathcal{G}} = \frac{df}{d\mathcal{X}}\frac{d\mathcal{X}}{dG} = \mu\frac{df}{d\mathcal{X}} = \mu\phi. \tag{25}$$

Hence, we end with the single scalar field point-like Lagrangian function

$$L^{A}_{f(R,\mathcal{G})}\left(N, a, \dot{a}, \phi, \dot{\phi}\right) = -\frac{6}{N}\phi a\dot{a}^2 - \frac{6}{N}a^2\dot{a}\dot{\phi} - \frac{8\mu}{N^3}\dot{\phi}\dot{a}^3 - Na^3V(\phi). \tag{26}$$

where now

$$V(\phi) = \mathcal{X}F_{,\mathcal{X}}(\mathcal{X}) - F(\mathcal{X}). \tag{27}$$

Evolution equations in terms of dimensionless variables

• For the phase-space analysis of $f(R+\mu\mathcal{G})$ models, it is suitable to introduce a new scalar field ϕ:

$$\phi = U(\sigma)\,, \tag{28}$$

therefore, $F(\sigma) = -\mu\phi$ and the effective potential becomes

$$V_{eff} = \frac{2}{3}F - \frac{U^2}{V} = -\frac{2}{3}\mu\phi - \frac{\phi^2}{V(\phi)}. \tag{29}$$

To obtain $V(\phi)$ one should invert Eq. (28) and solve with respect to σ.

• Using the variation of the point-like Lagrangian,

$$L^{A}_{f(R,G)}\left(N,a,\dot{a},\phi,\dot{\phi}\right)=-\frac{6}{N}\phi a\dot{a}^{2}-\frac{6}{N}a^{2}\dot{a}\dot{\phi}-\frac{8\mu}{N^{3}}\dot{\phi}\dot{a}^{3}-Na^{3}V\left(\phi\right). \quad (30)$$

we get evolution equations for the spatially flat FLRW metric

$$6\phi H^{2}+6H\frac{\dot{\phi}}{N}+24H^{3}\mu\frac{\dot{\phi}}{N}=V\left(\phi\right), \quad (31)$$

$$\begin{aligned} 0 &= \frac{4}{N}\left(\frac{d}{dt}\left(\phi H\right)+4\mu H^{3}\dot{\phi}\right)+\frac{2}{N^{2}}\left(8\mu H\dot{H}\dot{\phi}+\left(1+4\mu H^{2}\right)\ddot{\phi}\right) \\ &- \frac{2}{N^{3}}\left(1+4\mu H^{2}\right)\dot{N}\dot{\phi}+6\phi H^{2}-V\left(\phi\right), \quad (32) \end{aligned}$$

$$V_{,\phi}=6\left(2H^{2}+\frac{1}{N}\dot{H}\right)+24\mu H^{2}\left(H^{2}+\frac{1}{N}\dot{H}\right). \quad (33)$$

• We proceed with the analysis of the phase space for the latter set of nonlinear differential equations. Specifically, we examine the existence of equilibrium points. Each equilibrium point describes a specific era in the cosmological history. The stability properties of the equilibrium points give information about cosmic evolution.

• The Hubble function H can change its sign and take the value zero.

• We study the phase space dynamics by introducing appropriate dimensionless variables where the Hubble function can change its sign smoothly.

• In the following, without loss of generality, we assume the lapse function $N = 1$.

• We consider the following dimensionless variables [6]

$$x = \frac{8\dot{\phi}}{\phi\sqrt{M_{\rm Pl}^2 + H^2}}, \quad y = -\frac{V(\phi)}{\phi\left(M_{\rm Pl}^2 + H^2\right)}, \quad \eta = \frac{H}{\sqrt{M_{\rm Pl}^2 + H^2}}, \tag{34}$$

$$\lambda = \phi\frac{V_{,\phi}}{V}, \quad \tau = \int\sqrt{M_{\rm Pl}^2 + H^2}dt, \tag{35}$$

in which we assume τ to be a new independent variable (a parametric time), whereas x, y, η and λ to be new dependent variables.

[6] A. D. Millano, G. Leon, and A. Paliathanasis, Mathematics 11, 1408 (2023), arXiv:2302.09371 [gr-qc].
A. D. Millano, G. Leon, and A. Paliathanasis, Phys. Rev. D 108, 023519 (2023), arXiv:2304.08659 [gr-qc].
A. Paliathanasis, Gen. Rel. Grav. 56, 84 (2024).

