Exact multiparticle amplitudes from Landau method

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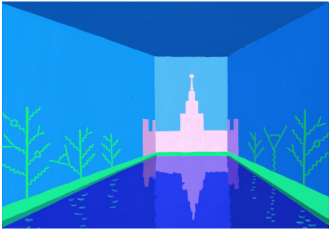


XXV-th International Workshop-School
on High Energy Physics and Quantum Field Theory

arXiv: 2307.11163, 2212.03268, 2111.04760 *& work in progress*

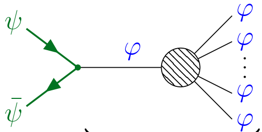
Difficulty with multiparticle production in QFT, $d = (3+1)$

$\varphi(t,\, \boldsymbol{x})$ $$S_{\varphi^4} = \frac{1}{2}\int dt\, d^3\boldsymbol{x}\left[(\partial_\mu\varphi)^2 - \varphi^2 - \frac{\lambda}{2}\varphi^4\right]$$ $(m=1)$

$\lambda \ll 1$ weak coupling

Production of $n \gg 1$ particles at threshold:

$\mathcal{A}_{1\to n} \sim$



important part

$= \langle n,\, \mathbf{p}=0\,|\hat{\mathcal{S}}\,\hat{\varphi}(0)\,|0\rangle$

Brown '92; Voloshin'92

Libanov et al '94

$$= \underbrace{n!\left(\frac{\lambda}{8}\right)^{\frac{n-1}{2}}}_{\mathcal{A}_{\text{tree}}}\Big[\underbrace{1}_{\text{tree}} + \underbrace{\lambda B(n^2 - 4n + 3)}_{\text{1 loop} \sim \lambda n^2} + \underbrace{\lambda^2 \frac{B^2}{2}(n^4 + \dots)}_{\text{2 loops} \sim (\lambda n^2)^2} + \dots\Big]$$

$\Rightarrow$ Blow-up at $n \gg \lambda^{-1/2}$!

$B = \frac{3^{3/2}}{(8\pi)^2}\left[i\pi - \ln(4 + 7\sqrt{3})\right]$

Description of multi–particle production?

Difficulty with $1 \to n$ transitions in QM, $d = (0+1)$

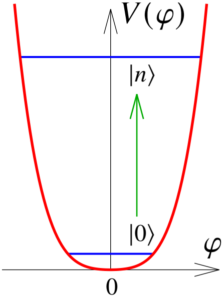
$\varphi(t)$ $$S_{\varphi^4} = \frac{1}{2}\int dt \left[(\partial_t\varphi)^2 - \varphi^2 - \frac{\lambda(t)}{2}\varphi^4\right]$$

quartic oscillator!

$\lambda_0 \ll 1$

weak coupling

- IR regularization: $\lambda(t) = \lambda_0\, \mathrm{e}^{-2\epsilon t}$ $\epsilon \ll m$



High-multiplicity transition:

$$\mathcal{A}_{1\to n} = \langle n|\hat{\mathcal{S}}\,\hat{\varphi}(0)\,|0\rangle \underset{\uparrow}{=} \mathrm{e}^{-i\int_0^\infty dt(E_n - E_0 - n)} \langle n|\hat{\varphi}\,|0\rangle\Big|_{t=0}$$

adiabatics at $\epsilon \to 0$!

Jaekel, Schenk, 2018

$$E_n - E_0 - n = \frac{\lambda}{8}(3n^2 + \dots) - \frac{\lambda^2}{64}(17n^3 + \dots) + \dots$$

$$\langle n|\hat{\varphi}|0\rangle = \mathcal{A}_{\mathrm{tree}}\left[1 - \frac{\lambda}{32}(17n^2 + 5n - 12) + \frac{\lambda^2}{2048}(289n^4 + \dots) + \dots\right]$$

The same blow-up at $n \gg 1/\sqrt{\lambda}$!

Exponentiation of high-multiplicity amplitudes

Conjecture *Libanov et al '94*

$$\mathcal{A}_{1\to n} = \mathcal{A}_{\text{tree}} \exp\left[\frac{1}{\lambda}F_0(\lambda n) + F_1(\lambda n) + \lambda F_2(\lambda n) + \dots\right]$$

resummation of all loops!

