$A_{2\pi}$ and f_0 contribution into the $K^+ \to \pi^0\pi^0\pi^0 e^+\nu$ decay

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PDG/OKA/ChPT results

- OKA recent result: $Br\left(K^+ \to \pi^0\pi^0\pi^0 e^+\nu\right) < 5.4 \times 10^{-8}$ at 90% confidence level. (arXiv:2409.08817)
- PDG result: $Br\left(K^+ \to \pi^0\pi^0\pi^0 e^+\nu\right) < 3.5 \times 10^{-6}$ at 90% confidence level.
- Chiral perturbation theory (Blaser): $Br\left(K^+ \to \pi^0\pi^0\pi^0 e^+\nu\right) = 2.5 \times 10^{-12}$.
- Can the OKA upper bound become closer to the theoretical prediction?

Four-particle vs five-particle phase volume

- It was shown by Blaser (Physics Letters B 345 (1995) 287-290) that

$$\left.\frac{\Gamma\left(K^+ \to \pi^0\pi^0 e^+\nu_e\right)}{\Gamma(K^+ \to \pi^0\pi^0\pi^0 e^+\nu_e)}\right|_{tree} \approx 3.4 \times 10^6.$$

- At the same time as it will be shown

$$m_K^2 \frac{V_4/2!}{V_5/3!} \approx 2.1 \times 10^6,$$

  where V_n is a n-particle phase volume.
- Consequently the main difference comes from the phase volumes ratio.

Pionium $A_{2\pi}$

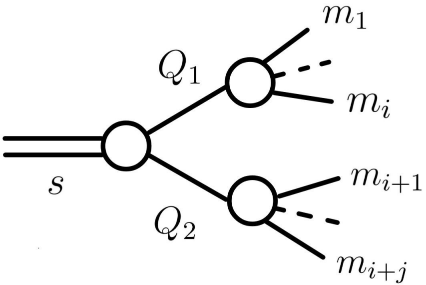
- $A_{2\pi}$ is a $\pi^+\pi^-$ atom bound by e/m interaction.
- Decay length $c\tau \sim 10^{-4}$cm is much larger than nuclear interaction scale.
- Main decay channel is $A_{2\pi} \to \pi^0\pi^0$.
- If the process $K^+ \to A_{2\pi}\pi^0 e^+\nu_e$ occurs then V_5 is replaced by V_4 and it enhances the width.

Recurrent formula

- For the V_n the following relation can be proved:

$$dV_n\left(s; m_1^2, m_2^2, \ldots, m_n^2\right) = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi} dV_2(s; Q_1^2, Q_2^2) \times dV_i\left(Q_1^2; m_1^2, \ldots, m_i^2\right) \times dV_j\left(Q_2^2; m_{i+1}^2, \ldots, m_{i+j}^2\right).$$

- It is convenient to use the following graphs:



V_4 and V_5

- For $K^+ \to \pi^0\pi^0 e^+\nu_e$ the recurrent formula leads to

$$V_4(s=m_K^2;0,0,m_{\pi^0}^2,m_{\pi^0}^2) = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi}\frac{\sqrt{\left[s-\left(\sqrt{Q_1^2}-\sqrt{Q_2^2}\right)^2\right]\left[s-\left(\sqrt{Q_1^2}+\sqrt{Q_2^2}\right)^2\right]}}{8\pi s}\times\frac{1}{8\pi}\times\frac{\sqrt{Q_2^2(Q_2^2-4m_\pi^2)}}{8\pi Q_2^2}.$$

- For the $K^+ \to \pi^0\pi^0\pi^0 e^+\nu_e$ in the same way we obtain

$$V_5(m_K^2;0,0,m_{\pi^0}^2,m_{\pi^0}^2,m_{\pi^0}^2) = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi}\frac{\sqrt{\left[s-\left(\sqrt{Q_1^2}-\sqrt{Q_2^2}\right)^2\right]\left[s-\left(\sqrt{Q_1^2}+\sqrt{Q_2^2}\right)^2\right]}}{8\pi s}\times\frac{1}{8\pi}\times\frac{\left(\sqrt{Q_2^2}-3m_{\pi^0}\right)^2}{2^6 3\sqrt{3}\pi^2}.$$

- Integration limits:

$$0 < Q_1^2 < (m_K - nm_{\pi^0})^2,$$

$$(nm_{\pi^0})^2 < Q_2^2 < \left(m_K - \sqrt{Q_1^2}\right)^2,$$

where $n = 2, 3$ is the number of pions in the final state.