It follows from Eqs. (33) and (34) that

$$H=\frac{M_{\rm Pl}\,\eta}{\sqrt{1-\eta^2}}\,, \tag{36}$$

$$\dot{H}=\frac{M_{\rm Pl}^2\left(24\mu_0\eta^4-12\eta^4-\eta^2\lambda\,y+12\eta^2+y\lambda\right)}{6\left(\eta^2-1\right)\left((4\mu_0-1)\eta^2+1\right)}\,. \tag{37}$$

- The deceleration parameter reads

$$q=-1-\frac{\dot{H}}{H^2}=\frac{(1-\lambda)\left(1-\eta^2\right)}{\left(1-(1-4\mu_0)\,\eta^2\right)}-\frac{\lambda}{8}\frac{x}{\eta}\,. \tag{38}$$

In terms of the dimensionless variables, Eq. (31) leads to the algebraic constraint

$$y=-6\eta^2-\frac{3}{4}\,\eta\,x\left(1+\frac{4\mu_0\eta^2}{1-\eta^2}\right), \tag{39}$$

- Evolution equations in terms of dimensionless variables:

$$\frac{d\eta}{d\tau} = \left(1-\eta^2\right)^2 \frac{\dot{H}}{M_{\rm Pl}^2} = -\frac{\left(1-\eta^2\right)\left(\lambda y\left(1-\eta^2\right)+12\eta^2\left(1-\left(1-2\mu_0\right)\eta^2\right)\right)}{6\left(1-\left(1-4\mu_0\right)\eta^2\right)}. \tag{40}$$

$$\begin{aligned}\frac{dx}{d\tau} \quad = \quad & -\frac{1}{8}x^2 + \frac{\eta x\left(\lambda y\left(1-\eta^2\right)+12\eta^2\left(1-\left(1-2\mu_0\right)\eta^2\right)\right)}{6\left(1-\left(1-4\mu_0\right)\eta^2\right)} \\ & -\frac{2\left(1-\eta^2\right)\left(y\left(6-4\lambda-2\lambda\mu_0 x\eta-\left(6-4\lambda-24\mu_0\right)\eta^2\right)\right)}{3\left(1-\left(1-4\mu_0\right)\eta^2\right)^2} \\ & -\frac{2\left(1-\eta^2\right)\eta\left(4\left(1+4\mu_0\right)\eta^3-4\eta+x\left(1-\eta^2\right)\right)}{\left(1-\left(1-4\mu_0\right)\eta^2\right)^2}. \end{aligned} \tag{41}$$

$$\frac{dy}{d\tau} = \frac{\left(\lambda-1\right)yx}{8} + \frac{\left(24\mu_0\eta^4-12\eta^4-\eta^2\lambda\, y+12\eta^2+y\lambda\right)\eta\, y}{3\left(1+\left(4\mu_0-1\right)\eta^2\right)}. \tag{42}$$

and

$$\frac{d\lambda}{d\tau} = \frac{\lambda}{8}x\left(1-\lambda+\lambda\Gamma\left(\lambda\right)\right) \ \text{ where } \quad \Gamma\left(\lambda\right) = \frac{V_{,\phi\phi}V}{V_{,\phi}^2}. \tag{43}$$

By definition, the dynamical variable η is governed by the constraint $|\eta| \leqslant 1$. However, the rest of the dynamical variables are not subject to constraints and assume values in whole set of the real numbers.

Due to the algebraic constraint equation (39), the order of the dynamical system is reduced by one. We substitute the dynamical variable y from (39) in the rest of the field equations and obtain a third-order dynamical system. It is better to eliminate y instead of x, because at $\eta = 0$, Eq. (39) gives $y = 0$, whereas there is no restriction on the values of x. We proceed with the derivation of the equilibrium points for this dynamical system for specific functional forms of the potential.

Phase-space analysis of $f(R+\mu\mathcal{G})$ model

We consider

$$f(R+\mu\mathcal{G}) = U_0(R+\mu\mathcal{G}) + \alpha(R+\mu\mathcal{G})^{n/(n-1)} - \Lambda, \qquad (44)$$

where U_0, α, Λ, and $n \neq 1$ are constants. At $\alpha = 0$ or $n = 0$, we retrieve the General Relativistic model with cosmological constant. We explore the case of modified gravity, when $\alpha \neq 0$ and $n \neq 0$.