Explicitly resumed in: *Libanov et al '94; Jaeckel, Schenk '18*

- QFT $(3+1)$: $F_0 = B(\lambda n)^2 + \mathcal{O}(\lambda n)^3$
- QM $(0+1)$: $F_0 = -(\frac{3i}{16\epsilon} + \frac{17}{32})(\lambda_0 n)^2 + (\frac{17i}{256\epsilon} + \frac{125}{256})(\lambda_0 n)^3 + \mathcal{O}(\lambda_0 n)^4$

$$F_1 = -(\frac{3i}{16\epsilon} + \frac{5}{32})\,\lambda_0 n + (\frac{51i}{512\epsilon} + \frac{99}{512})\,(\lambda_0 n)^2 + \mathcal{O}(\lambda_0 n)^3$$

$$F_2 = \frac{3}{8} + \mathcal{O}(\lambda_0 n)$$

We need a method to compute $F_l(\lambda n)$

- as power series in λn
  but no loop resummation!
- in the double-scaling limit $\lambda \to 0$, $\lambda n = \mathcal{O}(1)$
  nonperturbative

Results-I: correspondence

To compute $\mathcal{A}_{1\to n}$, **consider the same theory, but**

- with a source $S_j = \frac{1}{2}\int d^d x \left[(\partial_\mu\varphi)^2 - \varphi^2 - \frac{\lambda(t)}{2}\varphi^4\right] + ij\varphi(0)$

$$\lambda(t) \stackrel{\uparrow}{=} \lambda_0\, \mathrm{e}^{-2\epsilon t}, \; \epsilon \to 0$$

- on a singular background $\varphi(x) = \underbrace{\varphi_B(t - t_*)}_{\text{classical \& singular at } \boxed{t = t_*}} + \delta\hat{\varphi}(t, \boldsymbol{x})$

Brown solution:
$$\varphi_B = \frac{i\sqrt{2}}{\sqrt{\lambda}\sin(t - t_* + i\epsilon t)}$$

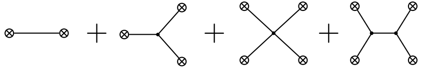
Then a perturbatively exact relation is valid:

$$\mathcal{A}_{1\to n} = -i\mathcal{A}_{\text{tree}} \cdot \sqrt{\frac{\lambda_0}{2}} \underbrace{\lim_{\phi_0\to\infty} \phi_0^2}_{\text{limit}} \underbrace{\int \frac{dj\,dt_*}{2\pi}\, \mathrm{e}^{j\phi_0 + it_* n}}_{\text{Laplace transform}} \underbrace{\mathcal{A}^{(j,\,\varphi_B)}_{\text{vac}\to\text{vac}}}_{\text{with source}}$$

Results-II: series for $F_l(\lambda n)$

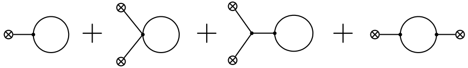
Our formula gives: $$\mathcal{A}_{1\to n} = \mathcal{A}_{\text{tree}} \cdot \exp\Big[\underbrace{\frac{1}{\lambda}F_0 + F_1 + \lambda F_2 + \dots}_{\text{connected graphs}}\Big]$$

$$F_0 = \underbrace{\Delta F_0}_{\text{extra int-ls}} + \dots = \text{tree graphs}$$



$\mathcal{O}(\lambda n)^2$, $\mathcal{O}(\lambda n)^3$, $\mathcal{O}(\lambda n)^4$ $\quad (\otimes = j)$

$$F_1 = \underbrace{\Delta F_1}_{\text{extra int-ls}} + \dots = \text{1-loop, etc}$$



$\mathcal{O}(\lambda n)$, $\mathcal{O}(\lambda n)^2$

Series for $F_l(\lambda n)$: l — number of loops

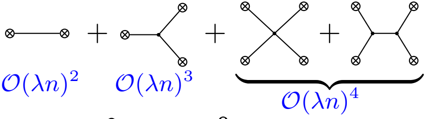
- extra integrals via saddle-point method: $j_s \sim \lambda n/\phi_0^2$ $\quad t_{*,s} \sim \phi_0^{-1}$
- $\phi_0 \to \infty \Rightarrow j \to 0$, singular φ_B $(t_* \to 0)$ *cf. Son '95*
- ϕ_0 cancels in diagrams, λn remains

  $\Rightarrow$ **expansion in j produces series for $F_l(\lambda n)$**

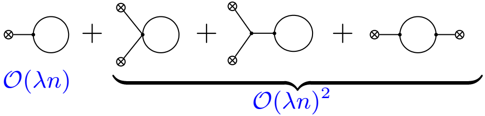
Results-III: double-scaling limit

Our formula gives:

$$\mathcal{A}_{1\to n} = \mathcal{A}_{\text{tree}} \cdot \exp\Big[\underbrace{\frac{1}{\lambda}F_0 + F_1 + \lambda F_2 + \dots}_{\text{connected graphs}}\Big]$$

