Chern-Simons theory, Knots and Topological quantum computer

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Chern-Simons theory and Wilson-loop averages

Chern-Simons theory is a 3d topological gauge theory

$$S = \frac{k}{4\pi}\int \mathrm{Tr}_{\mathrm{adj}}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

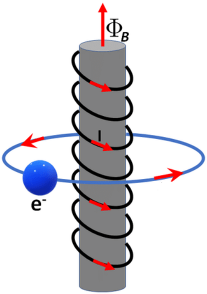
We study Wilson-loop averages in such a theory:

$$\left\langle W_Y^K \right\rangle_{CS(N,q)} = \left\langle \mathrm{Tr}_Y\, Pexp\left(\oint_K \mathcal{A}\right)\right\rangle_{CS(N,q)}$$

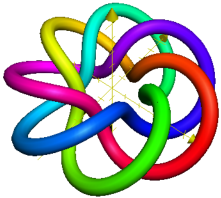
K - contour in 3d space - knots and links.

Wilson-loops

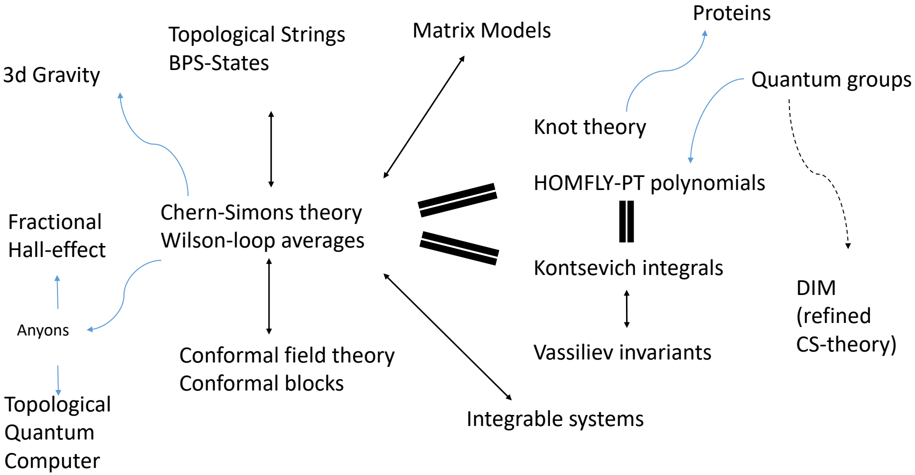
In 4d theory we need “singularities” to get non-trivial Wilson-loops



In 3d they can be entangled and non-trivial by themselves



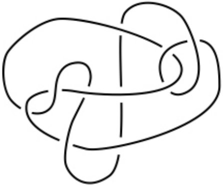
Chern-Simons theory and other theories



Knot theory

The goal of the knot theory is to distinguish different knots – embeddings

$$S^1 \hookrightarrow S^3$$



$\stackrel{?}{=}$

Knot invariants

To solve this problem knot invariants was suggested.

$$\text{Knot} \quad K \to In(K)$$

$$K_1 = K_2 \Rightarrow In(K_1) = In(K_2)$$

The goal is to construct **complete** knot invariant

$$In(K_1) \neq In(K_2) \Rightarrow K_1 \neq K_2$$

Promising type of invariants are polynomial invariants.

HOMFLY-PT polynomials

$$H(3_1) = -A^4 + A^2q^2 + \frac{A^2}{q^2}$$
$$H(4_1) = \frac{1}{A^2} + 1 + A^2 - q^2 - \frac{1}{q^2}$$
$$H(5_1) = A^4 - A^6q^2 - \frac{A^6}{q^2} + A^4q^4 + \frac{A^4}{q^4}$$
$$H(5_2) = -A^2 - A^4 - A^6 + A^4q^2 + A^2q^2 + \frac{A^4}{q^2} + \frac{A^2}{q^2}$$
$$H(6_1) = \frac{1}{A^2} + A^2 + A^4 - A^2q^2 - q^2 + 2 - \frac{A^2}{q^2} - \frac{1}{q^2}$$
$$H(6_2) = -2A^2 - A^4 + A^4q^2 + A^2q^2 + q^2 + \frac{A^4}{q^2} + \frac{A^2}{q^2} + \frac{1}{q^2} - A^2q^4 - \frac{A^2}{q^4}$$

