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based on JETP Lett. 116 (2022) 1, 61-69

J.Phys.G 50 (2023) 1, 015001 [2203.09307 [hep-ph]]

Phys.Rev.D 107 (2023) 9, 094034 [2302.12171 [hep-ph]]

JETP Lett. 118 (2023) 7, 478-482 [2307.16225 [hep-ph]]

JETP Lett. 120 (2024) 11, 804-809 [2408.16804 [hep-ph]]

Int.J.Mod.Phys.A 40 (2025) 04, 2450175 [2404.01873 [hep-ph]]

Particles 8 (2025) 1, 29

The XXV International Workshop-School "High Energy Physics and Quantum Field Theory" (QFTHEP-270), Moscow, Russia, June 30–July 5, 2025

Analytic QCD: The Recent Results

0. History. QED.

Consider so-called polarization operator $D(k^2)$ in QED. Leading logarithmic terms of $D(k^2)$ in the n order of perturbation theory with $|k^2| >> m^2$ (m is the electron mass) have the following form:

$$(e^2F(K^2,m^2))^n/K^2,\ K^2=-k^2\geq 0,\ F(K^2,m^2)=\frac{1}{3\pi}\ln\left(\frac{K^2}{4m^2}\right).$$

Resummation of the large logarithms leads to (Landau,Abrikosov,Khalatnikov:1954):

$$D_{\rm per}(k^2)=\frac{1}{K^2}\frac{1}{1-\frac{e^2}{3\pi}\ln\left(\frac{K^2}{4m^2}\right)}.$$

Then, there is the pole *(so-called Landau pole)* at K_p^2:

$$K_p^2=4m^2e^{3\pi/e^2}$$

and QED is not applicable at $K^2\geq K_p^2$ (Landau,Pomeranchuk:1955).

With another side, there is so-called Kallen-Lehmann representation:

$$D(k^2) = \frac{1}{K^2} + \int_{4m^2}^{\infty} dz\, \frac{I(z)}{z + K^2}, \quad I(z) = ImD(i\varepsilon - K^2)$$

and $D_{\mathrm{per}}(k^2)$ is not in agreement with the Kallen-Lehmann representation.

Combination of the Kallen-Lehmann representation and perturbation theory *(or same, perturbation theory for $I(z)$)* has been considered in (Redmond:1958), (Redmond,Uretsky:1958), (Bogolyubov,Logunov,Shirkov:1959): $I(z) \to I_{\mathrm{per}}(z)$.

We follow (Bogolyubov,Logunov,Shirkov:1959).

From calculation (Landau,Abrikosov,Khalatnikov:1954) they obtained that $I_{\rm per}(z) = 0$ for $z < 4m^2$ and for $z \geq 4m^2$:

$$I_{\rm per}(z) = \frac{e^2}{3\pi z} \frac{1}{\left(\left(1 - \frac{e^2}{3\pi} \ln\left(\frac{z-4m^2}{4m^2}\right)\right)^2 + \frac{e^2}{9}\right)}\,.$$

Using $I_{\rm per}(z)$ in the Kallen-Lehmann representation they obtained at $|k^2| >> m^2$

$$D(k^2) = \frac{1}{K^2} \frac{1}{1 - \frac{e^2}{3\pi} \ln\left(\frac{K^2}{4m^2}\right)} + \frac{(3\pi)/e^2}{K^2 - K_p^2}\,.$$

The additional term cancels exacly Landau pole at $K^2 = K_p^2$. Moreover, it cannot be obtained in the framework of perturbation theory, since it cannot be expanded in e^2-series.

Thus, the combination of perturation theory and Kallen-Lehmann representation (i.e. perturbation theory for spectral function) does not lead to the Landau problem in QED.

In the general case the QCD couplant is defined as a product of propagators and a vertex function. Therefore, one might pose a question concerning the analytic properties of this quantity. This matter has been examined (Ginzburg,Shirkov:1965).

It was shown that in this case the integral representation of the Kallen-Lehmann type holds for the running coupling, too. Proceeding from these motivations, the analytic approach was lately extended to Quantum Chromodynamics by D.V. Shirkov and I.L. Solovtsov.

1. Analytic coupling constant

According to the general principles of (local) quantum field theory (QFT) (Bogolyubov,Shirkov:1959); (Oehme:1994) observables in the spacelike domain can have singularities only with negative values of their argument Q^2.

On the other hand, for large values of Q^2, these observables are usually represented as power series expansion by the running coupling constant (couplant) $\alpha_s(Q^2)$, which, in turn, has a ghost singularity, the so-called Landau pole, for $Q^2 = \Lambda^2$.

To restore analyticity, this pole must be removed.

1.1 Strong coupling constant

Strong couplant $\alpha_s(Q^2)$ obeys the renormalized group equation

$$L \equiv \ln \frac{Q^2}{\Lambda^2} = \int^{\overline{a}_s(Q^2)} \frac{da}{\beta(a)}, \quad \overline{a}_s(Q^2) = \frac{\alpha_s(Q^2)}{4\pi}, \quad a_s(Q^2) = \beta_0 \, \overline{a}_s(Q^2)$$

with some boundary condition and the QCD β-function:

$$\beta(a_s) \; = \; - \sum_{i=0} \beta_i \overline{a}_s^{i+2} = -\beta_0 \overline{a}_s^2 \, (1 + \sum_{i=1} b_i a_s^i), \quad b_i = \frac{\beta_i}{\beta_0^{i+1}} \, ,$$

where the first fifth coefficients, i.e. β_i with $i \leq 4$, are exactly known (Baikov,Chetyrkin,Kuhn: 2017).

So, already at leading order (LO), when $a_s(Q^2) = a_s^{(1)}(Q^2)$, we have

$$a_s^{(1)}(Q^2) = \frac{1}{L} \, ,$$

i.e. $a_s^{(1)}(Q^2)$ does contain a pole at $Q^2 = \Lambda^2$.

1.2 Beyond LO

Following (Cvetic, Valenzuela: 2006), we introduce here the derivatives (in the k-order of perturbation theory (PT))

$$\tilde{a}_{n+1}^{(k)}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n a_s^{(k)}(Q^2)}{(dL)^n}, a_s^{(k)}(Q^2) = \frac{\beta_0 \alpha_s^{(k)}(Q^2)}{4\pi} = \beta_0 \, \overline{a}_s^{(k)}(Q^2),$$

which are very convenient in the case of analytic QCD. β_0 is the first coefficient of the QCD β-function:

$$\beta(\overline{a}_s^{(k)}) = -\left(\overline{a}_s^{(k)}\right)^2 (\beta_0 + \sum_{i=1}^{k} \beta_i \left(\overline{a}_s^{(k)}\right)^i),$$

where β_i are known up to $k = 4$ (Baikov, Chetyrkin, Kuhn: 2008).