$F_0 = \underbrace{\Delta F_0}_{\text{extra int-ls}} +$  $+ \dots =$ tree graphs

$(\otimes = j)$

$F_1 = \underbrace{\Delta F_1}_{\text{extra int-ls}} +$  $+ \dots =$ 1-loop, etc

Nonperturbative limit $\lambda \to 0$, $\lambda n = \mathcal{O}(1)$

- $F_0(\lambda n) = \sum$ tree graphs $= \lim\limits_{j\to 0} F_0[\varphi_{\text{cl}}]$ *Rubakov et al '92, Son '95*

  $\varphi_{\text{cl}}(t, \boldsymbol{x}) \in \mathbb{C}$ — classical solution at $j \neq 0$ $\leftarrow$ D.T. Son's method
- $F_1(\lambda n) = \sum$ one-loop graphs $= \lim\limits_{j\to 0} \ln\det[\text{fluctuations around } \varphi_{\text{cl}}]$

$\Rightarrow$ **Semiclassical methods work at finite λn, $j \neq 0$**

Derivation I: Exact Landau method for φ^4 oscillator in QM

$$U = \frac{1}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4$$

see B. Farkhtdinov's talk

$\lambda \propto \hbar$ is the semiclassical parameter: $\varphi \to \varphi/\sqrt{\lambda} \Rightarrow S_{\varphi^4} \to S_{\varphi^4}/\lambda$

Landau method: $\langle n|\hat{\varphi}|0\rangle = \int d\varphi\, \varphi\, \Psi_n(\varphi)\, \Psi_0(\varphi)$ at large n!

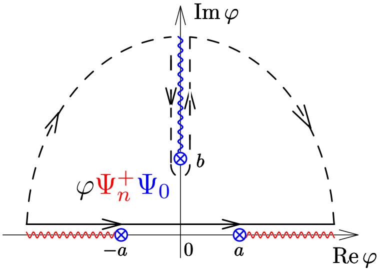
- $\Psi_n(\varphi) = \underbrace{\Psi_n^+(\varphi)}_{p>0} + \underbrace{\Psi_n^-(\varphi)}_{p<0}$ — to any order in λ!

  Semiclassically: $\Psi^\pm = \mathrm{e}^{\pm i S_\pm(\varphi)/\lambda}$, $S_\pm = S_\pm^{(0)} + \lambda S_\pm^{(1)} + \dots$
- Asymptotics: $\Psi_n^+ \Psi_0 \to \text{const}/\varphi^2$, $\varphi \to \infty$
- Deform the contour:

$$\langle n|\hat{\varphi}|0\rangle = \sum_\pm \int_{\mathcal{C}_\infty} d\varphi\, \varphi\, \Psi_n^\pm\, \Psi_0$$
$$= \text{residual at infinity}$$

$$\langle n|\hat{\varphi}|0\rangle = 2\pi \sum_\pm \lim_{\varphi=\pm i\infty} \varphi^2\, \Psi_n^\pm\, \Psi_0$$

perturbatively exact!



Derivation II: Landau method for amplitudes

- **Separate** $\varphi(\boldsymbol{x} = t = 0)$:

$$\mathcal{A}_{1\to n} \equiv \langle n|\hat{\mathcal{S}}\,\hat{\varphi}(0)\,|0\rangle = \int \mathcal{D}\varphi(x)\;\overbrace{\varphi(0)}^{1 = \int d\phi_0\,\delta(\phi_0 - \varphi(0)) = \int \frac{d\phi_0\,dj}{2\pi i}\,\mathrm{e}^{j(\phi_0 - \varphi(0))}}\;\mathrm{e}^{iS[\varphi]}\,\langle n|\varphi(t_f)\rangle$$

$$= N \underbrace{\int d\phi_0 \phi_0}_{\lim\limits_{\phi_0\to\infty} \phi_0^2} \underbrace{\int \mathcal{D}\varphi(x)\; dj\; \mathrm{e}^{j\phi_0 - j\phi(0) + iS[\varphi]}\,\langle n|\varphi(t_f)\rangle}_{\Psi_n^*(\phi_0)\Psi_0(\phi_0)\Big|_{t=0}}$$

