Discription of the dynamics of fermions by the influence functional method

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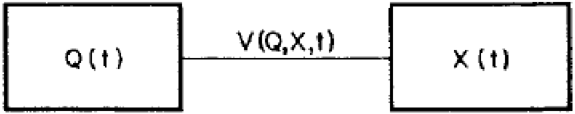
The Feynman-Vernon Influence Functional approach

R.P. Feynman, F.L. Vernon, Jr. **The Theory of a General Quantum System Interacting with a Linear Dissipative System**, *Annals of Physics* **24**, 118–173.



General quantum systems Q and X coupled by a potential $V(Q, X, t)$.

«. . . It is shown that the effect of the external systems in such a formalism [paths integral formalism] can always be included in a general class of functionals (influence functionals) of the coordinates of the system only . . . »

QED Lagrangian,Hamiltonian. Classical model.

$$\mathcal{L}(x) = \overline{\psi}(x)(\imath\gamma^{\mu}\partial_{\mu} - m)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - ej^{\mu}(x)A_{\mu}(x) \tag{1}$$

where

$$F_{\mu\nu}(x) = \partial_{\mu}A_{\mu}(x) - \partial_{\nu}A_{\mu}(x) \tag{2}$$

$$j^{\mu}(x) = \overline{\psi}(x)\gamma^{\mu}\psi(x) \tag{3}$$

$$H_{full} = H_{sys} + H_{field} + H_{int} \tag{4}$$

Quantum model.Second quantization. Evolution equation for statistical operator $\hat{\rho}(t_f)$

$$\hat{\rho}(t_f) = \hat{U}(t_f, t_{in})\hat{\rho}(t_{in})\hat{U}^{\dagger}(t_f, t_{in}) \tag{5}$$

where $\hat{\rho}(t_{in})$ is statistical operator, describing initial state at moment t_{in}, $\hat{U}(t_f, t_{in})$ — evolution operator.

$$\hat{U}(t_f, t_{in}) = \hat{T}\exp[-\frac{\imath}{\hbar}\int_{t_{in}}^{t_f} \hat{H}_{full}(\tau)d\tau]. \tag{6}$$

where $\hat{H}_{full}$:

$$\hat{H}_{full} = \hat{H}_{sys} + \hat{H}_{field} + \hat{H}_{int} \tag{7}$$

$$\hat{\psi}(\mathbf{x},t) = \sum_{\mathbf{p},\sigma=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{p}}^{(f)}}} \left(\hat{b}_{\mathbf{p}\sigma} u_\sigma(p) e^{\imath px} + \hat{c}^\dagger_{\mathbf{p}\sigma} u_\sigma(-p) e^{-\imath px} \right), \tag{8}$$

$$\hat{\overline{\psi}}(\mathbf{x},t) = \sum_{\mathbf{p},\sigma=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{p}}^{(f)}}} \left(\hat{b}^\dagger_{\mathbf{p}\sigma} \overline{u}_\sigma(p) e^{-\imath px} + \hat{c}_{\mathbf{p}\sigma} \overline{u}_\sigma(-p) e^{\imath px} \right), \tag{9}$$

$$\hat{j}_\mu(\mathbf{x},t) = \hat{\overline{\psi}}(\mathbf{x},t)\gamma_\mu \hat{\psi}(\mathbf{x},t) \tag{10}$$

$$\hat{A}^\mu(\mathbf{x},t) = \sum_{\mathbf{k},\lambda=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}^{(b)}}} \varepsilon^\mu_\lambda \left(\hat{a}_{\mathbf{k}\lambda} e^{\imath kx} + \hat{a}^\dagger_{\mathbf{k}\lambda} e^{-\imath kx} \right) \tag{11}$$