The series of derivatives $\tilde{a}_n(Q^2)$ can successfully replace the corresponding series of a_s-powers (see, e.g. (Kotikov, Zemlyakov: 2022)). Indeed, each derivative reduces the a_s power but is accompanied by an additional β-function $\sim a_s^2$. Thus, each application of a derivative yields an additional a_s, and thus it is indeed possible to use a series of derivatives instead of a series of a_s-powers.

In LO, the series of derivatives $\tilde{a}_n(Q^2)$ are exactly the same as a_s^n. Beyond LO, the relationship between $\tilde{a}_n(Q^2)$ and a_s^n was established in (Cvetic, Valenzuela: 2006), (Cvetic, Kogerler, Valenzuela: 2010) and extended to the fractional case, where $n \to$ is a non-integer ν, in (Cvetic, Kotikov: 2012)

Now we consider the $1/L$ expansion of $\tilde{a}_{\nu}^{(k)}(Q^2)$. After some calculatins, we have

$$\tilde{a}_{\nu,0}^{(1)}(Q^2) = \left(a_{s,0}^{(1)}(Q^2)\right)^{\nu} = \frac{1}{L_0^{\nu}},$$
$$\tilde{a}_{\nu,i}^{(i+1)}(Q^2) = \tilde{a}_{\nu,i}^{(1)}(Q^2) + \sum_{m=1}^{i} C_m^{\nu+m}\, \tilde{\delta}_{\nu,i}^{(m+1)}(Q^2),$$
$$\tilde{\delta}_{\nu,i}^{(m+1)}(Q^2) = \hat{R}_m \frac{1}{L_i^{\nu+m}}, \quad C_m^{\nu+m} = \frac{\Gamma(\nu+m)}{m!\Gamma(\nu)},$$

where

$$\hat{R}_1 = b_1[\hat{Z}_1(\nu) + \frac{d}{d\nu}], \quad \hat{R}_2 = b_2 + b_1^2[\frac{d^2}{(d\nu)^2} + 2\hat{Z}_1(\nu+1)\frac{d}{d\nu} + \hat{Z}_2(\nu+1)].$$

The representation of the $\tilde{\delta}_{\nu,i}^{(m+1)}(Q^2)$ corrections as $\hat{R}_m$-operators is very important to use. This will make it possible to present high-order results for the analytic couplant in a similar way.

Here

$$Z_2(\nu) = S_1^2(\nu) - S_2(\nu),$$
$$Z_1(\nu) \equiv S_1(\nu) = \Psi(1+\nu) + \gamma_{\rm E}, \quad S_2(\nu) = \zeta_2 - \Psi'(1+\nu),$$

and

$$S_m(N) = \sum_{k=1}^{N} \frac{1}{k^m}, \quad \hat{Z}_1(\nu) = Z_1(\nu) - 1, \quad \hat{Z}_2(\nu) = Z_2(\nu) - 2Z_1(\nu) + 1.$$

1.3. The case $\nu = 1$

For the case $\nu = 1$,

$$a_{s,0}^{(1)}(Q^2) = \frac{1}{L_0}, \quad a_{s,i}^{(i+1)}(Q^2) = a_{s,i}^{(1)}(Q^2) + \sum_{m=2}^{i} \delta_{s,i}^{(m)}(Q^2), \quad L_i = \ln \frac{Q^2}{\Lambda_i^2},$$

where the corrections $\delta_{s,k}^{(m)}(Q^2)$ can be represented as follows

$$\delta_{s,k}^{(2)}(Q^2) = -\frac{b_1 \ln L_k}{L_k^2}, \quad \delta_{s,k}^{(3)}(Q^2) = \frac{1}{L_k^3}\left[b_1^2(\ln^2 L_k - \ln L_k - 1) + b_2\right].$$

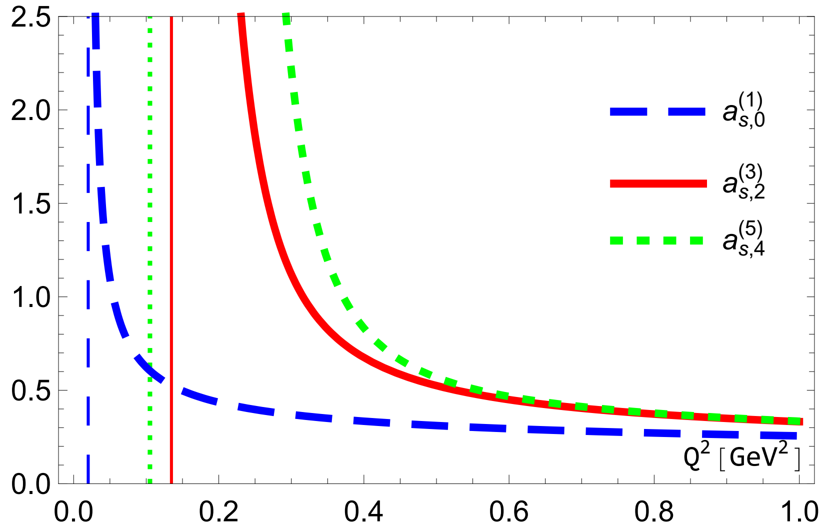

Figure 1: The results for $a_{s,i}^{(i+1)}(Q^2)$ and $(\Lambda_i^{f=3})^2$ (vertical lines) with $i = 0, 2, 4$.

In Fig. 1 one can see that the strong couplants $a_{s,i}^{(i+1)}(Q^2)$ become to be singular at $Q^2 = \Lambda_i^2$. The Λ_0 and Λ_i $(i \geq 1)$ values are rather different (Chen,Liu,Wang,Waqas,Peng: 2021):

$$\Lambda_0^{f=3} = 142 \;\; \text{MeV}, \;\; \Lambda_1^{f=3} = 367 \;\; \text{MeV}, \;\; \Lambda_2^{f=3} = 324 \;\; \text{MeV},$$
$$\Lambda_3^{f=3} = 328 \;\; \text{MeV}\,.$$

2. MA coupling

There are several ways to obtain analytical versions of the strong couplant a_s (see, e.g. (Bakulev: 2008)).

In a series of papers (Shirkov,Solovtsov: 1996,1997); (Milton,Solovtsov,Solovtsova: 1997); (Shirkov: 2001) authors have developed an effective approach to eliminate the Landau singularity without introducing extraneous IR regulators.