- **Simplify final state**: $|n\rangle = (\hat{a}^\dagger_{\boldsymbol{p}=0})^n|0\rangle = N\int dt_*\;\mathrm{e}^{it_* n}\,\underbrace{\exp(\mathrm{e}^{-it_*}\hat{a}^\dagger)|0\rangle}_{\text{coherent}}$

Coherent state $\equiv$ new BCs for $\varphi(x)$

OR: $\varphi(x) = \underbrace{\varphi_B(t - t_*)}_{\text{satisfies BCs}} + \underbrace{\delta\varphi(x)}_{\text{vacuum BCs}}$

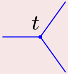
Derivation III: The relation

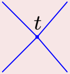
$$\mathcal{A}_{1\to n} = N \underbrace{\lim_{\phi_0\to\infty} \phi_0^2}_{\text{limit}} \underbrace{\int dj dt_* \, \mathrm{e}^{j\phi_0 + it_* n}}_{\text{Laplace transform}} \underbrace{\int \mathcal{D}\delta\varphi(x) \, \mathrm{e}^{j\varphi(0) + i\tilde{S}[\delta\varphi;\varphi_B]}}_{\mathcal{A}^{(j,\varphi_B)}_{\text{vac}\to\text{vac}}}$$

$\varphi = \varphi_B + \delta\varphi$, $\qquad \varphi_B(t - t_*) = \frac{i\sqrt{2}}{\sqrt{\lambda}\sin(t - t_* + i\epsilon t)}$ — Brown solution

Check for $\lambda\varphi^4$ oscillator, $d = (0+1)$

Quantize the theory: $\varphi(t) = \varphi_B + \delta\varphi$

$= G_B(t,\, t')$,


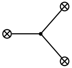
$= -6i\int dt\, \varphi_B \sqrt{\lambda}$,


$= -6i\int dt\, \lambda$,

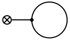
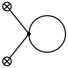
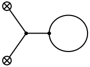
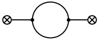

$= -j\Big|_{t=0}$

$$j_s\sqrt{\lambda_0} = -\frac{\lambda_0 n\sqrt{2}}{\phi_0^2\lambda_0} + \mathcal{O}(\lambda_0 n)^2, \qquad t_{*,s} = -\frac{i\sqrt{2}}{\phi_0\sqrt{\lambda_0}} + \mathcal{O}(\lambda_0 n)$$

$$\boxed{\mathcal{A}_{1\to n} = \mathcal{A}_{\rm tree}\cdot \mathrm{e}^{\frac{1}{\lambda}F_0(\lambda n) + F_1(\lambda n) + \lambda F_2(\lambda n) + \ldots}}$$

$F_0 = it_*\lambda_0 n + j\phi_0\lambda_0 +$  $+$  $+ \mathcal{O}(\lambda_0 n)^4$

$$= -\left(\frac{3i}{16\epsilon} + \frac{17}{32}\right)(\lambda_0 n)^2 + \left(\frac{17i}{256\epsilon} + \frac{125}{256}\right)(\lambda_0 n)^3 + \mathcal{O}(\lambda_0 n)^4$$

$F_1 = (j,\, t_*$ prefactors$)$ $+$  $+$  $+$  $+$  $+ \mathcal{O}(\lambda_0 n)^3$

$$= -\left(\frac{3i}{16\epsilon} + \frac{5}{32}\right)\lambda_0 n + \left(\frac{51i}{512\epsilon} + \frac{99}{512}\right)(\lambda_0 n)^2 + \mathcal{O}(\lambda_0 n)^3$$

$F_2 = (j,\, t_*$ contributions$) + O(\lambda_0 n) = \frac{3}{8} + O(\lambda_0 n)$

Coincide with explicit resummation! *(Jaeckel, Schenk '18)*

Semiclassical solution in $\lambda\varphi^4$ QFT, $d = (3+1)$

$\mathcal{A}_{1\to n} \sim \mathcal{A}_{\text{tree}} \cdot \mathrm{e}^{F_0/\lambda}$

$$F_0(\lambda n) = \sum \text{tree graphs}^j = \lim_{j\to 0} F_0^{(j)}[\varphi_{\text{cl}}]$$

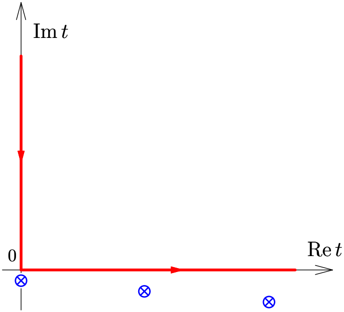
Rubakov–Son-Tinyakov conjecture *(Rubakov et al '92)*