$$\hat{H}_{full} = \sum_{\mathbf{p},\sigma=1,2} \omega_{\mathbf{p}}^{(f)} \left(\hat{b}^{\dagger}_{\mathbf{p}\sigma} \hat{b}_{\mathbf{p}\sigma} + \hat{c}^{\dagger}_{\mathbf{p}\sigma} \hat{c}_{\mathbf{p}\sigma} \right) + \sum_{\mathbf{k},\lambda=1,2} \omega_{\mathbf{k}}^{(b)} \hat{a}^{\dagger}_{\mathbf{k}\lambda} \hat{a}_{\mathbf{k}\lambda} + $$
$$+ e \sum_{\mathbf{k},\lambda=1,2} \frac{\imath}{\sqrt{2\omega_{\mathbf{k}}^{(b)} V}} \left(\varepsilon^{\mu}_{\lambda} \hat{j}^{+}_{\mu}(\mathbf{k},t) \hat{a}_{\mathbf{k}\lambda} + \varepsilon^{*\mu}_{\lambda} \hat{j}^{-}_{\mu}(\mathbf{k},t) \hat{a}^{\dagger}_{\mathbf{k}\lambda} \right) \tag{12}$$

where

$$\hat{j}^{+}_{\mu}(\mathbf{k},t) = \int \hat{j}_{\mu}(\mathbf{x},t) e^{\imath \mathbf{k}\mathbf{x}} d\mathbf{x}, \qquad \hat{j}^{-}_{\mu}(\mathbf{k},t) = \int \hat{j}_{\mu}(\mathbf{x},t) e^{-\imath \mathbf{k}\mathbf{x}} d\mathbf{x} \tag{13}$$

Coherent states for electromagetic field

$$\hat{a}_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle = \alpha_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle, \qquad \langle\alpha_{\mathbf{k}\lambda}|\hat{a}^{\dagger}_{\mathbf{k}\lambda} = \langle\alpha_{\mathbf{k}\lambda}|\alpha^{*}_{\mathbf{k}\lambda}, \tag{14}$$

where $\alpha_{\mathbf{k}\lambda}$ — complex value, which describe states $\mathbf{k}$ mode of quantum electromagnetic field. These states ($|\alpha\rangle$) are non-orthogonal:

$$\langle\alpha'_{\mathbf{k}'\lambda'}|\alpha_{\mathbf{k}\lambda}\rangle = \delta_{\mathbf{k}'\mathbf{k}}\delta_{\lambda'\lambda}\exp\left\{-\frac{1}{2}\left(|\alpha'_{\mathbf{k}'\lambda'}|^2 + |\alpha_{\mathbf{k}\lambda}|^2 - 2\alpha'^{*}_{\mathbf{k}'\lambda'}\alpha_{\mathbf{k}\lambda}\right)\right\}. \tag{15}$$

There is resolution of the identity operator:

$$\int |\alpha_{\mathbf{k}\lambda}\rangle\langle\alpha_{\mathbf{k}\lambda}|\frac{d^2\alpha_{\mathbf{k}\lambda}}{\pi} = \hat{1}. \tag{16}$$

Grassman states for Dirac field

$$\hat{b}_{\mathbf{p},\sigma}|\theta_{\mathbf{p},\sigma}\rangle = \theta_{\mathbf{p},\sigma}|\theta_{\mathbf{p},\sigma}\rangle, \qquad \langle\overline{\theta}_{\mathbf{p},\sigma}|\hat{b}^{\dagger}_{\mathbf{p},\sigma} = \langle\overline{\theta}_{\mathbf{p},\sigma}|\overline{\theta}_{\mathbf{p},\sigma}, \tag{17}$$

where $\theta_{\mathbf{p},\sigma}$ — grassman variable. These states ($|\theta\rangle$) are non-orthogonal:

$$\langle\overline{\theta}'_{\mathbf{p}'\sigma'}|\theta_{\mathbf{p}\sigma}\rangle = \delta_{\mathbf{p}'\mathbf{p}}\delta_{\sigma'\sigma}\exp\left\{-\frac{1}{2}\left(\overline{\theta}'_{\mathbf{p}'\sigma'}\theta'_{\mathbf{p}'\sigma'} + \overline{\theta}_{\mathbf{p}\sigma}\theta_{\mathbf{p}\sigma} - 2\overline{\theta}'_{\mathbf{p}'\sigma'}\theta_{\mathbf{p}\sigma}\right)\right\}. \tag{18}$$

There is resolution of the identity operator:

$$\int |\theta_{\mathbf{p}\sigma}\rangle\langle\overline{\theta}_{\mathbf{p}\sigma}|\frac{d^2\theta_{\mathbf{p}\sigma}}{\pi} = \hat{1}. \tag{19}$$

