Scalar quantum field theory in the background of Ellis wormhole

Mikhail Smolyakov

SINP MSU

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M.N. Smolyakov, "Peculiarities of quantum field theory in the presence of a wormhole", arXiv:2506.21256

Setup

The metric of the Ellis wormhole is

$$ds^2 = dt^2 - dr^2 - (r^2 + b^2)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right),$$

where $r \in (-\infty, \infty)$ and $b^2 > 0$.

A real massive scalar field $\phi(t, r, \theta, \varphi)$:

$$S = \int \sqrt{-g}\left(\frac{1}{2}\, g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{M^2}{2}\phi^2\right) d^4x.$$

Radial solutions

Decomposition in spherical harmonics $e^{\pm iEt}Y_{lm}(\theta,\varphi)f_l(k,r)$ leads to the radial equation (here $k=\sqrt{E^2-M^2}$)

$$\left(E^2-M^2\right)f_l(k,r)+\frac{1}{r^2+b^2}\frac{d}{dr}\left(\left(r^2+b^2\right)\frac{df_l(k,r)}{dr}\right)-\frac{l(l+1)}{r^2+b^2}f_l(k,r)=0.$$

The new function

$$\psi_l(k,r)=\sqrt{r^2+b^2}f_l(k,r)$$

satisfies a one-dimensional Schrödinger equation

$$-\frac{d^2\psi_l(k,r)}{dr^2}+V_l(r)\psi_l(k,r)=E^2\psi_l(k,r)$$

with the potential

$$V_l(r)=M^2+\frac{b^2}{\left(r^2+b^2\right)^2}+\frac{l(l+1)}{r^2+b^2}.$$

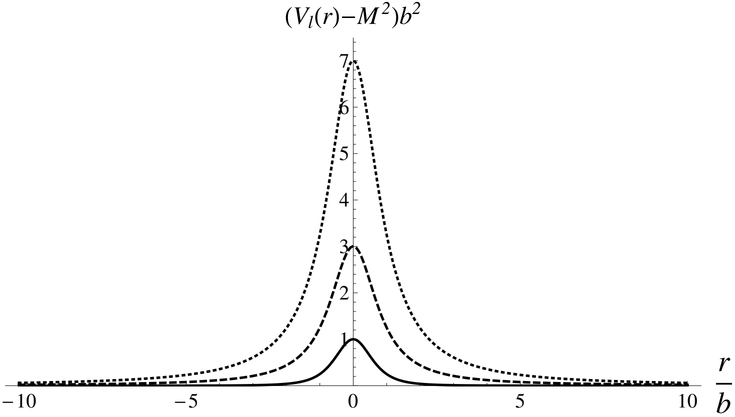

Figure 1: $V_l(r)$ for $l = 0$ (solid line), $l = 1$ (dashed line), and $l = 2$ (dotted line).

Two linearly independent solutions:

$$\psi_{l,s}(k,-r) = \psi_{l,s}(k,r),$$
$$\psi_{l,a}(k,-r) = -\psi_{l,a}(k,r).$$

For each l, these solutions form a complete set of orthogonal eigenfunctions.

Asymptotic behavior of the solutions:

$$\psi_{l,s}(k,r) \approx \frac{1}{\sqrt{\pi}} \sin\left(k|r| + \kappa_{l,s}(k)\right),$$
$$\psi_{l,a}(k,r) \approx \frac{\mathrm{sign}(r)}{\sqrt{\pi}} \sin\left(k|r| + \kappa_{l,a}(k)\right).$$

Isotropic coordinates

$$r = R - \frac{b^2}{4R}, \qquad R \in \left(-\infty, -\frac{b}{2}\right) \cup \left[\frac{b}{2}, \infty\right).$$

The metric takes the form

$$ds^2 = dt^2 - \left(1 + \frac{b^2}{4R^2}\right)^2 \left(dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right).$$

The asymptotics can be rewritten as

$$f_{l,s}(k,R) \approx \frac{1}{\sqrt{\pi}\,|R|} \sin\left(k|R| - \frac{\pi l}{2} + \delta_{l,s}(k)\right),$$

$$f_{l,a}(k,R) \approx \frac{\mathrm{sign}(R)}{\sqrt{\pi}\,|R|} \sin\left(k|R| - \frac{\pi l}{2} + \delta_{l,a}(k)\right),$$

where $\delta_{l,s}(k) = \kappa_{l,s}(k) + \frac{\pi l}{2}$ and $\delta_{l,a}(k) = \kappa_{l,a}(k) + \frac{\pi l}{2}$.

The coordinates

$$\vec{X} = \left[\begin{array}{ll} \vec{x} = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta), & R > \frac{b}{2}, \\ \vec{y} = (|R|\sin\theta\cos\varphi, |R|\sin\theta\sin\varphi, |R|\cos\theta), & R < -\frac{b}{2} \end{array} \right.$$

tend to Cartesian coordinates for $R \to \pm\infty$.