The idea: the dispersion relation, which connects the new analytic couplant $A_{\rm MA}(Q^2)$ with the spectral function $r_{\rm pt}(s)$, obtained in the framework of perturbative theory. In LO

$$A^{(1)}_{\rm MA}(Q^2) = \frac{1}{\pi} \int_0^{+\infty} \frac{ds}{(s+t)}\, r^{(1)}_{\rm pt}(s)\,, \quad r^{(1)}_{\rm pt}(s) = {\rm Im}\; a^{(1)}_s(-s - i\epsilon)\,,$$

So, let's repeat once again: the spectral function is taken directly from perturbation theory, but the analytic couplant $A_{\rm MA}(Q^2)$ is restored using dispersion relations.
This approach is called *Minimal Approach* (MA) (Cvetic, Valenzuela: 2008) or *Analytic Perturbation Theory* (APT) (Shirkov, Solovtsov:1996,1997); (Milton,Solovtsov,Solovtsova:1997); (Shirkov:2001)

Thus, MA QCD is a very convenient approach that combines the general (analytical) properties of quantum field quantities and the results obtained within the framework of perturbative QCD, leading to the appearance of the MA couplant $A_{\rm MA}(Q^2)$, which is close to the usual strong couplant $a_s(Q^2)$ in the limit of large values of its argument and completely different at $Q^2 \leq \Lambda^2$.

A further development of APT is the so-called fractional APT (FAPT), which extends the principles of constructing to non-integer powers of couplant, which arise for many quantities having non-zero anomalous dimensions (Bakulev,Mikhailov,Stefanis: 2005,2008,2010), with some privious study (Karanikas,Stefanis: 2001) and reviews (Bakulev: 2008), (Stefanis: 2013).

The results in FATP have a very simple form in LO perturbation theory, but they are quite complicated in higher orders.

2.1 LO

The LO minimal analytic coupling $A^{(1)}_{\mathrm{MA},\nu}$ have the form (Bakulev,Mikhailov,Stefanis: 2005)

$$A^{(1)}_{\mathrm{MA},\nu,0}(Q^2) = \left(a^{(1)}_{\nu,0}(Q^2)\right)^{\nu} - \frac{\mathrm{Li}_{1-\nu}(z_0)}{\Gamma(\nu)} \equiv \frac{1}{L_0^{\nu}} - \Delta^{(1)}_{\nu,0}\,,$$

where

$$\mathrm{Li}_{\nu}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^{\nu}} = \frac{z}{\Gamma(\nu)} \int_0^{\infty} \frac{dt\ t^{\nu-1}}{(e^t - z)}, \quad z_i = \frac{\Lambda_i^2}{Q^2}$$

is the Polylogarithmic function.

For $\nu = 1$ we recover the famous Shirkov-Solovtsov result (Shirkov, Solovtsov: 1996)

$$A^{(1)}_{\mathrm{MA},0}(Q^2) \equiv A^{(1)}_{\mathrm{MA},\nu=1,0}(Q^2) = a^{(1)}_{s,0}(Q^2) - \frac{z_0}{1-z_0} = \frac{1}{L_0} - \frac{z_0}{1-z_0}\,.$$

2.2 Beyond LO

Following to the LO analytic couplant, we consider the difference between the derivatives of usual and MA couplants:

$$\tilde{A}_{\mathrm{MA},n+1}(Q^2) = \frac{(-1)^n}{n!} \frac{d^n A_{\mathrm{MA}}(Q^2)}{(dL)^n} \,.$$

For the differences of fracted derivatives of usual and MA couplants

$$\tilde{\Delta}^{(i+1)}_{\nu,i} \equiv \tilde{a}^{(i+1)}_{\nu,i} - \tilde{A}^{(i+1)}_{\mathrm{MA},\nu,i}$$

we have the following results

$$\tilde{\Delta}^{(i+1)}_{\nu,i} = \tilde{\Delta}^{(1)}_{\nu,i} + \sum_{m=1}^{i} C^{\nu+m}_{m} \, \hat{R}_m \left(\frac{\mathrm{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right) \,,$$

where the operators $\hat{R}_i$ $(i = 1, 2, 3, 4)$ are shown above.

After some evaluations, we obtain

$$\tilde{\Delta}^{(i+1)}_{\nu,i} = \tilde{\Delta}^{(1)}_{\nu,i} + \sum_{m=1}^{i} C^{\nu+m}_{m} \, \overline{R}_m(z_i) \left(\frac{\mathrm{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right) \,,$$

where

$$\overline{R}_1(z) = b_1[\gamma_{\mathrm{E}} - 1 + \mathrm{M}_{-\nu,1}(z)],$$
$$\overline{R}_2(z) = b_2 + b_1^2[\mathrm{M}_{-\nu-1,2}(z) + 2(\gamma_{\mathrm{E}} - 1)\mathrm{M}_{-\nu-1,1}(z) + (\gamma_{\mathrm{E}} - 1)^2 - \zeta_2],$$

and

$$\mathrm{Li}_{\nu,k}(z) = (-1)^k \frac{d^k}{(d\nu)^k} \mathrm{Li}_\nu(z) = \sum_{m=1}^{\infty} \frac{z^m \ln^k m}{m^\nu}, \quad \mathrm{M}_{\nu,k}(z) = \frac{\mathrm{Li}_{\nu,k}(z)}{\mathrm{Li}_\nu(z)}.$$

So, we have for MA analytic couplants $\tilde{A}^{(i+1)}_{\mathrm{MA},\nu}$ the following expressions:

$$\tilde{A}^{(i+1)}_{\mathrm{MA},\nu,i}(Q^2) = \tilde{A}^{(1)}_{\mathrm{MA},\nu,i}(Q^2) + \sum_{m=1}^{i} C^{\nu+m}_{m} \tilde{\delta}^{(m+1)}_{\mathrm{MA},\nu,i}(Q^2)$$

where

$$\tilde{A}^{(1)}_{\mathrm{MA},\nu,i}(Q^2) = \tilde{a}^{(1)}_{\nu,i}(Q^2) - \frac{\mathrm{Li}_{1-\nu}(z_i)}{\Gamma(\nu)},$$

$$\tilde{\delta}^{(m+1)}_{\mathrm{MA},\nu,i}(Q^2) = \tilde{\delta}^{(m+1)}_{\nu,i}(Q^2) - \overline{R}_m(z_i)\,\frac{\mathrm{Li}_{-\nu+1-m}(z_i)}{\Gamma(\nu+m)}$$

and $\tilde{\delta}^{(k+1)}_{\nu,m}(Q^2)$ are given above.

There are three more representations for $\tilde{A}^{(1)}_{\mathrm{MA},\nu,i}(Q^2)$ (see (Kotikov, Zemlyakov: 2005)) that give exactly the same numerical results.
Each of the representations is useful in its own kinematic range.