Grassman variables properties

For two grassman variables θ and η

$$\theta\eta + \eta\theta = 0 \qquad \text{or} \qquad \theta\eta = -\eta\theta \qquad \text{so} \qquad \theta^2 = 0 \tag{20}$$

Then

$$\int d\theta f(\theta) = \int d\theta (A + B\theta) = B. \tag{21}$$

and

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = b \tag{22}$$

and

$$\left(\prod_i \int d\theta_i^* d\theta \right) \theta_k \theta_l^* e^{-\theta_i^* B_{ij} \theta_j} = \frac{\det B}{B_{kl}} \tag{23}$$

Evolution equation for density matrix in holomorphic representation $|\theta_{\mathbf{p}\sigma}, \alpha_{\mathbf{k}\lambda}\rangle = |\theta_{\mathbf{p}\sigma}\rangle \otimes |\alpha_{\mathbf{k}\lambda}\rangle$

The density matrix:

$$\rho(\alpha_f^*, \overline{\theta}_f, \alpha'_f, \theta'_f; t_f) = \langle \overline{\theta}_f, \alpha_f | \hat{\rho}(t_f) | \theta'_f, \alpha'_f \rangle \tag{24}$$

The kernel of evolution operator:

$$U(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \langle \overline{\theta}_f, \alpha_f | \hat{U}(t_f, t_{in}) | \theta_{in}, \alpha_{in} \rangle \tag{25}$$

The evolution equation:

$$\rho(\alpha_f^*, \overline{\theta}_f, \alpha'_f, \theta'_f; t_f) = \int \frac{d^2\alpha'_{in}}{\pi} \frac{d^2\theta'_{in}}{\pi} \frac{d^2\alpha_{in}}{\pi} \frac{d^2\theta_{in}}{\pi} \times$$

$$\times U(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) \rho(\alpha_{in}^*, \overline{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in}) U^*(\alpha'_f, \theta'_f, t_f | \alpha'^*_{in}, \overline{\theta}'_{in}; t_{in}) \tag{26}$$

The kernel of evolution operator as a path integral

$$U(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_f, \theta_f, t_{in}) = \int \mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau)\mathfrak{D}\overline{\theta}(\tau)\mathfrak{D}\theta(\tau)\times$$
$$\times \exp\left\{\imath S_{full}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right]\right\}, \quad (27)$$

where action

$$S_{full}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right] =$$
$$= S_f\left[\overline{\theta}(\tau), \theta(\tau)\right] + S_b\left[\alpha^*(\tau), \alpha(\tau)\right] + S_{int}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right]. \quad (28)$$

Action of fermionic field:

$$S_f\left[\overline{\theta}(\tau),\theta(\tau)\right]=\int_{t_{in}}^{t_f}\left(\frac{\dot{\overline{\theta}}(\tau)\theta(\tau)-\overline{\theta}(\tau)\dot{\theta}(\tau)}{2\imath}-\omega^{(f)}\overline{\theta}(\tau)\theta(\tau)\right)d\tau \tag{29}$$

Action of bosonic field:

$$S_b\left[\alpha^*(\tau),\alpha(\tau)\right]=\int_{t_{in}}^{t_f}\left(\frac{\dot{\alpha}^*(\tau)\alpha(\tau)-\alpha^*(\tau)\dot{\alpha}(\tau)}{2\imath}-\omega^{(b)}\alpha^*(\tau)\alpha(\tau)\right)d\tau \tag{30}$$

Action of interaction part:

$$S_{int}\left[\alpha^*(\tau),\alpha(\tau),\overline{\theta}(\tau),\theta(\tau)\right]=\int_{t_{in}}^{t_f}ej^{\mu}(\overline{\theta}(\tau),\theta(\tau))(\varepsilon_{\mu}^{*}\alpha^*(\tau)+\varepsilon_{\mu}\alpha(\tau))]d\tau; \tag{31}$$