For the function

$$f(\sigma) = U_0\sigma + \alpha\sigma^{n/(n-1)} - \Lambda,$$

we get

$$\phi = f_{,\sigma} = U_0 + \frac{\alpha n}{n-1}\sigma^{1/(n-1)}, \qquad (45)$$

$$V = \sigma f_{,\sigma} - f = \frac{\alpha}{n-1}\sigma^{n/(n-1)} + \Lambda = V_0\left(\phi - U_0\right)^n + \Lambda, \qquad (46)$$

where

$$V_0 = \frac{\alpha}{n-1}\left(\frac{n-1}{\alpha n}\right)^n. \qquad (47)$$

The effective potential is the

$$V_{eff} = -\frac{2}{3}\mu\left(U_0 + \frac{\alpha n\,\sigma^{n/(n-1)}}{n-1}\right) - \frac{\left(\alpha\, n\sigma^{n/(n-1)} + (n-1)U_0\,\sigma\right)^2}{(n-1)\,\sigma^2\left(\Lambda(n-1) + \alpha\,\sigma^{n/(n-1)}\right)}.$$

Power-law potential $V(\phi) = V_0\phi^n$

- In the case of $U_0 = 0$ and $\Lambda = 0$, we get
- the power-law potential $V(\phi) = V_0\phi^n$

and

$$f(\sigma) = \alpha\,\sigma^{\frac{n}{n-1}}. \tag{48}$$

- The effective potential is

$$V_{eff}(\sigma) = -\frac{n\alpha}{3(n-1)}\left(3n\sigma^{-(n-2)/(n-1)} + 2\mu\sigma^{1/(n-1)}\right), \tag{49}$$

- de Sitter solutions,

$$\sigma_{dS} = \frac{3(n-2)n}{2\mu}. \tag{50}$$

- The solution H_{dS}

$$H_{dS}^2 = \frac{V(\sigma_{dS})}{6U(\sigma_{dS})} = \frac{(n-2)M_{\mathrm{Pl}}^2}{4\mu_0}. \tag{51}$$

existing at $(n-2)\mu_0 > 0$.

- The value of η_{dS} is

$$\eta_{dS} = \sqrt{\frac{2-n}{2-n-4\mu_0}}, \tag{52}$$

where we use $H_{dS} > 0$.

- At a de Sitter point, we get

$$y_{dS} = -\frac{\sigma_{dS}}{n\,(M_{\rm Pl}^2 + H^2)} = -\frac{3\,(n-2)\,M_{\rm Pl}^2}{2\mu_0\,(M_{\rm Pl}^2 + H^2)}\,. \tag{53}$$

$$x_{dS} = 0\,. \tag{54}$$

The search of fixed points of the equations in terms of dimensionless variable allows to get this de Sitter solution as well, it is the point P_4^+. Formula (35) implies that a monomial potential $V(\phi)$ corresponds to a constant $\lambda = n$. Therefore, we examine the phase-space for the two-dimensional space with dynamical variables x and η.

Solutions in dimensionless coordinates

It follows that the field equations possess nine equilibrium points $P = (x(P), \eta(P))$. The physical properties and the stability of the asymptotic solutions near to the stationary points are described below:

- The fixed point

$$P_0 = (0, 0),$$

describes the Minkowski spacetime, where the scale factor is a constant. The eigenvalues of the linearized system around the point are zero. Hence, the point is a center and the solution is unstable.

- The fixed points

$$P_1^{\pm} = (0, \pm 1),$$

describe scaling solutions where the Gauss-Bonnet term dominates, that is $q\left(P_1^{\pm}\right) = 0$. The corresponding eigenvalues of the linearized system are calculated for point $\{\pm 4, \pm 2\}$, from where we infer that P_1^{+} is always a source, and point P_1^{-} is always an attractor.

- The fixed points

$$P_2^{\pm} = \left(\pm \frac{32}{1+3n}, \pm 1 \right)$$

exist for $n \neq -\frac{1}{3}$. The points describe scaling solutions with $q\left(P_2^{\pm}\right) = -\frac{4n}{1+3n}$, that is, acceleration is occurred for $n < -\frac{1}{3}$ and $n > 0$. For large values of $|n|$, it follows that $q\left(P_2^{\pm}\right) \to -\frac{4}{3}$. The eigenvalues of the linearized system near to the stationary points are $\left\{ \frac{2(1-n)}{1+3n}, -4 \right\}$ for point P_2^+ and $\left\{ -\frac{2(1-n)}{1+3n}, 4 \right\}$ for point P_2^-. Hence, for $n < -\frac{1}{3}$or $n > 0$, point P_2^+ is an attractor and P_2^- is a source. Nevertheless, for n within the region $-\frac{1}{3} < n < 0$, the equilibrium points are saddle points.

