On the analytical properties of multiparticle amplitudes

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Contents

Multiparticle amplitudes are necessary for:

direct description of processes with a large multiplicity (whose role increases with the energy of colliding particles)

calculating amplitudes with a smaller number of particles using unitarity relations.

In both cases, their analytical properties are important.

Without being distracted by the spin structure, we shall call the amplitude of any process a function of kinematic invariants lying in the complex plane whose boundary values for physical values of these invariants coincide with the amplitudes of physical processes in different reaction channels.

The analytical properties of elastic scattering amplitudes have been well known since the middle of the last century, when a consistent theory of strong interactions did not exist, and the main tools for their description were dispersion relations and the Regge-Gribov theory of complex moments J

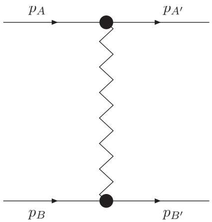
Gribov, V. N., The theory of complex angular momenta: Gribov lectures on theoretical physics, 2003
Cambridge Monographs on Mathematical Physics Cambridge University Press.

In this theory, the asymptotics of the scattering amplitude $A(s,t)$ for $s \to \infty$ and a fixed t is determined by the position of the poles (called Reggeons) in the j-plane of the partial wave $A_l(t)$ analytically continued to complex j. The analytic properties of the scattering amplitudes allow continuation from either even or odd l, so reggeons have an additional quantum number compared to particles, the signature. The contribution of a reggeon with trajectory $\alpha \equiv \alpha(t)$ and signature σ to the amplitude of the process $AB \to A'B'$ is given by the expression

$$\mathcal{A}_{AB}^{A'B'} = \Gamma_{AA'}(t)(s)^{\alpha}\xi_{\alpha}\Gamma_{BB'}(t), \qquad (1)$$

where $\Gamma_{AA'}(t)$ and $\Gamma_{AA'}(t)$ are the vertices of the reggeon-particle interaction,

$s=(p_A+p_B)^2, \quad t=(p_A-p'_A)^2$, $\xi_\alpha = \frac{e^{-i\pi\alpha}+\sigma}{\sin\pi\alpha}$ – signature factor and is presented by the picture



The important thing is that the vertices of the reggeon-particle interaction are real in the region of physical momentum transfer t, so that the analytical properties of the amplitudes are ehibited explicitly in the expression (1).
The situation with the amplitudes of many-particle processes is not so good, although the increasing role of multiple production processes in strong interactions with increasing energy was recognized already in those days, see, for example,

K. A. Ter-Martirosyan, Asymptotic behaviour of essentially inelastic cross sections, Nucl.Phys, Vol. 68 (1965) p. 591-608. and investigation of their analytical properties was started already in the sixties of the last century. Since there was no acceptable field theory of strong interactions before the advent of QCD, the analytic properties of amplitudes were studied in axiomatic field theory.

These properties are important for construction of Regge theory of multiparticle amplitudes.

I must say that I am not an expert in the analytical properties of multiparticle amplitudes and they interest me mainly in connection with the BFKL equation.

V,S.F. E. A. Kuraev, L. N. Lipatov, On the Pomeranchuk Singularity in Asymptotically Free Theories, 1975. Phys. Lett. B, Vol. 60, p. 50-52

I. I. Balitsky, L. N. Lipatov, The Pomeranchuk Singularity in Quantum Chromodynamics, 1978, Sov. J. Nucl. Phys. , Vol. 28, p. 822-829

This equation was derived using the unitarity relation.

In the unitarity relations, multiple production amplitudes in the multi-Regge kinematics (MRK) were taken into account.

MRK is the kinematics where all particles have limited transverse momenta (with respect to momenta of colliding particle) and are combined into jets with limited invariant mass of each jet and large (increasing with s) invariant mass of any pair of jets.

The muli-Regge form was used for these amplitudes.

It seems that one of the analytical properties of the multiple amplitudes used in construction of this form is not correct.