Knot polynomials in CS

Wilson-loop averages in Chern-Simons theory are equal to knot polynomials.
Depending on the gauge group they are equal to

- SU(2) – Jones polynomials $\left\langle W_Y^K \right\rangle_{CS(2,q)} = J(q)$
- SU(N) – HOMFLY-PT polinomials $\left\langle W_Y^K \right\rangle_{CS(N,q)} = H(A,q)$
- SO(N) – Kauffmann polynomials

Variables in polynomials are functions of theory parameters:

$$q = exp(\tfrac{4\pi i}{k+N})$$

$$A = q^N$$

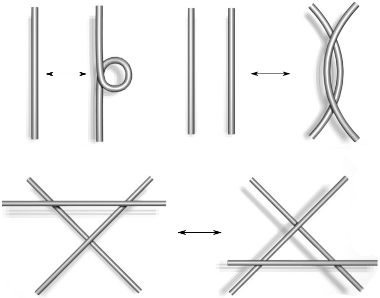
Gauge in CS

For the temporal gauge $\mathcal{A}_0 = 0$ in Chern-Simons theory one can get the following propagator
$D(x-y) \sim \delta(x_1-y_1)\delta(x_2-y_2)\theta(x_0-y_0)$

Therefore contributions to Wilson loop given only by crossings on 2-dimensional diagram of a knot.

Reidemeister moves

For Wilson-loops to be topologically invariant they should satisfy Reidemeister moves:



Third move is (quantum) Yang-Baxter equation, its solutions are called $\mathcal{R}$-matrices. For the CS theory they are $\mathcal{R}$-matrices of quantum groups.

$\mathcal{R}$-matrix in the space of intertwiners

$\mathcal{R}$-matrix commutes with coproduct of a quantum group.

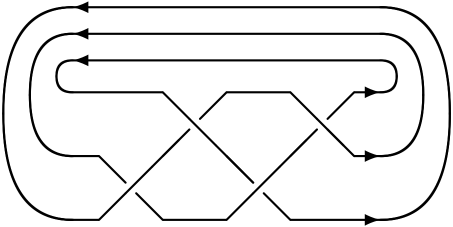
$$\mathcal{R}\Delta(g)\mathcal{R}^{-1} = \Delta(g)$$

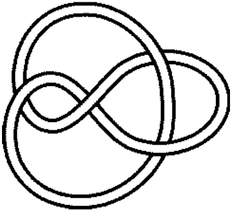
This means that it acts in the same way on the elements of irreducible representation in an expansion of a tensor product of two irreps.

For $SU(2)$ group it works like this $[1] \otimes [1] = [2] + [1,1]$

$$\mathcal{R} = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & q - q^{-1} & \\ & & & q \end{pmatrix} \rightarrow \begin{pmatrix} q & & & \\ & q & & \\ & & -q^{-1} & \\ & & & q \end{pmatrix} \begin{matrix} [2] \\ [2] \\ [1,1] \\ [2] \end{matrix}$$

Braids and $\mathcal{R}$-matrices





$\mathcal{R}$-matrices can be used in a braid.