2.3. The case $\nu = 1$

For the case $\nu = 1$,

$$A^{(i+1)}_{\mathrm{MA},i}(Q^2) \equiv \tilde{A}^{(i+1)}_{\mathrm{MA},\nu=1,i}(Q^2) = A^{(1)}_{\mathrm{MA},i}(Q^2) + \sum_{m=1}^{i} \tilde{\delta}^{(m+1)}_{\mathrm{MA},1,i}(Q^2)$$

where

$$A^{(1)}_{\mathrm{MA},i}(Q^2) = \tilde{a}^{(1)}_{\nu=1,i}(Q^2) - \mathrm{Li}_0(z_i) = a^{(1)}_{s,i}(Q^2) - \mathrm{Li}_0(z_i),$$

$$\tilde{\delta}^{(m+1)}_{\mathrm{MA},1,i}(Q^2) = \tilde{\delta}^{(m+1)}_{1,i}(Q^2) - \overline{R}_m(z_i)\,\frac{\mathrm{Li}_{-m}(z_i)}{m!}$$

and

$$\mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \quad \mathrm{Li}_{-2}(z) = \frac{z(1+z)}{(1-z)^3}.$$

The results can be used for phenomenological studies beyond LO in the framework of the minimal analytic QCD.

Here we apply the inverse logarithmic expansion of the MA couplants, recently obtained in (Kotikov, Zemlyakov: 2023) for any PT order. This approach is very convenient: for LO the MA couplants have simple representations (see (Bakulev, Mikhailov, Stefanis: 2007,2007,2010)), while beyond LO the MA couplants are very close to LO ones, especially for $Q^2 \to \infty$ and $Q^2 \to 0$, where the differences between MA couplants of various PT orders become insignificant. Moreover, for $Q^2 \to \infty$ and $Q^2 \to 0$ the (fractional) derivatives of the MA couplants with $n \geq 2$ tend to zero, and therefore only the first term in perturbative expansions makes a valuable contribution.

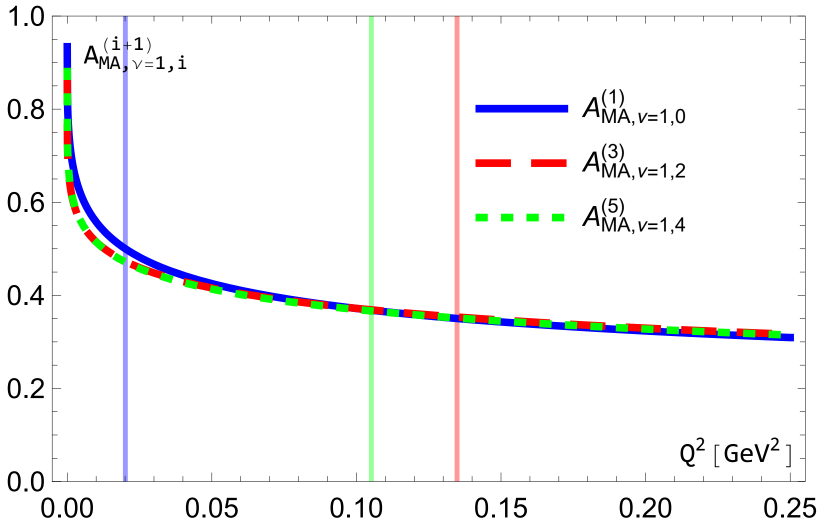

Figure 2: The results for $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ with $i = 0, 2, 4$.

From Fig. 2 we can see differences between $A^{(i+1)}_{\mathrm{MA},\nu=1,\mathrm{i}}(Q^2)$ with $i = 0, 2, 4$, which are rather small and have nonzero values around the position $Q^2 = \Lambda_i^2$.

2.4. Minimal analytic coupling. Another form

The above results for analytic couplant is very convenient at the range at large and at small Q^2 values. For $Q^2 \sim \Lambda_i^2$ the both part, standard ones and additional ones have singularities, which are cancelled in its sum. So, some numerical applications of these results can be complicated. So, here we present another form, which is very useful at $Q^2 \sim \Lambda_i^2$ and can be used also for any Q^2 values, excepting the ranges of very large and very small Q^2 values. As in the previous section, we will present firstly LO results taken from (Bakulev,Mikhailov,Stefanis: 2005)
and later to extend them beyond LO.

2.4.1 LO

The LO minimal analytic coupling $A^{(1)}_{\mathrm{MA},\nu}(Q^2)$ (Shirkov, Solovtsov: 1996), (Milton,Solovtsov,Solovtsova: 1997), (Shirkov: 2001)

have also the another form (Bakulev,Mikhailov,Stefanis: 2005)

$$A^{(1)}_{\mathrm{MA},\nu}(Q^2) = \frac{(-1)}{\Gamma(\nu)} \sum_{r=0}^{\infty} \zeta(1-\nu-r)\, \frac{(-L)^r}{r!} \quad (L < 2\pi),$$

where Euler functions $\zeta(\nu)$ are

$$\zeta(\nu) = \sum_{m=1}^{\infty} \frac{1}{m^{\nu}} = \mathrm{Li}_{\nu}(z=1)$$

This result considering the property of the Lerch function, which can be considered as a generalization of Polylogarithms. This form is very convenient at low L values, t.e. at $Q^2 \sim \Lambda^2$. Moreover, we can use the relation between $\zeta(1-\nu-r)$ and $\zeta(\nu+r)$ functions

$$\zeta(1-\nu-r) = \frac{2\Gamma(\nu+r)}{(2\pi)^{\nu+r}}\, Sin\left[\frac{\pi}{2}(1-\nu-r)\right] \zeta(\nu+r)$$

For $\nu = 1$ we have

$$A_{\mathrm{MA}}^{(1)}(L) = - \sum_{r=0}^{\infty} \zeta(-r) \frac{(-L)^r}{r!}$$

with

$$\zeta(-r) = (-1)^r \frac{B_{r+1}}{r+1}$$

and B_{r+1} are Bernoulli numbers.

Using the properties of Bernoulli numbers (δ_m^0 is Croneker symbol), we have for even $r = 2m$ and for odd $r = 1 + 2l$ values

$$\zeta(-2m) = -\frac{\delta_m^0}{2}, \quad \zeta(-(1+2l)) = -\frac{B_{2(l+1)}}{2(l+1)} .$$

Thus, we have for $A_{\mathrm{MA}}^{(1)}(Q^2)$ the following results

$$A_{\mathrm{MA}}^{(1)}(Q^2) = \frac{1}{2} \left(1 + \sum_{l=0}^{\infty} \frac{B_{2(l+1)}}{l+1} \frac{(-L)^{2l+1}}{(2l+1)!} \right) = \frac{1}{2} \left(1 + \sum_{s=1}^{\infty} \frac{B_{2s}}{s} \frac{(-L)^{2s-1}}{(2s-1)!} \right) ,$$

with $s = l + 1$.