Evolution of density matrix in paths integral formulation

We have

$$\rho(\alpha_f^*, \overline{\theta}_f, \alpha_f', \theta_f'; t_f) = \int \frac{d^2\alpha_{in}'}{\pi}\frac{d^2\theta_{in}'}{\pi}\frac{d^2\alpha_{in}}{\pi}\frac{d^2\theta_{in}}{\pi}\rho(\alpha_{in}^*, \overline{\theta}_{in}, \alpha_{in}', \theta_{in}'; t_{in})\times$$
$$\times\mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau)\mathfrak{D}\overline{\theta}(\tau)\mathfrak{D}\theta(\tau)\mathfrak{D}\alpha'^*(\tau)\mathfrak{D}\alpha'(\tau)\mathfrak{D}\overline{\theta}'(\tau)\mathfrak{D}\theta'(\tau)\times$$
$$\times\exp\left\{\imath\left(S_{full}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right] - S_{full}\left[\alpha'^*(\tau), \alpha'(\tau), \overline{\theta}'(\tau), \theta'(\tau)\right]\right)\right\}, \qquad (32)$$

Fermionic density matrix and influence functional

$$\rho(\overline{\theta}_f,\theta'_f;t_f)=Sp_{\alpha_f=\alpha'_f}\rho(\alpha^*_f,\overline{\theta}_f,\theta'_f,\alpha'_f;t_f)=\int d\theta'_f d\theta_f \mathfrak{D}\overline{\theta}(\tau)\mathfrak{D}\theta(\tau)\mathfrak{D}\overline{\theta}'(\tau)\mathfrak{D}\theta'(\tau)d\theta'_{in}d\theta_{in}\times$$
$$\times\exp\left\{\imath\left(S_f[\overline{\theta}(\tau),\theta(\tau)]-S_f[\overline{\theta}'(\tau),\theta'(\tau)]\right)\right\}F[\theta(\tau),\theta'(\tau)] \quad (33)$$

where $F[\overline{\theta}(\tau),\theta'(\tau)]$ is influence functional of electromagnetic field on fermionic subsystems.

$$F[\overline{\theta}(\tau),\theta'(\tau)]=Sp_{\alpha_f=\alpha'_f}\int \mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau)\mathfrak{D}\alpha'^*(\tau)\mathfrak{D}\alpha'(\tau)\frac{d^2\alpha_{in}}{\pi}\frac{d^2\alpha'_{in}}{\pi}\rho(\alpha^*_{in},\overline{\theta}_{in},\alpha'_{in},\theta'_{in};t_{in})\times$$
$$\times\exp\left\{\imath\left(S_b\left[\alpha^*(\tau),\alpha(\tau)\right]+S_{int}\left[\alpha^*(\tau),\alpha(\tau),\overline{\theta}(\tau),\theta(\tau)\right]-S_b\left[\alpha'^*(\tau),\alpha'(\tau)\right]-S_{int}\left[\alpha'^*(\tau),\alpha'(\tau),\overline{\theta}'(\tau),\theta'(\tau)\right]\right)\right\} \quad (34)$$

In many cases, we can choose at initial moment t_{in}

$$\rho(\alpha^*_{in},\overline{\theta}_{in},\alpha'_{in},\theta'_{in};t_{in})=\rho_f(\overline{\theta}_{in},\theta'_{in};t_{in})\times\rho_b(\alpha^*_{in},\alpha'_{in};t_{in}) \quad (35)$$

Influence functional of electromagnetic field

$$F[\overline{\theta}(\tau), \theta'(\tau)] = \int Sp_{\alpha_f=\alpha'_f} \frac{d^2\alpha_{in}}{\pi} \frac{d^2\alpha'_{in}}{\pi} \times$$

$$\times U_{infl}(\alpha^*_f, \overline{\theta}_f, t_f|\alpha_{in}, \theta_{in}, t_{in}) \rho(\alpha^*_{in}, \overline{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in}) U^*_{infl}(\alpha'_f, \theta'_f, t_f|\alpha'^*_{in}, \overline{\theta}'_{in}; t_{in}) \quad (36)$$

where $U_{infl}(\alpha^*_f, \overline{\theta}_f, t_f|\alpha_{in}, \theta_{in}, t_{in})$ is electromagnetic field transition amplitude from initial state $|\alpha_{in}\rangle$ to final state $|\alpha^*_f\rangle$ inducing by external source j:

$$U_{infl}(\alpha^*_f, \overline{\theta}_f, t_f|\alpha_{in}, \theta_{in}, t_{in}) = \int \mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau) \exp\{\imath S_{infl}[\alpha^*(\tau), \alpha(\tau), x(\tau)]\} \quad (37)$$

where $S_{infl}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right] = S_b\left[\alpha^*(\tau), \alpha(\tau)\right] + S_{int}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right]$. In general, influence functional (26) describes the influence (action) of electromagnetic field on fermionic field.