To create a Regge theory of multiparticle processes, knowledge of the analytical properties of many-particle amplitudes was required. In the absence of any reliably established properties of these amplitudes, various models were used: the ladder model

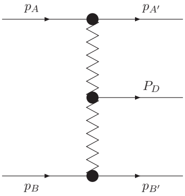
I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Signature in production amplitudes, Phys. Lett. B, 28:676–678, 1969,

the hybrid Gribov model,

I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, The two-reggeon/particle coupling, Nucl. Phys. B, 11:383–405, 1969,

I. T. Drummond, Multi-reggeon behavior of production amplitudes, Phys. Rev., 176:2003–2013, 1968.,

the dual resonance model
J. H. Weis, Factorization of multi-Regge amplitudes, Phys. Rev. D, 4:1777–1787, 1971,
Carleton E. DeTar and J. H. Weis, Analytic structure of the triple-regge vertex, Phys. Rev. D, 4:3141–3161, 1971.
It was recognized that the reggeon-reggeon-particle vertex $V^{D}_{R_1 R_2}(q_1, q_2)$ in the direct generalization of the regge pole contribution to the elastic amplitude (1) for the case of the process $AB \to A' DB'$



$$\mathcal{A}_{AB}^{A'DB'} = \Gamma_{AA'}(t_1) s_1^{\alpha_1} \xi_{\alpha_1} V^D_{R_1 R_2}(q_1, q_2) s_2^{\alpha_2} \xi_{\alpha_2} \Gamma_{BB'}(t_2), \qquad (2)$$

in the Multi-Regge kinematics

$$s \gg s_i \gg |t_i|, \;\; i = 1, 2, \;\; s = (p_A + p_B)^2, \;\; s_1 = (p'_A + p_D)^2,$$

$$s_2 = (p'_B + p_D)^2, \;\; t_1 = (p_A - p'_A)^2, t_2 = (p_B - p'_B)^2) \qquad (3)$$

has a complicated analytical structure. Finally, Regge theory for multiparticle amplitudes was built on the system of postulates. R. C. Brower, Carleton E. DeTar, and J. H. Weis, Regge Theory for Multiparticle Amplitudes, Phys. Rept., 14:257, 1974.
One of the most important postulated properties of many-particle amplitudes is the absence of simultaneous discontinuities in the squares of the invariant masses of overlapping channels
(Recall that two channels are said to overlap when they have common particles but are not subchannels of each other).

This property allows one to write a multi-Regge representation for many-particle amplitudes in a form that explicitly shows all their analytical properties. Contribution of Reggeons with trajectories $\alpha_i(t)$ and signatures τ_i to the amplitude of the process $AB \to A'CB'$
is presented in the form

$$\mathcal{A}_{AB}^{A'DB'} = \Gamma_{AA'}(t_1)\left[s^{\alpha_2}\xi_{\alpha_2}s_1^{\alpha_1-\alpha_2}\xi_{\alpha_1\alpha_2}\right]V_R(t_1,t_2,\kappa)$$

$$\times\left[s^{\alpha_1}\xi_{\alpha_1}s_2^{\alpha_2-\alpha_1}\xi_{\alpha_2\alpha_1}\right]V_L(t_1,t_2,\kappa)\Big]\Gamma_{BB'}(t_2), \qquad (4)$$

where $\kappa = \frac{s_1 s_2}{s}$, $\xi_{\alpha_1\alpha_2} = \frac{e^{-i\pi(\alpha_1-\alpha_2)}+\tau_1\tau_2}{\sin(\pi(\alpha_1-\alpha_2))}$.
The advantage of this representation is that for $t_i < 0$ the functions V_L and V_R are real, which allows us to uniquely separate the amplitude into real and imaginary parts. Similar representations exist for the production of a larger number of particles (with the number of vertices increasing with the number of particles).

These representations, which arose before the advent of QCD, currently considered as the theory of strong interactions, are still used today. As already mentioned, they are based on the hypothesis of the absence of simultaneous discontinuities in overlapping channels. To justify this hypothesis, reference is made usually to the Steinmann relations
O. Steinmann, Uber den Zusammenhang zwischen den Wightmanfunktionen und den retardierten Kommutatoren, 1960, Helv. Phys. Acta , Vol. 33, p. 267-298;
Wightmanfunktionen und den retardierten Kommutatoren, 1960, Helv. Phys. Acta , Vol. 33, p. 347-362.
The justification can be done with a big reservation, since the Steinmann relations refer to the vacuum average value of retarded commutators, not for S-matrix elements. Moreover, the hypothesis imposes much stronger restrictions on the amplitudes than the Steinmann relations.