2.4.2 Beyond LO

Now we consider the derivatives of MA couplants $\tilde{A}^{(1)}_{\mathrm{MA},\nu}$,

$$\tilde{A}^{(i+1)}_{\mathrm{MA},\nu,i}(Q^2) = \tilde{A}^{(1)}_{\mathrm{MA},\nu,i}(Q^2) + \sum_{m=1}^{i} C_m^{\nu+m} \tilde{\delta}^{(m+1)}_{\mathrm{MA},\nu,i}(Q^2) \qquad (1)$$

where $\tilde{A}^{(1)}_{\mathrm{MA},\nu,i} = A^{(1)}_{\mathrm{MA},\nu}$ with $L \to L_i$ and

$$\tilde{\delta}^{(m+1)}_{\mathrm{MA},\nu,i}(Q^2) = \hat{R}_m \, A^{(1)}_{\mathrm{MA},\nu+m,i} \, , \qquad (2)$$

where operators $\hat{R}_m$ are given above.

After some calculations, we have

$$\tilde{\delta}^{(m+1)}_{\mathrm{MA},\nu,k}(Q^2) = \frac{(-1)}{\Gamma(\nu+m)} \sum_{r=0}^{\infty} \tilde{R}_m(\nu+r) \frac{(-L_k)^r}{r!} \qquad (3)$$

where

$$\tilde{R}_1(\nu+r) = b_1[(\gamma_{\mathrm{E}}-1)\zeta(-\nu-r) + \zeta_1(-\nu-r)],$$
$$\tilde{R}_2(\nu+r) = b_2\zeta(-\nu-r-1) + b_1^2[\zeta_2(-\nu-r-1) + 2(\gamma_{\mathrm{E}}-1)\zeta_1(-\nu-r-1)$$
$$+[(\gamma_{\mathrm{E}}-1)^2 - \zeta_2]\zeta(-\nu-r-1)],$$

and

$$\zeta_k(\nu) = \mathrm{Li}_{\nu,k}(z=1) = \sum_{m=1}^{\infty} \frac{\ln^k m}{m^\nu} \, .$$

Strictly speaking, the functions $\zeta_n(-m-\nu-r-k)$ are not so good defined at large r values and we can replace them by $\zeta_n(m+\nu+r+k)$. However, the results are long and can be found in (Kotikov,Zemlyakov: 2022).

At the point $L_k = 0$, i.e. $Q^2 = \Lambda_k^2$, we have

$$A_{\mathrm{MA}}^{(1)} = \frac{1}{2}, \quad \delta_s^{(2)} = \frac{2}{(2\pi)^2} Q_{1a}(2) = -\frac{b_1}{2\pi^2} \left(\zeta_1(2) + l\zeta(2)\right),$$

$$\delta_s^{(3)} = -\frac{\pi}{(2\pi)^3} Q_{2b}(3) = \frac{b_1^2}{4\pi^2} \left(\zeta_1(3) + (2l-1)\zeta(3)\right),$$

where $\zeta_k(\nu)$ are given abobe and

$$l = \ln(2\pi) \, .$$

2.5 Integral representations for minimal analytic coupling

As already discussed in Introduction, the MA couplant $A^{(1)}_{\rm MA}(Q^2)$ is constructed as follows:
the LO spectral function $r^{(1)}_{\rm pt}(s)$ is taken directly from perturbation theory but the MA couplant $A^{(1)}_{\rm MA}(Q^2)$ itself was built using the correct integration counter (see also Introduction):

$$A^{(1)}_{\rm MA}(Q^2) = \frac{1}{\pi} \int_0^{+\infty} \frac{ds}{(s+t)} r^{(1)}_{\rm pt}(s)\,, \quad r^{(1)}_{\rm pt}(s) = {\rm Im}\; a^{(1)}_s(-s-i\epsilon)\,.$$

For the ν-derivative of $A^{(1)}_{\rm MA}(Q^2)$, i.e. $\tilde{A}^{(1)}_{\rm MA,\nu}(Q^2)$, (Cvetic,Kotikov: 2012)

$$\tilde{A}^{(1)}_{\rm MA,\nu}(Q^2) = \frac{(-1)}{\Gamma(\nu)} \int_0^{\infty} \frac{ds}{s} r^{(1)}_{\rm pt}(s) {\rm Li}_{1-\nu}(-sz)\,,$$

where ${\rm Li}_{1-\nu}(-sz)$ is the Polylogarithmic function.

Above MA case, it is correct too:

$$\tilde{A}^{(1)}_{\mathrm{nMA},\nu}(Q^2) = \frac{(-1)}{\Gamma(\nu)} \int_0^\infty \frac{ds}{s} r^{(1)}(s) \mathrm{Li}_{1-\nu}(-sz)\,,$$

Beyond LO, this representation can be extended in two ways, which will be shown in following subsections.

2.5.1 Modification of spectral functions

The first possibility is the modification of the spectral function:

$$\tilde{A}^{(i+1)}_{\mathrm{MA},\nu,k}(Q^2) = \frac{(-1)}{\Gamma(\nu)} \int_0^\infty \frac{ds}{s} r^{(i+1)}_{\mathrm{pt}}(s) \mathrm{Li}_{1-\nu}(-sz_k)\,,$$

i.e. it is similar to the LO one with the replacement the LO spectral function $r^{(1)}_{\mathrm{pt}}(s)$ by $i+1$-order one $r^{(i+1)}_{\mathrm{pt}}(s)$:

$$r^{(i+1)}_{\mathrm{pt}}(s) = r^{(1)}_{\mathrm{pt}}(s) + \sum_{m=1}^{i} \delta^{(m+1)}_{\mathrm{r}}(s)$$

and (Nesterenko,C.Simolo: 2010), (Nesterenko: 2017)

$$y = \ln s,\ \ r_{\rm pt}^{(1)}(y) = \frac{1}{y^2+\pi^2},$$
$$\delta_{\rm r}^{(2)}(y) = -\frac{b_1}{(y^2+\pi^2)^2}\left[2yf_1(y) + (\pi^2 - y^2)f_2(y)\right],$$
$$\delta_{\rm r}^{(3)}(y) = \frac{b_1^2}{(y^2+\pi^2)^3}\Big[(3y^2-\pi^2)\{\frac{b_2}{b_1^2} + f_1(y)(f_1(y)-1) - \pi^2 f_2^2(y) - 1\}$$
$$-y(y^2-3\pi^2)f_2(y)(2f_1(y)-1)\Big],$$

with

$$f_1(y) = \frac{1}{2}\ln(y^2+\pi^2), \quad f_2(y) = \frac{1}{2} - \frac{1}{\pi}arctan\left(\frac{y}{\pi}\right).$$

For the couplant itself, we have

$$A_{\rm MA,k}^{(i+1)}(Q^2) \equiv \tilde{A}_{{\rm MA},\nu=1,k}^{(i+1)}(Q^2) = \int_0^{+\infty} \frac{ds\, r_{\rm pt}^{(i+1)}(s)}{(s+t_k)}.$$

2.5.2 Modification of Polylogaritms

Beyond LO, the LO results can be extended also by using the $\hat{R}_m$ operators.