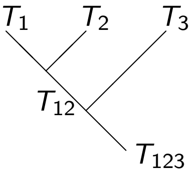
$$\mathcal{R}_1 = \widetilde{\mathcal{R}} \otimes I, \quad \mathcal{R}_2 = I \otimes \widetilde{\mathcal{R}}$$

Yang-Baxter equation looks like

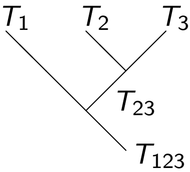
$$\mathcal{R}_1\mathcal{R}_2\mathcal{R}_1 = \mathcal{R}_2\mathcal{R}_1\mathcal{R}_2$$

Racah matrices (6-j symbols)

Different $\mathcal{R}_i$-matrices are related by bases rotation in the space of irreps – Racah matrices.



$$= \sum_{T_{23}} \begin{bmatrix} T_1 & T_2 & T_{12} \\ T_3 & T_{123} & T_{23} \end{bmatrix}$$



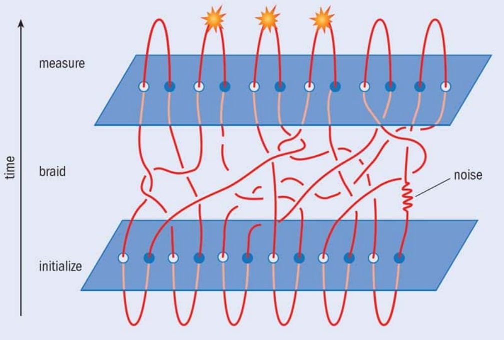
Problem of finding general answers for Racah matrices is an open one.

Known Racah

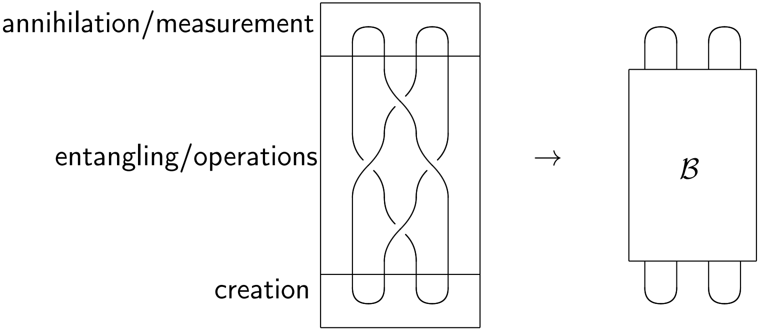
- $SU(2)$
- 3-strand symmetric representations
- multiplicity-free
- 3-strands for representations [2,1] and [3,1]

Topological quantum computer

Chern-Simons theory is an effective theory for the anyon pseudoparticles. These particles give rise to an idea of topological quantum computer.



One-qubit



One-qubit operations are given by 4-plat knots, with space of Hilbert space of a qubit provided by representations [2] and [1, 1], and operations by different $\mathcal{R}$-matrices acting in the braid.

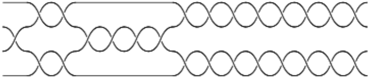
One-qubit operations

There are two matrices acting in braids of this type for $SU(2)$. They are unitary and therefore can be used as quantum gates or be modeled on the quantum computer.

$$\mathcal{R} = \begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} = \begin{pmatrix} exp(\frac{4\pi i}{k+N}) & \\ & -exp(-\frac{4\pi i}{k+N}) \end{pmatrix}$$

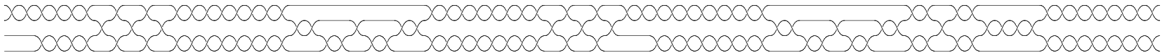
$$S = \frac{1}{q+q^{-1}} \begin{pmatrix} 1 & \sqrt{q^2+1+q^2} \\ \sqrt{q^2+1+q^{-2}} & -1 \end{pmatrix}$$

Approximation by R-matrices



Zero-th approximation of $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

precision $\epsilon = 0.043$



First approximation, precision $\epsilon = 0.0068$
$N = 2,\ k = 13$

2-qubit operations

This braid gives a 2-qubit operation $\mathrm{diag}(1, 1, 1, e^{0.14\pi i})$ for $N = 2$, $k = 42$ with precision $\epsilon = 0.99$.

THANK YOU FOR YOUR ATTENTION!