In particular, these relations deal with discontinuities in the physical region of one of the reaction channels, whereas the hypothesis extends to the non-physical region as well. In the review

R. C. Brower, Carleton E. DeTar, and J. H. Weis, Regge Theory for Multiparticle Amplitudes, Phys. Rept., 14:257, 1974.

it was also noted that the original formulation of the Steinmann relations is not covariant, it uses energies, not energy invariants. This remark is not essential, since a covariant generalization of the relations was carried out in

Kevin E. Cahill and Henry P. Stapp, Optical Theorems and Steinmann Relations, Annals Phys., 90:438, 1975.

Thus, the Regge theory of multiparticle amplitudes, which arose before the advent of QCD, was based on the hypothesis of the absence of simultaneous discontinuities in the energy invariants of overlapping channels.
But it is easy to see that this hypothesis does not agree with the factorization of infrared singularities.
The factorization of infrared singularities is well known and most simply seen in quantum electrodynamics (QED). In [Yennie:1961ad]
D. R. Yennie, Steven C. Frautschi, and H. Suura, The infrared divergence phenomena and high-energy processes, Annals Phys., 13:379–452, 1961
closed-form formulas were obtained for infrared singular contributions from both virtual corrections in scattering amplitudes and from the emission of real photons into cross sections.

According to [Yennie:1961ad], the amplitudes of processes with an arbitrary number of particles with momenta p_i (all momenta are considered incoming) are represented as

$$A(\{p_i\}) = \exp\left\{ -\sum_{i<j} Q_i Q_j V\left(p_i, p_j\right) \right\} A_{ns}(\{p_i\}) \,,$$

$$V\left(p_i, p_j\right) = -\frac{e^2}{2}\int \frac{d^4k}{i(2\pi)^4}\frac{1}{k^2-\lambda^2+i0}\left(\frac{2p_i-k}{k^2-2(kp_i)+i0}+\frac{2p_j+k}{k^2+2(kp_j)+i0}\right)^2 , \tag{5}$$

where $Q_i = 1$ for an electron (positron) in the initial (final) state and $Q_i = -1$ for an electron (positron) in the final (initial) state, λ –introduced to regularize the infrared divergence of the "photon mass", and the amplitude $A_{ns}(\{p_i\})$ is finite at $\lambda \to 0$.

The expression for $V\left(p_i, p_j\right)$ integrated over d^4k is well known. Its infrared singular part is quite simple, especially in the high-energy region of interest to us:

$$V_{sing}\left(p_i, p_j\right) \simeq \frac{\alpha}{2\pi}\left(\ln\left(\frac{-s_{ij}}{m^2}\right) - 1\right)\ln\left(\frac{m^2}{\lambda^2}\right), \qquad (6)$$

where $s_{ij} = (p_i + p_j)^2$. Since the exponent in the infrared singular factor contains the sum $\sum_{i<j} \ln(-s_{ij})$ over all channels, then when expanding the exponent we obtain products of powers of $\ln(-s_{ij})$ over all channels, including overlapping ones, i.e. terms that violate the prohibition on the existence of simultaneous discontinuities in overlapping channels.

In quantum chromodynamics (QCD), factorization is complicated by the non-Abelian nature of the theory, which leads to both additional singularities and to a matrix structure of the emission vertices. The masslessness of gluons, which have a colour charge and therefore emit, leads to collinear (or mass) singularities arising from divergences in integrals over the angles between the momenta of the emitting and emitted particles at zero emission angles. Following the generally accepted terminology, we will use the term ”infrared divergences” not only for the divergences in frequencies but also for the divergences in radiation angles, as is usually done for brevity. By now, many papers have been published in which these singularities have been studied in QCD amplitudes. The standard set of references on this topic includes the papers

[Catani1998] Stefano Catani, The Singular behavior of QCD amplitudes at two loop order, Phys. Lett. B, 427:161–171, 1998,
[Sterman2003] George F. Sterman and Maria E. Tejeda-Yeomans, Multiloop amplitudes and resummation, Phys. Lett. B, 552:48–56, 2003,
[Dixon2008] Lance J. Dixon, Lorenzo Magnea, and George F. Sterman, Universal structure of subleading infrared poles in gauge theory amplitudes, JHEP, 08:022, 2008,
[Becher2009] Thomas Becher and Matthias Neubert, On the Structure of Infrared Singularities of Gauge-Theory Amplitudes, JHEP, 06:081, 2009, Erratum: JHEP 11, 024 (2013),
[Gardi2009] Einan Gardi and Lorenzo Magnea, Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes, JHEP, 03:079, 2009.