Here, the application of the operators $\hat{R}_m$ for the LO results leads to the following result:

$$\tilde{A}^{(i+1)}_{\mathrm{MA},\nu,i}(Q^2) = \int_0^{\infty} \frac{ds}{s} r^{(1)}_{\mathrm{pt}}(s) \tilde{\Delta}^{(i+1)}_{\nu,i} ,$$

where the results for $\tilde{\Delta}^{(i+1)}_{\nu,i}$ can be found above. They are

$$\tilde{\Delta}^{(i+1)}_{\nu,i} = \tilde{\Delta}^{(1)}_{\nu,i} + \sum_{m=1}^{i} C^{\nu+m}_{m} \hat{R}_m \left(\frac{\mathrm{Li}_{-\nu-m+1}(z_i)}{\Gamma(\nu+m)} \right) ,$$

where the operators $\hat{R}_i$ $(i = 1, 2, 3, 4)$ are shown above.

For MA couplant itself, we have beyond LO

$$A^{(i+1)}_{\mathrm{MA,i}}(Q^2) \equiv \tilde{A}^{(i+1)}_{\mathrm{MA},\nu=1,i}(Q^2) = \int_0^{+\infty} \frac{ds}{s} r^{(1)}_{\mathrm{pt}}(s) \tilde{\Delta}^{(i+1)}_{\nu=1,i} .$$

It can be extended to nonMinimal case as

$$\tilde{A}^{(i+1)}_{\mathrm{nMA},\nu,i}(Q^2) = \int_0^\infty \frac{ds}{s} r^{(1)}(s) \tilde{\Delta}^{(i+1)}_{\nu,i} \,,$$

2.6. Other models of analytic couplant

MA coupling (Shirkov,Solovtsov: 1996,1997); (Milton,Solovtsov,Solovtsova: 1997); (Shirkov: 2001):

$$A^{(1)}_{SS}(Q^2) = \frac{1}{L} + \frac{1}{1-t}, \quad t = \frac{Q^2}{\Lambda^2}, \quad L = ln(t) \,,$$

with the infrared finite value:

$$A^{(1)}_{SS}(0) = 1 \,.$$

There are several other models of analytic couplant. We will show a few of them.

1. Let us start with the model developed by (Alekseev,Arbuzov:1998), (Alekseev:1998). By making use of a special solution to the Schwinger-Dyson equations for the gluon propagator, these authors proposed the following expression for the QCD running coupling

$$A_{AA}^{(1)}(Q^2) = \frac{1}{L} + \frac{1}{1-t} + \frac{c}{t} + \frac{1-c}{t+m_g^2/\Lambda^2},$$

where m_g is the gluon mass and c denotes a dimensionless parameter fixed by the phenomenological value of the gluon condensate. The running coupling has enhancement in the infrared domain.

2. Another similar model for the QCD analytic coupland:

$$A^{(1)}_{Latt}(Q^2) = \frac{1}{L} + \frac{1}{1-t} + \frac{\nu}{t},$$

cames from an analysis of the lattice simulation data on the lowenergy behavior of the QCD coupland (Boucaud et al.:2000), (Burgio et al.:2002). The model also possesses the infrared enhancement.

3. By making use of a certain phenomenological reasoning, Webber suggested the coupland of the following form (Webber:1998)

$$A^{(1)}_{W}(Q^2) = \frac{1}{L} + \frac{1}{1-t}\frac{t+b}{1+b}\left(\frac{1+c}{t+c}\right)^p,$$

with a specific choice of the parameters: $b = 1/4$, $c = 4$, $p = 4$. This model has infrared finite value:

$$A^{(1)}_{W}(0) = \frac{1}{2}.$$

4. Nesterenko model (Nesterenko:2000,2001) at LO:

$$\frac{d\ln[A_N^{(1)}(Q^2)]}{d\ln Q^2} = -A_{MA}^{(1)}(Q^2) = -\frac{1}{\pi}\int_0^{+\infty} \frac{ds}{(s+t)}\, r_{\rm pt}^{(1)}(s)\,,$$

that leads to

$$A_N^{(1)}(Q^2) = \frac{t-1}{tL}$$

The model also possesses the infrared enhancement.

5. A generalization of the Nesterenko model (Srivastava et al.:2001) $(0 < p \leq 1)$:

$$A_{SPPW}^{(1)}(Q^2) = \left[\frac{1}{A_{SPPW}^{(1)}(\Lambda^2)} + \int_0^{\infty} d\sigma\, \frac{(t-1)t^p}{(\sigma+t)((\sigma+1)(1+t^p)}\right]^{-1},$$

which equal to $A_N^{(1)}(Q^2)$, when $p = 1$. The model also possesses the infrared enhancement.

5. 2δ and 2δ models of analytic QCD (Ayala,Contreras,Cvetic:2012), (Ayala,Cvetic:2015), (Ayala,Cvetic,Kogerler:2017):

$$r_{\rm pt}^{(1)}(\sigma) \to r_n^{(1)}(\sigma) = \pi \sum_{j=1}^{n} F_j \, \delta(\sigma - M_j^2) + \theta(\sigma - M_0^2) \, r_{\rm pt}^{(1)}(\sigma) \, ,$$

where $M_1^2 < M_2^2 < ... < M_n^2 < M_0^2$ and F_j are some constants. Here $(n=2)$ and $(n=3)$ for 2δ and 2δ models.