V_4 and V_5

Approximations used:

1. $m_e = 0 \to V_2 = \frac{1}{8\pi}$.
2. Pions in K_{e5} decay are non-relativistic $\to V_3 = \frac{\left(\sqrt{s} - 3m_{\pi^0}\right)^2}{2^6 3\sqrt{3}\pi^2}$.

As a result of integration:

- $V_4 = 7.1 \times 10^{-3} \frac{m_K^4}{2^{11}\pi^5}$.
- $V_5 = 1.37 \times 10^{-6} \frac{m_K^6}{2^{14} 3\sqrt{3}\pi^6}$.
- Taking into account identity of neutral pions we obtain

$$m_K^2 \frac{V_4/2!}{V_5/3!} = 2.1 \times 10^6.$$

$A_{2\pi}\pi^0 e\nu_e$ in the final state

- In this case we have again the V_4 with substitution $m_{\pi^0} \to 2m_{\pi^+}$:

$$V_A = \int \frac{dQ_1^2}{2\pi}\frac{dQ_2^2}{2\pi}\frac{\sqrt{\left[s-\left(\sqrt{Q_1^2}-\sqrt{Q_2^2}\right)^2\right]\left[s-\left(\sqrt{Q_1^2}+\sqrt{Q_2^2}\right)^2\right]}}{8\pi s}\times\frac{1}{8\pi}\times$$
$$\times\frac{\sqrt{\left[Q_2^2-\left(\sqrt{4m_{\pi^+}^2}-\sqrt{m_{\pi^0}^2}\right)^2\right]\left[Q_2^2-\left(\sqrt{4m_{\pi^+}^2}+\sqrt{m_{\pi^0}^2}\right)^2\right]}}{8\pi Q_2^2},$$

where the integration limits are

$$0 < Q_1^2 < (m_K - 2m_{\pi^+} - m_{\pi^0})^2,$$
$$(2m_{\pi^+} + m_{\pi^0})^2 < Q_2^2 < \left(m_K - \sqrt{Q_1^2}\right)^2,$$

- After performing integration $V_A = 0.83 \times 10^{-4}\frac{m_K^4}{2^{11}\pi^5}$.

Ratio estimation and $\psi(0)$

- The increase of the width is

$$\frac{m_K^2 V_A}{V_5/3!} \approx 5 \times 10^4.$$

- But the pionium is a bound system $\to$ we need to modify the amplitude as well:

$$< A_{2\pi}\pi^0 e^+\nu_e|K >\sim \int d\vec{p}\phi(\vec{p}) < \pi^+\pi^-\pi^0 e^+\nu_e|K > \Big|_{|\vec{p}|\to 0} \to \psi(0) < \pi^+\pi^-\pi^0 e^+\nu_e|K > .$$

- As a result

$$\frac{\Gamma(K^+ \to A_{2\pi}\pi^0 e^+\nu_e)}{\Gamma(K^+ \to \pi^0\pi^0\pi^0 e^+\nu_e)} \sim |\psi(0)|^2 \frac{m_K^2 V_A}{V_5/3!} \sim 7.5 \times 10^{-4},$$

  where $\psi(0) = 1/\sqrt{\pi a_B^3} \sim \alpha^{3/2}/(2\sqrt{2\pi})$.

- We see the suppression of the decay via pionium rather than enhancement.