As usual in QCD, the analysis is carried out with the regularization of divergences (both ultraviolet and infrared) by the space-time dimension $D = 4 + 2\epsilon$. At present, factorization formulas representing amplitudes as a product $\mathcal{Z}\,\mathcal{H}$ are considered well established, where $\mathcal{H}$ is the so-called hard amplitude, which has no singularities in ϵ, and all singularities are contained in the factor $\mathcal{Z}$. This representation is valid in both QCD and QED, for both massless and massive electrons. In the latter case, it is equivalent to the representation of [Yennie:1961ad] with the regularization of infrared divergences by the photon mass λ. But unlike QED, in QCD the factors $\mathcal{Z}$ and $\mathcal{H}$ have a matrix structure.

For amplitudes with total number n of participating partons the factorization is written as

$$\mathcal{M}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right)\mathcal{H}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) . \quad (7)$$

Here $\mathcal{H}$ is a colour vector, which is finite as $\epsilon \to 0$, and represents a matching condition, to be determined order by order in perturbation theory after the subtraction of divergent contributions. The infrared operator $\mathcal{Z}_n$, on the other hand, is an $r \times r$ matrix in colour space, generating all infrared and collinear singularities of the amplitude; it satisfies a (matrix) renormalization group equation, whose general solution can be written in the form

$$\mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \mathcal{P}\exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right)\right] , \quad (8)$$

where $\Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right)$ is the soft anomalous dimension matrix and $\mathcal{P}$ denotes path ordering in colour space. All poles in ϵ are generated through the integration of the d-dimensional running coupling down to vanishing scale, $\lambda \to 0$.

For massless particles, up to two loops, the n-parton soft anomalous dimension matrix has a remarkably simple form, proportional to the one-loop result, regardless of the number of partons involved. The "dipole formula" has the form

$$\Gamma_n^{\rm dip}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s(\lambda^2)\right)\sum_{(i,j)} \ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{n}\gamma_i\left(\alpha_s(\lambda^2)\right). \quad (9)$$

The basic feature of this representation is that the colour structure remains the same as at leading order.

All dependence on on the coupling constant is contained in the colourless anomalous dimensions

$$\hat{\gamma}_K = \frac{\gamma_K^{[i]}}{\mathcal{C}_{[i]}}, \tag{10}$$

$\gamma_K^{[i]}$ is the cusp anomalous dimension in representation $[i]$, $\mathcal{C}_{[i]}$ is the corresponding quadratic Casimir eigenvalue.
The dipole formula is exact at least up to two loops.
The functions γ_i are collinear anomalous dimensions which can be extracted from form factor data. Note that in higher orders the "dipole formula" for Γ_n must be supplemented by contributions of higher "multipoles". These are contributions containing sums over combinations of k particles with $k > 2$. The quadrupole correction first appearing in three loops was calculated relatively recently.

The common feature of these formulas is the representation of $\mathcal{Z}$ as an exponential with an index containing $\sum_{i<j}\ln(-(p_i+p_j)^2)$ (all momenta are considered incoming) over emitting particles i, j. Such a representation contradicts the absence of in overlapping channels. Since the exponent in the infrared singular factor $\mathcal{Z}$ contains the sum $\sum_{i<j}\ln(-(p_i+p_j)^2)$ over all channels, then when expanding the exponential we obtain products of powers of $\ln(-(p_i+p_j)^2$ over all channels, including overlapping ones, i.e. terms having discontinuities in overlapping channels, which contradicts the hypothesis of the absence of such discontinuities. Since they appear in physical region, it contradicts also the Steinmann relations in their usual interpretation.