3. Bjorken sum rule

The polarized (nonsinglet) BSR is defined as the difference between the proton and neutron polarized SFs, integrated over the entire interval x

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 dx\,[g_1^p(x,Q^2) - g_1^n(x,Q^2)].$$

Theoretically, the quantity can be written in the OPE form (Shuryak, Vainshtein: 1982), (Balitsky, Braun, Kolesnichenko: 1990)

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}\left(1 - D_{\rm BS}(Q^2)\right) + \sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}}\,,$$

where g_A=1.2762 $\pm$ 0.0005 (PDG: 2020) is the nucleon axial charge, $(1{-}D_{BS}(Q^2))$ is the leading-twist contribution, and μ_{2i}/Q^{2i-2} $(i \geq 1)$ are the higher-twist (HT) contributions.

Since we include very small Q^2 values here, this representation of the HT contributions is inconvenient. It is much better to use the so-called "massive" representation for the HT part (introduced in (Teryaev: 2013), (Khandramai, Teryaev, Gabdrakhmanov: 2016)):

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}(1 - D_{\rm BS}(Q^2)) + \frac{\hat{\mu}_4 M^2}{Q^2 + M^2},$$

where the values of $\hat{\mu}_4$ and M^2 have been fitted in (Ayala et al.: 2018) in the different analytic QCD models.

Up to the k-th PT order, the perturbative part has the form

$$D_{\rm BS}^{(1)}(Q^2) = \frac{4}{\beta_0} a_s^{(1)}, D_{\rm BS}^{(k\geq 2)}(Q^2) = \frac{4}{\beta_0} a_s^{(k)} \left(1 + \sum_{m=1}^{k-1} d_m (a_s^{(k)})^m\right),$$

where d_1, d_2 and d_3 are known from exact calculations. The exact d_4 value is not known, but it was recently estimated in (Ayala, Pineda: 2022))

Converting the powers of couplant into its derivatives, we have

$$D^{(1)}_{\rm BS}(Q^2) = \frac{4}{\beta_0}\,\tilde{a}_1^{(1)}\,, D^{(k\geq 2)}_{\rm BS}(Q^2) = \frac{4}{\beta_0}\,\left(\tilde{a}_1^{(k)} + \sum_{m=2}^{k}\tilde{d}_{m-1}\tilde{a}_m^{(k)}\right),$$

where $b_i = \beta_i/\beta_0^{i+1}$ and

$$\tilde{d}_1 = d_1, \quad \tilde{d}_2 = d_2 - b_1 d_1, \quad \tilde{d}_3 = d_3 - \frac{5}{2} b_1 d_2 - (b_2 - \frac{5}{2} b_1^2)\, d_1,$$
$$\tilde{d}_4 = d_4 - \frac{13}{3} b_1 d_3 - (3b_2 - \frac{28}{3} b_1^2)\, d_2 - (b_3 - \frac{22}{3} b_1 b_2 + \frac{28}{3} b_1^3)\, d_1\,.$$

For the case of 3 active quark flavors ($f=3$), we have

$$d_1 = 1.59, \quad d_2 = 3.99, \quad d_3 = 15.42 \quad d_4 = 63.76,$$
$$\tilde{d}_1 = 1.59, \quad \tilde{d}_2 = 2.73, \quad \tilde{d}_3 = 8.61, \quad \tilde{d}_4 = 21.52\,,$$

i.e., the coefficients in the series of derivatives are slightly smaller.

In MA QCD, the results for BSR become as follows

$$\Gamma_{\mathrm{MA},1}^{p-n}(Q^2) = \frac{g_A}{6}\,(1 - D_{\mathrm{MA,BS}}(Q^2)) + \frac{\hat{\mu}_{\mathrm{MA},4}M^2}{Q^2+M^2},$$

where the perturbative part $D_{\mathrm{BS,MA}}(Q^2)$ takes the form

$$\begin{aligned} D_{\mathrm{MA,BS}}^{(1)}(Q^2) &= \frac{4}{\beta_0}\,A_{\mathrm{MA}}^{(1)}, \\ D_{\mathrm{MA,BS}}^{k\geq 2}(Q^2) &= \frac{4}{\beta_0}\,(A_{\mathrm{MA}}^{(k)} + \sum_{m=2}^{k}\,\tilde{d}_{m-1}\,\tilde{A}_{\mathrm{MA},\nu=\mathrm{m}}^{(k)})\,. \end{aligned}$$

4. Results

The fitting results of experimental data obtained only with statistical uncertainties are presented in Table 1 and shown in Figs. 3 and 4. For the fits we use Q^2-independent M^2 and $\hat{\mu}_4$ and the two-twist parts for regular PT and APT, respectively.

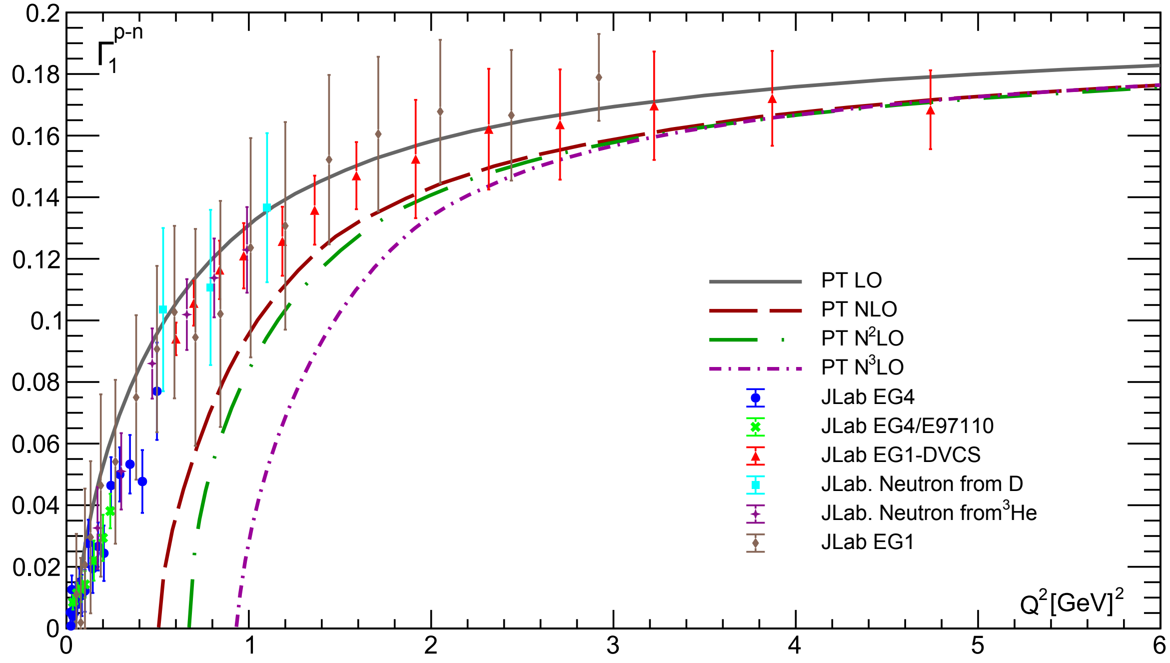

Figure 3: The results for $\Gamma_1^{p-n}(Q^2)$ in the first four orders of PT.