In these coordinates, equation for the spatial part of the scalar field $\phi(k, \vec{X})$ takes the form

$$\left(1 + \frac{b^2}{4\vec{X}^2}\right)^3 \left(E^2 - M^2\right) \phi(k, \vec{X}) - \eta^{ij} \frac{\partial}{\partial X^i} \left(\left(1 + \frac{b^2}{4\vec{X}^2}\right) \frac{\partial \phi(k, \vec{X})}{\partial X^i} \right) = 0.$$

Choosing the quantum states. Step 1

The scattering states defined as

$$\phi_s(\vec{k},\vec{X}) = \frac{1}{4\pi k}\sum_{l=0}^{\infty}(2l+1)e^{i\left(\frac{\pi l}{2}+\delta_{l,s}(k)\right)}P_l\left(\frac{\vec{k}\vec{X}}{k|R|}\right)f_{l,s}\left(k,R\right),$$

$$\phi_a(\vec{k},\vec{X}) = \frac{1}{4\pi k}\sum_{l=0}^{\infty}(2l+1)e^{i\left(\frac{\pi l}{2}+\delta_{l,a}(k)\right)}P_l\left(\frac{\vec{k}\vec{X}}{k|R|}\right)f_{l,a}\left(k,R\right)$$

satisfy the equation of motion for the scalar field.

At large $|R|$

$$\phi_s(\vec{k},\vec{X}) \approx \frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}} \left(e^{i\vec{k}\vec{X}} + A_s\left(\vec{k},\frac{\vec{X}}{|R|}\right)\frac{e^{ik|R|}}{|R|} \right),$$

$$\phi_a(\vec{k},\vec{X}) \approx \frac{\mathrm{sign}(R)}{\sqrt{2}(2\pi)^{\frac{3}{2}}} \left(e^{i\vec{k}\vec{X}} + A_a\left(\vec{k},\frac{\vec{X}}{|R|}\right)\frac{e^{ik|R|}}{|R|} \right)$$

with the scattering amplitudes

$$A_s\left(\vec{k},\vec{n}\right) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l\left(\frac{\vec{k}\vec{n}}{k}\right)\left(e^{i2\delta_{l,s}(k)}-1\right),$$

$$A_a\left(\vec{k},\vec{n}\right) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l\left(\frac{\vec{k}\vec{n}}{k}\right)\left(e^{i2\delta_{l,a}(k)}-1\right).$$

Choosing the quantum states. Step 2

$$\phi_+(\vec{k},\vec{X}) = \frac{1}{\sqrt{2}}\left(\phi_s(\vec{k},\vec{X}) + \phi_a(\vec{k},\vec{X})\right),$$
$$\phi_-(\vec{k},\vec{X}) = \frac{1}{\sqrt{2}}\left(\phi_s(\vec{k},\vec{X}) - \phi_a(\vec{k},\vec{X})\right).$$

These new states satisfy the orthogonality conditions

$$\int \left(1+\frac{b^2}{4R^2}\right)^3 \phi_+^*(\vec{k},\vec{X})\phi_-(\vec{k}',\vec{X})\,d^3X = 0,$$
$$\int \left(1+\frac{b^2}{4R^2}\right)^3 \phi_+^*(\vec{k},\vec{X})\phi_+(\vec{k}',\vec{X})\,d^3X = \delta^{(3)}(\vec{k}-\vec{k}'),$$
$$\int \left(1+\frac{b^2}{4R^2}\right)^3 \phi_-^*(\vec{k},\vec{X})\phi_-(\vec{k}',\vec{X})\,d^3X = \delta^{(3)}(\vec{k}-\vec{k}')$$

and the completeness relation

$$\int \left(\phi_-^*(\vec{k},\vec{X})\phi_-(\vec{k},\vec{X}') + \phi_+^*(\vec{k},\vec{X})\phi_+(\vec{k},\vec{X}')\right) d^3k = \frac{\delta^{(3)}(\vec{X}-\vec{X}')}{\left(1+\frac{b^2}{4R^2}\right)^3}.$$

Asymptotics for large positive R:

$$\phi_+(\vec{k},\vec{X}) \approx \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{i\vec{k}\vec{x}} + \frac{1}{2(2\pi)^{\frac{3}{2}}}\left(A_s\left(\vec{k},\frac{\vec{x}}{R}\right) + A_a\left(\vec{k},\frac{\vec{x}}{R}\right)\right)\frac{e^{ikR}}{R},$$

$$\phi_-(\vec{k},\vec{X}) \approx \frac{1}{2(2\pi)^{\frac{3}{2}}}\left(A_s\left(\vec{k},\frac{\vec{x}}{R}\right) - A_a\left(\vec{k},\frac{\vec{x}}{R}\right)\right)\frac{e^{ikR}}{R}.$$