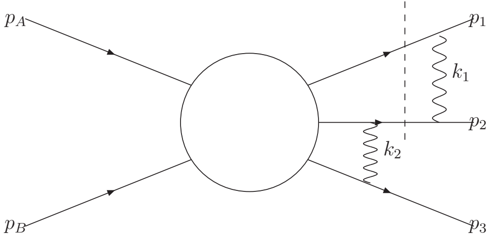
It's necessary to say here that the possibility of violation of the Steinmann relations in theories with massless particles was indicated by Steinmann himself.
In particular, he wrote in
O. Steinmann, The Infrared Problem in Electron Scattering, Acta Phys. Austriaca Suppl. **11** (1973), 167-198
"The axiomatic way of defining S also does not work. In the known proofs of asymptotic conditions it is assumed that the particles under consideration belong to isolated one-particle hyperboloids in the energy-momentum spectrum of the relevant superselection sector. This is not the case for electrons."
Nevertheless, these relations are used in quantum chromodynamics, and in supersymmetric theories, and even in cosmology:
P. Benincasa, A. J. McLeod and C. Vergu, Steinmann Relations and the Wavefunction of the Universe, Phys. Rev. D **102** (2020), 125004.

The existence of simultaneous discontinuities in the energy invariants of overlapping channels can be verified both in the presence of infrared singularities and in their absence by considering the two-loop radiative correction in QED to the process with three charged particles in the final state (as, for example, in the Bethe-Heitler process). We will consider only diagrams with insertions of photon vertices into the external lines of the Born approximation diagrams, neglecting the photon momenta in the internal lines. Note that according to the Landau criterion
[Landau:1959fi] L. D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys., 13(1):181–192, 1959.
the singularities of these diagrams are contained among the singularities of the total amplitude, since the latter include the singularities of diagrams in which some of the lines are missing, i.e. the vertices they connect merge.

Let us denote the momenta of the final particles as $p_1,\ p_2,\ p_3$. For simplicity, we will assume that the masses of the particles are equal. Let

$$s_{ij} = (p_i + p_j)^2, \quad S = (p_1 + p_2 + p_3)^2 = s_{12} + s_{13} + s_{23} - 3m^2. \quad (11)$$

Note that in the physical region (i.e. in the region where the momenta of all particles have physical meaning) the sign of s_{ij} coincides with the sign of $(p_i p_j)$, so the sign of the product $s_{12}s_{13}s_{23}$ is always positive. In this case, according to the accepted terminology, any two of these channels are overlapping. One of these channels and the s-channel are non-overlapping, since it contains all 3 particles.
Let us investigate the presence of simultaneous discontinuities in the channels s_{12} and s_{23}. Such discontinuities can only be given by diagrams in which one of the photons connects the lines of particles with momenta p_1 and p_2, and the other with momenta p_3 and p_2.



If the vertex of the first photon on the line of a particle with momentum p_2 is closer to the external end of this line, then the contribution of such a diagram is proportional to the Born amplitude by a factor of

$$I \equiv I(s_{12}, s_{23}, s_{13}) = \int \frac{d^4 k_1}{(2\pi)^4 i} \frac{1}{d_{10} d_{11} d_{12}} J_3(\tilde{s}_{23}; \tilde{m}^2), \qquad (12)$$

where

$$\tilde{s}_{23} = (p_2 + p_3 - k_1)^2, \ \tilde{m}^2 = (p_2 - k_1)^2;$$

$$J_3(\tilde{s}_{23}; \tilde{m}^2, m^2) = \int \frac{d^4 k_2}{(2\pi)^4 i} \frac{1}{d_{20} d_{21} d_{22}}, \qquad (13)$$

$$d_{10} = (k_1^2 - \lambda^2 + i0), \ \ d_{11} = ((k_1 + p_1)^2 - m^2 + i0),$$
$$d_{12} = ((k_1 - p_2)^2 - m^2 + i0), \ \ d_{20} = (k_2^2 - \lambda^2 + i0),$$
$$d_{21} = ((k_2 + p_3)^2 - m^2 + i0), \ \ d_{22} = ((k_1 + k_1 - p_2)^2 - m^2 + i0). \qquad (14)$$