As it can be seen in Fig. 3, with the exception of LO, the results obtained using conventional couplant are very poor. Moreover, the discrepancy in this case increases with the order of PT.

The LO results describe experimental points relatively well, since the value of $\Lambda_{\rm LO}$ is quite small compared to other Λ_i, and disagree-

	M^2	$\hat{\mu}_{\mathrm{MA},4}$	$\chi^2/(\mathrm{d.o.f.})$
LO	0.472 $\pm$ 0.035	-0.212 $\pm$ 0.006	0.667
NLO	0.414 $\pm$ 0.035	-0.206 $\pm$ 0.008	0.728
$\mathrm{N^2LO}$	0.397 $\pm$ 0.034	-0.208$\pm$ 0.008	0.746
$\mathrm{N^3LO}$	0.394 $\pm$ 0.034	-0.209 $\pm$ 0.008	0.754
$\mathrm{N^4LO}$	0.397 $\pm$ 0.035	-0.208 $\pm$ 0.007	0.753

Table 1: The values of the fit parameters.

ment with the data begins at lower values of Q^2.

Thus, using the "massive" twist-four form does not improve these results, since with $Q^2 \to \Lambda_i^2$ conventional couplants become singular, which leads to large and negative results for the twist-two part (see above). So, as the PT order increases, ordinary couplants become singular for ever larger Q^2 values, while BSR tends to negative values for ever larger Q^2 values.

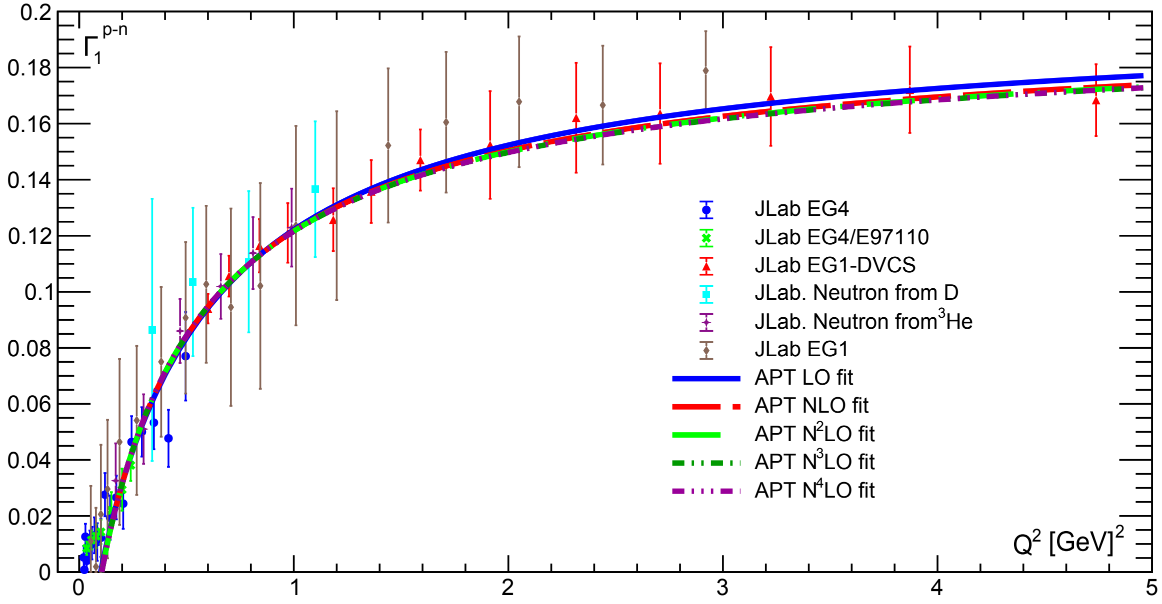

Figure 4: The results for $\Gamma_1^{p-n}(Q^2)$ in the first four orders of APT.

In contrast, our results obtained for different APT orders are practically equivalent: the corresponding curves become indistinguishable when Q^2 approaches 0 and slightly different everywhere else. As can be seen in Fig. 4, the fit quality is pretty high, which is demonstrated by the values of the corresponding $\chi^2/(\mathrm{d.o.f.})$ (see Table 1).

5. Conclusions

In this talk, we have focused on the introduction of the Shirkov-Solovtsov and Bakulev-Mikhailov-Stefanis approaches and their recent extension beyond the leading order of perturbation theory.

We have considered $1/L$-expansions of the ν-derivatives of the strong couplant a_s expressed as combinations of operators $\hat{R}_m$ applied to the LO couplant $a_s^{(1)}$.

Applying these operators to the ν-derivatives of the LO MA couplant $A_{\rm MA}^{(1)}$, we have got representations for the ν-derivatives of the MA couplant: $\tilde{A}_{\rm MA,\nu}^{(i)}$, i.e. , in each i-order of PT.

The high-order corrections are negligible in the $Q^2 \to 0$ and $Q^2 \to \infty$ asymptotics and are nonzero in a neighborhood of the point $Q^2 = \Lambda^2$. Thus, in fact, they are really only small corrections to the LO MA couplant $\mathcal{A}^{(1)}_{\mathrm{MA},\nu}(Q^2)$.

All our results have a compact form and do not contain complicated special functions, such as the Lambert W-function (Magradze: 1999), which already appears in two-loop order as an exact solution to the usual couplant and which was used to estimate the MA couplants in (Bakulev,Mikhailov,Stefanis: 2010).

I would like to point out that I have shown only one type of representations for the MA couplant of $A_{\rm MA}(Q^2)$ in the space-like domain. There are three other types of representations for $A_{\rm MA}(Q^2)$, but they are beyond the scope of the talk. I would like to note also that two of them are in integral form and can be extended to the non-minimal case, where the corresponding spectral functions become non-purely perturbative.

There are also two types of representations for $U_{\rm MA}(s)$ in the time-like domain. One of them is in integral form and can be extended to the non-minimal case.

As a example, we considered the Bjorken sum rule and obtained results similar to previous studies in (Pasechnik,Shirkov,Teryaev,Solovtsova,Khandramai: 2008,2009,2011), (Ayala,Cvetic,Kotikov,Shaikhatdenov: 2018)

The results based on usual perturbation theory do not not agree with the experimental data at $Q^2 \leq 1.5$ GeV^2. MA APT leads to good agreement with the data when we used the "massive" version for high-twist contributions.