- The fixed points

$$P_3^{\pm} = \frac{1}{\sqrt{1-2\mu_0}} (\pm 8, \pm 1)$$

are real and physically accepted for $\mu_0 < \frac{1}{2}$. For these points we calculate that $q\left(P_3^{\pm}\right) = -1$. Someone may infer that the asymptotic solutions describe the de Sitter universe, but on the other hand, $\eta^2 = 1$. Thus, these points are not physically accepted. The eigenvalues of the linearized system around the stationary points are $\left\{-\frac{1-n}{\sqrt{1-2\mu_0}}, -\frac{4}{\sqrt{1-2\mu_0}}\right\}$ for point P_3^+, and $\left\{\frac{1-n}{\sqrt{1-2\mu_0}}, \frac{4}{\sqrt{1-2\mu_0}}\right\}$ for point P_3^-. Hence, for parameter $n < 1$, point P_3^+ is an attractor and P_3^- is a source. Otherwise, for $n \geq 1$ the points are saddle points. In order to avoid these points, parameter n should be greater than one, $n > 1$. On the other hand, points $P_3^{\pm}$ do not exist at $\mu_0 > \frac{1}{2}$.

- The fixed points

$$P_4^{\pm} = \left(0, \pm\sqrt{\frac{2-n}{2-n-4\mu_0}}\right),$$

are real and physically accepted for $\{\mu_0 < 0 : n < 2\}$ or $\{\mu_0 > 0 : 2 < n \}$.
The asymptotic solutions have deceleration parameter $q\left(P_4^{\pm}\right) = -1$, so, P_4^{+} describes de Sitter universe, whereas P_4^{-} describes universe with a constant negative Hubble parameter. The analysis of the eigenvalues gives that point P_4^{+} is an attractor when $\{\mu_0 < 0,\ 1 < n < 2\ \}$, while point P_4^{-} is a source.

Compactified variables

In order to examine the existence of stationary points for very large values of the parameter x, we introduce the new dependent and independent variables

$$x = \frac{X}{\sqrt{1 - X^2}}, \ \sqrt{1 - X^2} dT = d\tau, \tag{55}$$

where the parameter X is constrained as $|X| \leq 1$.
The two-dimensional dynamical system takes the form

$$\frac{dX}{dT} = F_1(X, \eta; n), \tag{56}$$

$$\frac{d\eta}{dT} = F_2(X, \eta; n). \tag{57}$$

There are six equilibrium points which solve the algebraic equations $X^2 = 1$ and $\eta\left(\eta^2 - 1\right) = 0$.
The phase-space analysis at the infinity regime is summarized in the following lines.

- The fixed points

$$P_0^{(\infty)\pm} = (\pm 1, 0)\,,$$

describe Minkowski solutions, with eigenvalues $\left\{\pm\frac{n}{8}, \pm\frac{1}{4}\right\}$, from where infer that for $n > 0$, $P_0^{(\infty)+}$ is a source point, while $P_0^{(\infty)-}$ is an attractor.

- The fixed points

$$P_1^{(\infty)\pm} = (\pm 1, 1)\,.$$

correspond to universes with $q\left(P_1^{(\infty)\pm}\right) = -\frac{n}{8}\infty$. Hence, for $n > 0$, the points describe Big Rip singularities, however for $n < 0$, they describe Minkowski spacetimes. Because for $n < 0$, $q\left(P_1^{(\infty)\pm}\right) > 0$, but $\eta > 0$, which means that necessary the asymptotic solution is that of the Minkowski spacetime. The eigenvalues of the linearized system are determined $\left\{\mp\frac{n}{4}, \pm\frac{1}{4}(1+3n)\right\}$. Therefore only point $P_1^{(\infty)-}$ can be an attractor, for $-\frac{1}{3} < n < 0$.

• The fixed points

$$P_{-1}^{(\infty)\pm} = (\pm 1, -1)$$

correspond to universes with $q\left(P_{-1}^{(\infty)\pm}\right) = \frac{n}{8}\infty$. Hence for $n > 0$ the points describe Big Crunch singularities, however for $n < 0$, they describe Minkowski spacetimes, because $\eta < 0$. The eigenvalues are calculated $\left\{\mp\frac{n}{4}, \pm\frac{1}{4}(1+3n)\right\}$. Therefore only point $P_{-1}^{(\infty)-}$ can be an attractor, for $-\frac{1}{3} < n < 0$.

• In Figs. 1 and 2, we present the qualitative evolution of the phase-space for the two-dimensional system (41), (40) and the equilibrium points for different values of the free parameters. The phase-space portraits can be used to constraint the initial conditions such that the resulting attractor to describes the cosmic expansion and acceleration, as described by point P_4^+, as also to avoid singularities described by the stationary points at the infinite regime.