Let us consider the discontinuities of I with respect to the invariant s_{12}. There are only two such discontinuities: a two-particle one, which occurs due to the simultaneous vanishing of d_{11} and d_{12}, and a three-particle discontinuity, which occurs due to the simultaneous vanishing of d_{11}, d_{22}, and d_{20}. The second of them has no singularities, since the vanishing of any of the remaining denominators would contradict the law of conservation of energy-momentum. For a

$$\Delta_{s_{12}} I = \int \frac{d^4k}{(2\pi)^4 i} \frac{(2\pi i)^2 \delta((p_1+k)^2 - m^2 \delta((p_2-k)^2 - m^2))}{k^2 - \lambda^2}$$

$$\times J_3(\tilde{s}_{23}; \tilde{m}^2). \quad (15)$$

We will use the Sudakov parametrization (light-cone variables). Introducing the light-cone vectors l_1 and l_2 such that

$$p_1 = l_1 + \frac{m^2}{\tilde{s}} l_2, \;\; p_2 = l_2 + \frac{m^2}{\tilde{s}} l_1, \;\; l_1^2 = 0, \;\; l_2^2 = 0, \;\; (l_1 + l_2)^2 = \tilde{s},$$

$$s_{12} = \tilde{s}(1 + \frac{m^2}{\tilde{s}})^2, \;\; \tilde{s} = s_{12}\frac{(1+v_{12})^2}{4}, \;\; v_{12} = \sqrt{1 - \frac{4m^2}{s}}, \quad (16)$$

representing k as

$$k = -\beta l_1 + \alpha l_2 + k_\perp, \;\; (k_\perp l_1) = (k_\perp l_2) = 0, \;\; k_\perp^2 \equiv -\vec{k}^2 \le 0, \quad (17)$$

so that

$$k^2 = -\tilde{s}\alpha\beta - \vec{k}^2, \;\; (p_1+k)^2 - m^2 = \tilde{s}\alpha(1-\beta) - m^2\beta - \vec{k}^2,$$
$$(p_2-k)^2 - m^2 = \tilde{s}\beta(1-\alpha) - m^2\alpha - \vec{k}^2, \qquad (18)$$

and using $d^4k = \frac{\tilde{s}}{4} d\alpha d\beta d\vec{k}_\perp^2 d\phi$, where ϕ is the azimuth angle of $\vec{k}_\perp$, we obtain, passing to the integration variable $z = \beta/(1-m^2/\tilde{s})$,

$$\Delta_{s_{12}} I = -\frac{2i}{(4\pi)^2\sqrt{s_{12}(s_{12}-4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12}-4m^2}} \int_0^{2\pi} d\phi J_3(\tilde{s}_{23}; m^2), \qquad (19)$$

$$J((p_3+p_2-k)^2) = -\frac{1}{(4\pi)^2}\int_0^1 dx \int_0^1 \frac{ydy}{(y^2\tilde{p}_x^2 + \lambda^2(1-y))}, \qquad (20)$$

Here it should be said that $\tilde{s}_{23} = (p_3+p_2-k)^2$ we must calculate in the physical region, since we use Sudakov's parametrization.

where $\tilde{p}_x = x(p_2 - k) - (1 - x)p_3$, and

$$(p_2-k)^2 = m^2, \quad k = -z(p_1-p_2)+k_\perp, \quad \vec{k}_\perp^2 = (s_{12}-4m^2)z(1-z), \tag{21}$$

using these properties, we obtain

$$\tilde{p}_x^2 = m^2 - x(1 - x)(s_{13}z + s_{23}(1 - z) + 2(\vec{p}_{3\perp}\vec{k}_\perp)). \tag{22}$$

The most singular contribution to $\Delta_{s_{12}} I$ comes from $z = 0$ to $J((p_3 + p_2 - k)^2)$. Then

$$\Delta_{s_{12}} I = -\frac{4\pi i}{(4\pi)^2\sqrt{s_{12}(s_{12} - 4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12}-4m^2}} J_3(\tilde{s}_{23}), . \tag{23}$$

The presence of a discontinuity $\Delta_{s_{23}} J_3(s_{23})$ makes obvious the existence of a double discontinuity in overlapping channels. For positive s_{13} (i.e. in the physical domain) this is a violation of the Steinmann relations.

In this case

$$\Delta_{s_{23}}\Delta_{s_{12}}I = \Delta_{s_{12}}J_3(s_{12})\Delta_{s_{23}}J_3(s_{23}) \tag{24}$$

according to infrared factorization.
It is worth noting that the double discontinuity (by s_{12} and s_{23}) may be not only from the contribution I under consideration, but also from the contribution I' corresponding to the diagram in which the vertices of the interaction of photons with a particle with momentum p_2 change places. But the calculation of the double discontinuity depends on the order in which the discontinuities are calculated. As already mentioned, $\Delta_{s_{23}}I$ has no singularities; similarly $\Delta_{s_{12}}I'$. Therefore (24) gives the full discontinuity.