Functional integration over electromagnetic field paths

$$U_{infl}(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \exp\left\{ e^{-\imath\omega(t_f - t_{in})}\alpha_f^*\alpha_{in} - e^2 \int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau} \varepsilon^{\mu} j_{\mu}^{+}(\tau)\varepsilon^{*\nu} j_{\nu}^{-}(\tau')e^{\imath\omega(\tau-\tau')}d\tau d\tau' - \right.$$
$$\left. - \imath\alpha_{in} e\int_{t_{in}}^{t_f} \varepsilon^{\mu} j_{\mu}^{+}(\tau)e^{-\imath\omega(\tau - t_{in})}d\tau - \imath\alpha_f^* e\int_{t_{in}}^{t_f} \varepsilon^{*\mu} j_{\mu}^{-}(\tau)e^{-\imath\omega(t_f-\tau)}d\tau \right\} \quad (38)$$

For multimode field and two polarizations without interaction between modes

$$U_{infl} = \prod_{\mathbf{k},\lambda} U_{infl}^{(\mathbf{k},\lambda)} \quad (39)$$

$$U_{infl}(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) =$$

$$= \prod_{\mathbf{k},\lambda=1,2} \exp\left\{ e^{-\imath\omega_{\mathbf{k}}(t_f - t_{in})} \alpha_{\mathbf{k}\lambda}^{(f)*} \alpha_{\mathbf{k}\lambda}^{(in)} - \frac{e^2}{2\omega_{\mathbf{k}} V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) \varepsilon_\lambda^{*\nu} j_\nu^-(\mathbf{k},\tau') e^{\imath\omega_{\mathbf{k}}(\tau-\tau')} d\tau d\tau' - \right.$$

$$\left. - \imath\alpha_{\mathbf{k}\lambda}^{(in)} \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(\tau - t_{in})} d\tau - \imath\alpha_{\mathbf{k}\lambda}^{(f)*} \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^{*\mu} j_\mu^-(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(t_f - \tau)} d\tau \right\} =$$

$$= \exp\left\{ \sum_{\mathbf{k},\lambda} \left(e^{-\imath\omega_{\mathbf{k}}(t_f - t_{in})} \alpha_f^* \alpha_{in} - \frac{e^2}{2\omega_{\mathbf{k}} V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) \varepsilon_\lambda^{*\nu} j_\nu^-(\mathbf{k},\tau') e^{\imath\omega_{\mathbf{k}}(\tau-\tau')} d\tau d\tau' - \right.\right.$$

$$\left.\left. - \imath\alpha_{in} \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(\tau - t_{in})} d\tau - \imath\alpha_f^* \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^{*\mu} j_\mu^-(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(t_f - \tau)} d\tau \right) \right\} \quad (40)$$

Vacuum influence functional

For the case when initial and final states of electromagnetic field are vacuum:

$$\phi_{in}(\alpha_{in}) = \langle \alpha_{in}|0\rangle = \exp\left\{-\frac{1}{2}|\alpha_{in}|^2\right\}, \qquad \phi^*_f(\alpha_f) = \langle 0|\alpha_f\rangle = \exp\left\{-\frac{1}{2}|\alpha_f|^2\right\}. \quad (41)$$