Asymptotics for large negative R:

$$\phi_+(\vec{k},\vec{X}) \approx \frac{1}{2(2\pi)^{\frac{3}{2}}}\left(A_s\left(\vec{k},\frac{\vec{y}}{|R|}\right) - A_a\left(\vec{k},\frac{\vec{y}}{|R|}\right)\right)\frac{e^{ik|R|}}{|R|},$$

$$\phi_-(\vec{k},\vec{X}) \approx \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{i\vec{k}\vec{y}} + \frac{1}{2(2\pi)^{\frac{3}{2}}}\left(A_s\left(\vec{k},\frac{\vec{y}}{|R|}\right) + A_a\left(\vec{k},\frac{\vec{y}}{|R|}\right)\right)\frac{e^{ik|R|}}{|R|}.$$

In particular, for $R \to \infty$

$$\phi_+(\vec{k},\vec{X}) \to \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{i\vec{k}\vec{x}},$$
$$\phi_-(\vec{k},\vec{X}) \to 0,$$

whereas for $R \to -\infty$

$$\phi_+(\vec{k},\vec{X}) \to 0,$$
$$\phi_-(\vec{k},\vec{X}) \to \frac{1}{(2\pi)^{\frac{3}{2}}}\, e^{i\vec{k}\vec{y}}.$$

There is a good reason to believe that the states $\phi_+(\vec{k},\vec{X})$ live mostly in the universe with $R > \frac{b}{2}$, whereas the states $\phi_-(\vec{k},\vec{X})$ live mostly in the universe with $R < -\frac{b}{2}$; they penetrate into the "opposite" universes only to a small extent.

Canonical quantization

$$\phi(t,\vec{X}) = \int \frac{d^3k}{\sqrt{2\sqrt{\vec{k}^2+M^2}}} \Big(e^{-i\sqrt{\vec{k}^2+M^2}\,t} \left(\phi_-(\vec{k},\vec{X})a_-(\vec{k}) + \phi_+(\vec{k},\vec{X})a_+(\vec{k})\right)$$
$$+ e^{i\sqrt{\vec{k}^2+M^2}\,t} \left(\phi_-^*(\vec{k},\vec{X})a_-^\dagger(\vec{k}) + \phi_+^*(\vec{k},\vec{X})a_+^\dagger(\vec{k})\right)\Big),$$

where

$$[a_-(\vec{k}), a_-^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k}-\vec{k}'),$$
$$[a_+(\vec{k}), a_+^\dagger(\vec{k}')] = \delta^{(3)}(\vec{k}-\vec{k}'),$$

all other commutators being equal to zero. The canonical commutation relations are satisfied exactly:

$$[\phi(t,\vec{X}), \pi(t,\vec{X}')] = i\delta^{(3)}(\vec{X}-\vec{X}'),$$
$$[\phi(t,\vec{X}), \phi(t,\vec{X}')] = [\pi(t,\vec{X}), \pi(t,\vec{X}')] = 0,$$

where

$$\pi(t,\vec{X}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(t,\vec{X})} = \sqrt{-g(\vec{X})}\, g^{00}(\vec{X})\dot{\phi}(t,\vec{X}) = \left(1+\frac{b^2}{4R^2}\right)^3 \dot{\phi}(t,\vec{X}).$$

The Hamiltonian has the form

$$H = \int \sqrt{-g}\, g^{00} :T_{00}: d^3x = \frac{1}{2}\int \sqrt{-g}\, g^{00} :\left(\dot{\phi}^2 - \ddot{\phi}\,\phi\right): d^3x$$
$$= \int \sqrt{\vec{k}^2 + M^2}\left(a_-^\dagger(\vec{k})a_-(\vec{k}) + a_+^\dagger(\vec{k})a_+(\vec{k})\right) d^3k.$$

For comparison, the Hamiltonian of the scalar field in Minkowski spacetime is

$$H = \int \sqrt{k^2 + M^2}\, a^\dagger(\vec{k})a(\vec{k})\, d^3k.$$

Additional degeneracy is a purely topological effect (topological structure of the wormhole spacetime is $R^2 \times S^2$, topological structure of Minkowski spacetime is R^4). Exactly the same degeneracy exists in the Schwarzschild spacetime with the same topological structure $R^2 \times S^2$:

- V. Egorov, M. Smolyakov, I. Volobuev, Phys. Rev. D **107** (2023) 025001 [arXiv:2209.02067]
- M. Smolyakov, Phys. Rev. D **108** (2023) 105006 [arXiv:2309.06249]

Thank you for your attention!

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