Thus, infrared singularity destroys the hypothesis of the absence of simultaneous discontinuities in overlapping channels.
The question arises whether there is a contradiction with the hypothesis when the photon mass is different from zero.
A direct demonstration of the violation of the hypothesis would be existence a discontinuities of $\Delta_{s_{12}} I$ in one of the invariants s_{13} and s_{23} in the case where the second of the invariants is negative. It should be said that this is a non-physical region, and this does not mean a violation of the Steinmann relations in their usual interpretation.
It is not difficult to show that $m = 0$ there are no such discontinuities.
In this case

$$\vec{p}_3^{\,2} = \frac{s_{13}s_{23}}{s_{12}}, \quad \vec{k}^2 = s_{12}z(1-z) \tag{25}$$

and denoting

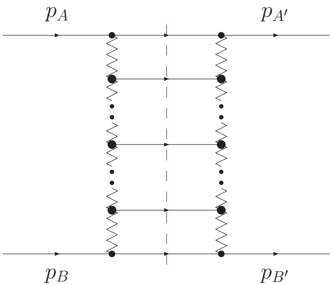
$$a = \lambda^2(1-y), \quad b = y^2x(1-x)s_{13}z, \quad c = y^2x(1-x)s_{23}(1-z) \tag{26}$$

we have

$$\int_0^{2\pi} d\phi J_3(\tilde{s}_{23}; m^2) = -\frac{1}{8\pi^2}\int_0^1 dx \int_0^1 \frac{ydy}{\sqrt{(a-b-c)^2-4bc}}, \tag{27}$$

which makes evident absence of the discontinuities.
It is most likely that in the absence of infrared singularities there is no violation of the hypothesis of the absence of simultaneous discontinuities in overlapping channels, although we don't have any strict proof yet.

The BFKL approach is founded on the gluon Reggeization.
In the dispersive method, used for the derivation of the BFKL equation, the unitarity relations are used for the calculation of imaginary parts of elastic amplitudes. Regge form of multiparticle amplitudes is used in unitarity relations.
In the unitarity relations, multiple production amplitudes in the multi-Regge kinematics (MRK) must be taken into account.
MRK is the kinematics where all particles have limited transverse momenta (with respect to momenta of colliding particle) and are combined into jets with limited invariant mass of each jet and large (increasing with s) invariant mass of any pair of jets.
The muli-Regge form was used for these amplitudes.



The s-channel discontinuity.

The fallacy of the hypothesis of the absence of simultaneous discontinuities in overlapping channels, and hence of the multi-Regge form of multiparticle amplitudes based on this hypothesis, may cast doubt on the derivation of the BFKL equation.

However, these doubts are unfounded both in the LLA and in the NLLA since in these approximations only the real part of the amplitudes included in the unitarity relations was used in deriving the BFKL equation.
It is quite clear in the LLA, where imaginary parts of the multiparticle amplitudes are neglected.
This is also true in the NLLA.
The reason is that in this approximation one of two amplitudes in the unitarity relations can lose $\ln s$, while the second one must be taken in the LLA. The LLA amplitudes are real, so that only real parts of the NLLA amplitudes are important in the unitarity relations.

Unfortunately, it is not so in the NNLLA. In this approximation two powers of $\ln s$ can be lost compared with the LLA in the product of two amplitudes in the unitarity relations. It can be done losing one $\ln s$ in each of the amplitudes, so that their analytical properties become important.

Summary

- Analytical properties of of multiparticle amplitudes are important for high energy theoretical physics.
- Regge theory of multiparticle amplitudes was constructed using the set of postulated properties of the amplitudes.
- One of important properties is the absence of simultaneous discontinuities in energy invariants of overlapping channels.
- This property is often considered as a consequence of the Steinmann relations.
- But this property does not agree with the infrared factorisation.
- Therefore, the commonly accepted Regge form of the multiparticle amplitudes is not valid in QCD.
- It must be taken into account in dispersive derivation of the BFKL equation in higher approximations.