We define influence functional of electromagnetic vacuum

$$F_{\langle vac|vac\rangle}[\overline{\theta}(\tau), \theta'(\tau)] = \int \frac{d^2\alpha_f}{\pi}\frac{d^2\alpha'_f}{\pi}\frac{d^2\alpha_{in}}{\pi}\frac{d^2\alpha'_{in}}{\pi}\rho_f(\overline{\theta}_{in}, \theta'_{in}; t_{in})\times$$
$$\times\phi^*_f(\alpha_f)U_{infl}(\alpha^*_f, \overline{\theta}_f, t_f|\alpha_{in}, \theta_{in}, t_{in})\phi_{in}(\alpha_{in})\phi^*_{in}(\alpha'_{in})U^*_{infl}(\alpha'_f, \theta'_f, t_f|\alpha'^*_{in}, \overline{\theta}'_{in}; t_{in})\phi^*_f(\alpha'_f) =$$
$$= \exp\left\{-\sum_{\mathbf{k},\lambda}\frac{e^2}{2\omega_{\mathbf{k}}V}\int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau}\left(\varepsilon^{\mu}_{\lambda}j^{+}_{\mu}(\mathbf{k},\tau)\varepsilon^{*\nu}_{\lambda}j^{-}_{\nu}(\mathbf{k},\tau')e^{\imath\omega_{\mathbf{k}}(\tau-\tau')}d\tau d\tau' + \varepsilon^{\mu}_{\lambda}j'^{+}_{\mu}(\mathbf{k},\tau)\varepsilon^{*\nu}_{\lambda}j'^{-}_{\nu}(\mathbf{k},\tau')e^{-\imath\omega_{\mathbf{k}}(\tau-\tau')}d\tau d\tau'\right)\right\} \quad (42)$$

From sum over $\mathbf{k}$ to integral: $\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3}\int d\mathbf{k}$

$$-\sum_{\mathbf{k}} \frac{e^2}{2\omega_{\mathbf{k}} V} \int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu}\varepsilon_{\lambda}^{nu}\right] j_{\mu}^{+}(\mathbf{k},\tau) j_{\nu}^{-}(\mathbf{k},\tau') e^{\imath\omega_{\mathbf{k}}(\tau-\tau')} d\tau d\tau' =$$

$$= -\frac{e^2}{(2\pi)^3} \int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau}\int \frac{1}{2\omega_{\mathbf{k}}} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu}\varepsilon_{\lambda}^{*\nu}\right] j_{\mu}^{+}(\mathbf{k},\tau) j_{\nu}^{-}(\mathbf{k},\tau') e^{\imath\omega(\tau-\tau')} d\mathbf{x}d\mathbf{x}'d\mathbf{k}d\tau d\tau' =$$

$$= -\frac{e^2}{(2\pi)^3} \int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau}\int \frac{1}{2\omega_{\mathbf{k}}} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu}\varepsilon_{\lambda}^{*\nu}\right] j_{\mu}(\mathbf{x},\tau) j_{\nu}(\mathbf{x}',\tau') e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{\imath\omega(\tau-\tau')} d\mathbf{x}d\mathbf{x}'d\mathbf{k}d\tau d\tau' =$$

$$= -\frac{e^2}{4\pi\imath} \int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau}\int \underbrace{\left[\frac{1}{(2\pi)^3}\int \frac{2\pi\imath d\mathbf{k}}{\omega_{\mathbf{k}}} \left(\sum_{\lambda} \varepsilon_{\lambda}^{\mu}\varepsilon_{\lambda}^{*\nu}\right) e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{\imath\omega(\tau-\tau')} d\mathbf{k}\right]}_{D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')} j_{\mu}(\mathbf{x},\tau) j_{\nu}(\mathbf{x}',\tau') d\mathbf{x}d\mathbf{x}'d\tau d\tau'$$

where $D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')$ is photon propagator [1].

[1] V.B. Berestetskii, E.M. Lifshitz, L.P. Pitaevskii **Quantum Electrodynamics**

$$F_{\langle vac|vac\rangle}[\overline{\theta}(\tau),\theta'(\tau)] = \exp\left\{-\frac{e^2}{4\pi\imath}\int\limits_{t_{in}}^{t_f}\int\limits_{t_{in}}^{\tau}\int j_\mu(\mathbf{x},\tau)D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')j_\nu(\mathbf{x}',\tau')d\mathbf{x}d\mathbf{x}'d\tau d\tau' - \right.$$
$$\left. - \frac{e^2}{4\pi\imath}\int\limits_{t_{in}}^{t_f}\int\limits_{t_{in}}^{\tau}\int j'_\mu(\mathbf{x},\tau)D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')j'_\nu(\mathbf{x}',\tau')d\mathbf{x}d\mathbf{x}'d\tau d\tau'\right\} \quad (43)$$