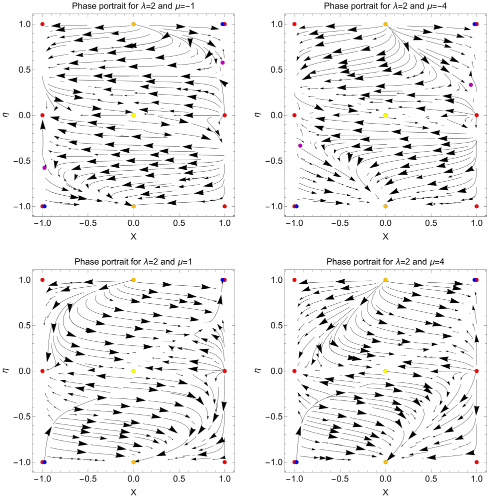

Рис.: Phase-space portrait for the two-dimensional dynamical system (41), (40) in the compactified variables X and η for $n = 2$ and various values of the parameter μ_0. Equilibrium points at the infinity are marked with red. Point P_0 is marked with yellow, orange points are $P_1^{\pm}$, blue points are $P_2^{\pm}$, and purple points are $P_3^{\pm}$.

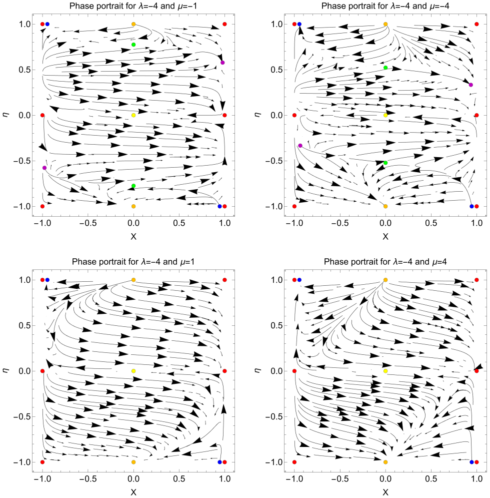

Рис.: Phase-space portrait for the two-dimensional dynamical system (41), (40) in the compactified variables X and η for $n = -4$ and various values of μ_0. Equilibrium points at the infinity are marked with red. Point P_0 is marked with yellow. Orange points are $P_1^{\pm}$, blue points are $P_2^{\pm}$, purple points are $P_3^{\pm}$, and green points are equilibrium points $P_4^{\pm}$.

CONCLUSIONS

• In this work, we perform a detailed analysis of the phase-space for the cosmological field equations within the fourth-order $f(R, \mathcal{G})$-gravity. In particular, we considered spatially flat FLRW cosmology within the context of $f(R, \mathcal{G}) = f(R + \mu\mathcal{G})$ theory of gravity. With the use of the Lagrange multiplier approach, this theory with a nonlinear function f is equivalent to a theory with a scalar field nonminimally coupled to the Ricci scalar and to the Gauss-Bonnet term.

• We investigate the stability of exact solutions of physical interest utilizing two methods. Firstly, we considered the effective potential approach to determine conditions for the stability of the de Sitter solution. However, this method cannot be used to determine the stability of scaling solutions or to understand the global dynamics. Therefore, we introduce dimensionless variables and wrote the cosmological field equations in equivalent form and get a system of first-order nonlinear differential equations of these variables.

• For the $f(R, \mathcal{G}) \simeq (R + \mu\mathcal{G})^{\frac{n}{n-1}}$ model, with and without a cosmological constant term, we have performed a detailed analysis of the stability of the stationary points.

CONCLUSIONS

• In this work, we perform a detailed analysis of the phase-space for the cosmological field equations within the fourth-order $f(R,\mathcal{G})$-gravity. In particular, we considered spatially flat FLRW cosmology within the context of $f(R,\mathcal{G}) = f(R+\mu\mathcal{G})$ theory of gravity. With the use of the Lagrange multiplier approach, this theory with a nonlinear function f is equivalent to a theory with a scalar field nonminimally coupled to the Ricci scalar and to the Gauss-Bonnet term.

• We investigate the stability of exact solutions of physical interest utilizing two methods. Firstly, we considered the effective potential approach to determine conditions for the stability of the de Sitter solution. However, this method cannot be used to determine the stability of scaling solutions or to understand the global dynamics. Therefore, we introduce dimensionless variables and wrote the cosmological field equations in equivalent form and get a system of first-order nonlinear differential equations of these variables.

• For the $f(R,\mathcal{G}) \simeq (R+\mu\mathcal{G})^{\frac{n}{n-1}}$ model, with and without a cosmological constant term, we have performed a detailed analysis of the stability of the stationary points.

Thank you for your attention