Another V_5 division

- Another V_5 division is V_2 for resonant $\pi^0\pi^0$ production and V_3 for the rest $\pi^0 e^+\nu$ system:

$$V_5 = \int \frac{dQ_2^2}{2\pi} \int \frac{dQ_1^2}{2\pi} V_2(m_K^2; Q_1^2, Q_2^2) V_3(Q_1^2) V_2(Q_2^2),$$

  where V_3 is

$$V_3(Q_1^2) = \frac{Q_1^2}{256\pi^3} \left[1 - \frac{m_\pi^4}{Q_1^4} + 2\frac{m_\pi^2}{Q_1^2} \ln \frac{m_\pi^2}{Q_1^2}\right]$$

- Integration limits are

$$\left(m_K - \sqrt{Q_2^2}\right)^2 > Q_1^2 > m_\pi^2,\ (m_K - m_\pi)^2 > Q_2^2 > (2m_\pi)^2.$$

- After the integration we obtain:

$$V_5 = 2.99 \times 10^{-6} \frac{m_K^6}{2^{16}\pi^7} \equiv \phi \frac{m_K^6}{2^{16}\pi^7}.$$

Resonance in two-pion system

- Assume a resonance R with mass M_R and the total width Γ_R which interacts only with $\pi\pi$. The decay mechanism is then $K^+ \to \pi^0\pi^0\pi^0 e^+\nu \to R\pi^0 e^+\nu \to \pi^0\pi^0\pi^0 e^+\nu$.
- The hadronic part A_μ of the K_{e5} decay can be written in the following way:

$$A_\mu = \int \frac{d^4k}{(2\pi)^4} G(k)G(Q_2 - k) < \pi^0\pi^0(k)\pi^0(Q_2 - k)|\bar{s}\gamma_\mu(1-\gamma_5)u|K^+ > \times$$
$$\times \frac{< \pi^0(k)\pi^0(Q_2 - k)|R >< R|\pi^0\pi^0 >}{Q_2^2 - M_R^2 + iM_R\Gamma_R}$$

- For the square of the matrix element $|< R|\pi^0\pi^0 > |^2$:

$$\Gamma_{R\to\pi^0\pi^0} \sim |< R|\pi^0\pi^0 > |^2,$$

we can estimate the total K_{e3} matrix element M:

$$|M|^2 \sim \frac{\Gamma^2_{R\to\pi^0\pi^0}}{\left(Q_2^2 - M_R^2\right)^2 + M_R^2\Gamma_R^2}.$$

Numerical estimation

- Introduction of the resonance R leads to the following replacement:

$$V_2(Q_2^2) \to \frac{\Gamma^2_{R\to\pi^0\pi^0}/4}{\left(\sqrt{Q_2^2} - M_R\right)^2 + \Gamma_R^2/4} V_2(Q_2^2).$$

- We assume that the resonance can decay only into pions so $\Gamma_R = \Gamma_{R\to\pi\pi}$. (this holds for both $A_{2\pi}$ and f_0). By isotopic invariance $\Gamma_{R\to\pi^0\pi^0} = \Gamma_R/3$.
- Assuming the R with pionium mass $M_{A_{2\pi}} \approx 280$ MeV and varying its width 0.01 $MeV < \Gamma_R < 1000$ MeV we obtain the following results:

Γ(MeV)	0.01	0.1	1	10	100	1000
ϕ	1.74×10^{-10}	1.74×10^{-9}	1.69×10^{-8}	1.24×10^{-7}	3.11×10^{-7}	3.31×10^{-7}

Numerical estimates

- It is clear that for $\Gamma_R \ll M_R$ the K_{e5} decay width grows linearly: $\Gamma_R \delta\left(\sqrt{Q_2^2} - M_R\right)$ - the case of $A_{2\pi}$.
- However for $\Gamma_R \sim M_R$ the K_{e5} decay width is not enhanced with respect to the non resonant final particles production - the case of f_0.
- For the f_0 parameters $\Gamma_{f_0} = M_{f_0} = 500\ MeV$ we obtain $\phi = 1.7 \times 10^{-6}$ instead of 2.99×10^{-6} for non resonant case.

Conclusion

- The impact of pionium into the K_{e5} decay was investigated.
- It was checked that in case of K_{e4} and K_{e5} the width ratio is determined by phase spaces ratio.
- The pionium contribution was estimated and the phase space enhancement was shown.
- It was pointed out that there is additional suppression due to the factor $\psi(0)$ of the pionium. Consequently the pionium contribution is negligible.
- The f_0 resonance contribution into K_{e5} decay was considered.
- The f_0 resonance does not considerably enhance the K_{e5} decay width in comparison with the non resonant contribution.