Semiclassical equations: $\varphi = \varphi_{\text{cl}}(t,\, \boldsymbol{x})$

$$\begin{cases} \Box\varphi_{\text{cl}} + \varphi_{\text{cl}} + \lambda\varphi_{\text{cl}}^3 = ij\delta^{(4)}(x) \\ \varphi_{\text{cl}} \to 0 \text{ as } t \to +i\infty \leftarrow \text{in: vacuum BC} \\ \varphi_{\text{cl}} \to \varphi_B + \text{negative/freq. as } t \to +\infty \end{cases}$$

— out-state: φ_B + vacuum

Son's method of singular solutions *(Son '95)*

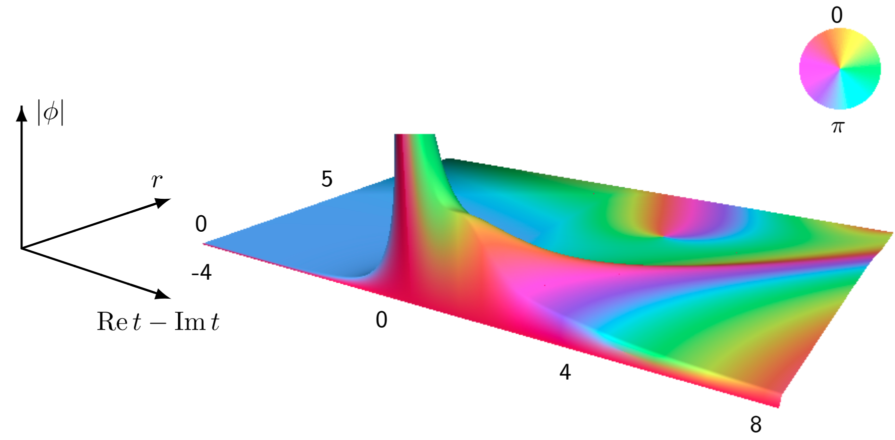


$$F_0(\lambda n) + \lambda \ln \mathcal{A}_{\text{tree}} = i\lambda \lim_{j\to 0} \left\{ n\, t_* + S_{\varphi^4}[\varphi_{\text{cl}}] + \text{boundary terms} \right\}$$

solutions become singular as $j \to 0$!

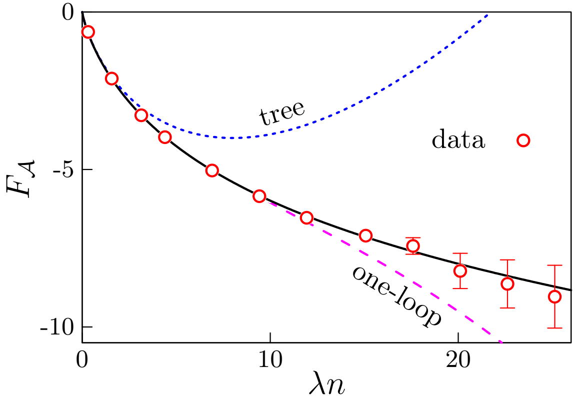
Semiclassical solution: an example

Demidov, Farkhtdinov, DL '23



Numerical results.

$$|\mathcal{A}_{1\to n}|^2 \sim n!\, e^{2F_{\mathcal{A}}(\lambda n)/\lambda}\,, \quad F_{\mathcal{A}} \equiv \mathrm{Re}\, F_0(\lambda n) + \frac{1}{2}\lambda n \ln(\lambda n/8e)$$



circles: nonperturbative result!

Conclusions and future plans

- **Correspondence for $\lambda\varphi^4$ theory:**

$$\mathcal{A}_{1\to n} \leftrightarrow \mathcal{A}_{\text{vac}\to\text{vac}}^{(j,\ \text{singular background})}$$

- **Correspondence describes the double-scaling limit**

$$\lambda \to 0, \qquad \lambda n = \text{const}$$

- We verified it in QM: $d = (0+1)$
- ... and applied to $(3+1)$–dimensional QFT

- Comparison to Feynman diagrams in field theory?
- Application to broken φ^4 theory? ("Higgs" model)
  ... check the "Higgsplosion" scenario! *cf. Khoze, Spannowsky '18*

Thank you!

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