For $t_f \to \infty$, $t_{in} \to -\infty$ we have relativistic invariant influence functional of electromagnetic vacuum:

$$F_{\langle vac|vac\rangle}[\overline{\theta}(\tau),\theta'(\tau)] = \exp\left\{-\frac{e^2}{4\pi\imath}\int\int\left(j_\mu(x)D^{\mu\nu}(x-x')j_\nu(x') + j'_\mu(x)D^{\mu\nu}(x-x')j'_\nu(x')\right)d^4xd^4x'\right\} \quad (44)$$

Fermionic density matrix evolution in coordinate representation

$$\rho(\overline{\psi}_f, \psi'_f; t_f) = \int \mathfrak{D}\overline{\psi}(\tau)\mathfrak{D}\psi(\tau)\mathfrak{D}\overline{\psi}'(\tau)\mathfrak{D}\psi'(\tau)\rho_f(\psi_{in}, \psi'_{in}; t_{in})\times$$
$$\times \exp\left\{\imath\left(S_{full}[\overline{\psi}(\tau), \psi(\tau)] - S_{full}[\overline{\psi}'(\tau), \psi'(\tau)]\right)\right\} \quad (45)$$

where $S_{full}\left[\overline{\psi}(x), \psi(x)\right]$ is given by

$$S_{full}\left[\overline{\psi}(\tau), \psi(\tau)\right] = \int \mathcal{L}_{fullvac}\left(\overline{\psi}(x), \psi(x)\right) dx \quad (46)$$

where Lagrangian density is

$$\mathcal{L}_{fullvac}\left(\overline{\psi}(x), \psi(x)\right) = \mathcal{L}_F\left(\overline{\psi}(x), \psi(x)\right) + \mathcal{L}_{vac}\left(\overline{\psi}(x), \psi(x)\right) \quad (47)$$

$$\mathcal{L}_F\left(\overline{\psi}(x), \psi(x)\right) = \overline{\psi}(x)\left(\imath\gamma^\mu \frac{\partial}{\partial x_\mu} - m + U(\vec{r})\right)\psi(x) \quad (48)$$

$$\mathcal{L}_{vac}\left(\overline{\psi}(x), \psi(x)\right) = \frac{e^2}{4\pi}\overline{\psi}(x)\gamma_\mu\psi(x)\int D^{\mu\nu}\left(x - x'\right) j_\nu(x')dx' \quad (49)$$

So we have effective Lagrangian

$$\mathcal{L} = \overline{\psi}(x)(\imath\gamma_\mu\partial^\mu - m)\psi(x) + \frac{e^2}{4\pi} j_\mu(x) \int D^{\mu\nu}(x - x')j_\nu(x')dx' \tag{50}$$

The Euler–Lagrange equation for this model is

$$(\imath\gamma_\mu\partial^\mu - m)\psi(x) + \frac{e^2}{4\pi}\left[\int D^{\mu\nu}(x - x')j_\nu(x')dx'\right]\gamma_\mu\psi(x) = 0 \tag{51}$$

where

$$D^{\mu\nu}(x - x') = \frac{1}{(2\pi)^3}\int \frac{2\pi\imath d\mathbf{k}}{\omega_{\mathbf{k}}}\left(\sum_\lambda \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\lambda\right) e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{\imath\omega(\tau-\tau')} \tag{52}$$

and

$$j_\nu(x') = \overline{\psi}(x')\gamma_\nu\psi(x') \tag{53}$$

We note that obtaibed equation is non-linear. We present equation (45) in two forms:

$$(\imath\gamma_\mu\partial^\mu - \mathcal{M})\,\psi(x) = 0 \tag{54}$$

where

$$\mathcal{M} = m - \frac{e^2}{4\pi}\left[\int D^{\mu\nu}(x-x')j_\nu(x')dx'\right]\gamma_\mu \tag{55}$$

Quantum transition amplitude of electron in this model presents as path integral

$$K\left(\psi(+\infty)|\psi(-\infty)\right)=\int \exp\left[\imath S_{full}\left(\overline{\psi}(x),\psi(x)\right)\right] D\overline{\psi(\tau)}D\psi(\tau) \tag{56}$$

We find an equation for $\psi(x)$ by the use of quasiclassal approximation [Feynman]

$$\delta S_{full}\left[\overline{\psi}(\tau),\psi(\tau)\right]=0 \tag{57}$$

We consider an electron in a bound state (with potential energy $U(\vec{r})$). We find an equation for $\psi(x)$ by the use of (57) in the following form

$$\left(\imath\gamma^{\mu}\frac{\partial}{\partial x_{\mu}}-m\right)\psi\left(x\right)+\frac{e^{2}}{4\pi}\gamma_{\mu}\psi(x)\int D^{\mu\nu}\left(x-x'\right)j_{\nu}\left(x'\right)dx'=0 \tag{58}$$

The equation (58) is nonlinear.

We write the equation in the form [2]

$$\imath\frac{\partial}{\partial t}\psi\left(\vec{r},t\right)=\left(c\vec{\alpha}\hat{p}+\beta mc^2+U\left(\vec{r}\right)\right)\psi\left(\vec{r},t\right)+\frac{e^2}{4\pi}\gamma_\mu\psi(x)\int D^{\mu\nu}\left(x-x'\right)j_\nu(x')dx' \tag{59}$$

We present $\psi(\vec{r},t)=e^{-\frac{\imath}{\hbar}Et}\psi_E(\vec{r})$.
Then

$$E\psi_E\left(\vec{r}\right)=\left(c\vec{\alpha}\hat{p}+\beta mc^2+U(\vec{r})\right)\psi_E(\vec{r})+\frac{e^2}{4\pi}\gamma_\mu\psi(x)\int D^{\mu\nu}\left(x-x'\right)j_\nu(x')dx' \tag{60}$$

The equation (60) allows us to find the energy E and wave function $\psi_E(r)$ of an electron with the influence of vacuum fluctuations.

[2] Quantum electrodynamics by A. I. Akhiezer, V. B. Berestetskii

In nonlinear equation (60) the second term is small. Then (60) we present as

$$E\psi_E(\vec{r}) = \left(c\vec{\alpha}\hat{p} + \beta mc^2 + U(\vec{r})\right)\psi_E(\vec{r}) \tag{61}$$

For a spherically symmetric potential $U = -\frac{Ze^2}{r}$ we have E_{nj} and ψ_{nj} where $j = l + 1/2$, l — orbital quantum number, n — principal quantum number.

$$E_{nj}\psi_{nj}(\vec{r}) = \left(c\vec{\alpha}\hat{p} + \beta mc^2 + U(\vec{r})\right)\psi_{nj}(\vec{r}) \tag{62}$$

We assume that the values found by solving the equation (62) are the first approximation in the solution of equation (60). We find electron energy with the influence of vacuum fluctuations from equation (60)

$$E_{nj}^{vac}\psi_{nj} = \left(c\vec{\alpha}\hat{p} + \beta mc^2 + U(r)\right)\psi_{nj} + \frac{e^2}{4\pi}\gamma_\mu\psi_{nj}(r,\theta,\phi)\int D^{\mu\nu}(x - x')\, j_\nu(x')dx' \tag{63}$$

We multiply the equation from the left by $\overline{\psi}(r,\theta,\phi)$ and using the equation (62). Then we integrate over r,θ,ϕ ($\int \overline{\psi}_{nj}(x)\psi_{nj}(x)dx = 1$). After that we find E_{nj}^{vac}:

$$E_{nj}^{vac} = E_{nj} + \frac{e^2}{4\pi}\int\int \overline{\psi}_{nj}(x)\gamma_\mu\psi_{nj}(x)D^{\mu\nu}\left(x - x'\right)j_\nu(x')dx' \tag{64}$$

The equation (64) describe electron energy E_{nj}^{vac} in quantum state $\psi_{nj}(x)$ with the influence of vacuum fluctuations. The corresponding energy shift has the following form

$$\Delta E_{nj}^{vac} = \frac{e^2}{4\pi}\int\int \overline{\psi}_{nj}(x)\gamma_\mu\psi_{nj}(x)D^{\mu\nu}\left(x - x'\right)j_\nu(x')dx' \tag{65}$$

Thanks for your attention!