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Some all-loop results for the renormalization of supersymmetric theories

Supersymmetry and high energy physics

Investigating of quantum corrections can shed a light to the structure of the surrounding world. For instance, the very precise agreement of the theoretical prediction of the electron anomalous magnetic moment with the experimental data tells us that nature is described by quantum field theory.

The unification of running couplings and absence of divergent quantum corrections to the Higgs boson mass can be considered indirect indications to the existence of supersymmetry and Grand Unification.

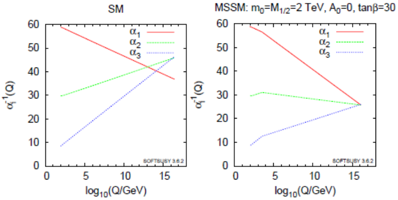


Figure 94.1: Running couplings in SM and MSSM using two-loop RG evolution. The SUSY threshold at 2 TeV is clearly visible on the MSSM side. (We thank Ben Allanach for providing the plots created using SOFTSUSY [61].)

Some important information about new physics can be obtained from the detailed analysis of quantum corrections to (light) Higgs boson in supersymmetric theories, anomalous magnetic moment of muon, etc.

$\mathcal{N} = 1$ supersymmetric gauge theories are described by the manifestly supersymmetric action

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\, d^2\theta\, W^a W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\, d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \mbox{c.c.}\Big\}.$$

Here V is the gauge superfield, ϕ_i are the chiral matter superfields in the representation R of the gauge group G, and

$$W_a = \frac{1}{8}\bar{D}^2\Big(e^{-2V}D_a e^{2V}\Big)$$

is the supersymmetric gauge field strength.

The gauge invariant theory is obtained if the Yukawa couplings and masses satisfy the constraints

$$m_0^{im}(T^A)_m{}^j + m_0^{mj}(T^A)_m{}^i = 0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k + \lambda_0^{imk}(T^A)_m{}^j + \lambda_0^{mjk}(T^A)_m{}^i = 0,$$

where $(T^A)_i{}^j$ are the generators of the gauge group G in the representation R.

In $\mathcal{N}=1$ supersymmetric gauge theories the β-function is related to the anomalous dimensions of the matter superfields by the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) equation

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381; Phys. Lett. **166B**(1986), 329;
D. R. T. Jones, Phys. Lett. **123B** (1983), 45;
M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986), 456.

For a general $\mathcal{N}=1$ supersymmetric gauge theory with a single gauge coupling it can be written in the form

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2\delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

For the pure $\mathcal{N}=1$ supersymmetric Yang–Mills (SYM) theory this relation gives the exact β-function in the form of the geometric series.

Scheme dependence of the NSVZ equation

The NSVZ equation is valid only for certain renormalization prescriptions. For instance, in the $\overline{\text{DR}}$-scheme the NSVZ equation is not valid starting from the order $O(\alpha^4)$ (the three-loop approximation for the β-function and the two-loop approximation for the anomalous dimension)

I. Jack, D. R. T. Jones and C. G. North, Phys.Lett. B **386** (1996) 138; Nucl.Phys. B **486** (1997) 479.

The all-loop derivation of the exact NSVZ β-function by direct summation of the perturbative series and the construction of the all-loop prescription giving the NSVZ scheme have been done with the help of the higher covariant derivative regularization, proposed by A. A. Slavnov

A.A.Slavnov, Nucl.Phys. **B31**, (1971), 301;
Theor. Math. Phys. **13** (1972), 1064; **33** (1977), 977.

(Unlike dimensional reduction) this regularization is sefconsistent. It can be formulated in terms of $\mathcal{N}=1$ superfields and does not break supersymmetry

V. K. Krivoshchekov, Theor. Math. Phys. **36** (1978), 745;
P.West, Nucl.Phys. B268, (1986), 113.

The higher covariant derivative regularization

For constructing the regularized theory we first add to its action terms with higher derivatives,

$$S_{\text{reg}} = \frac{1}{2e_0^2}\,\text{Re}\,\text{tr}\int d^4x\, d^2\theta\, W^a\left(e^{-2\boldsymbol{V}} e^{-2\mathcal{F}(V)}\right)_{Adj} R\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big)_{Adj}$$

$$\times\left(e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}\right)_{Adj} W_a + \frac{1}{4}\int d^4x\, d^4\theta\, \phi^{*i}\Big[F\Big(-\frac{\bar\nabla^2\nabla^2}{16\Lambda^2}\Big) e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}\Big]_i{}^j\phi_j$$

$$+\Big[\int d^4x\, d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j + \frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big) + \text{c.c.}\Big],$$

where the covariant derivatives are defined as

$$\nabla_a = D_a; \qquad \bar\nabla_{\dot a} = e^{2\mathcal{F}(V)} e^{2\boldsymbol{V}}\,\bar D_{\dot a} e^{-2\boldsymbol{V}} e^{-2\mathcal{F}(V)}.$$

Gauge is fixed by adding the term

$$S_{\text{gf}} = -\frac{1}{16\xi_0 e_0^2}\,\text{tr}\int d^4x\, d^4\theta\, \boldsymbol{\nabla}^2 V K\Big(-\frac{\bar{\boldsymbol{\nabla}}^2\boldsymbol{\nabla}^2}{16\Lambda^2}\Big)_{Adj}\bar{\boldsymbol{\nabla}}^2 V.$$

It is also necessary to introduce the Faddeev-Popov and Nielsen–Kalosh ghosts. The regulator functions $R(x)$, $F(x)$, and $K(x)$ should rapidly increase at infinity and satisfy the condition $R(0) = F(0) = K(0) = 1$.

For regularizing the residual one-loop divergences we insert into the generating functional two Pauli–Villars determinants,

$$Z = \int D\mu\, \mbox{Det}(PV, M_\varphi)^{-1} \mbox{Det}(PV, M)^c \times \exp\Big\{i\Big(S_{\mbox{reg}} + S_{\mbox{gf}} + S_{\mbox{FP}} + S_{\mbox{NK}} + S_{\mbox{sources}}\Big)\Big\},$$

where $D\mu$ is the functional integration measure, and

$$\mbox{Det}(PV, M_\varphi)^{-1} \equiv \int D\varphi_1\, D\varphi_2\, D\varphi_3\, \exp(iS_\varphi);$$

$$\mbox{Det}(PV, M)^{-1} \equiv \int D\Phi\, \exp(iS_\Phi).$$

(Here we use chiral commuting Pauli–Villars superfields.)

The superfields $\varphi_{1,2,3}$ belong to the adjoint representation and cancel one-loop divergences coming from gauge and ghost loops. The superfields Φ_i lie in a representation $R_{\mbox{PV}}$ and cancel divergences coming from a loop of the matter superfields if $c = T(R)/T(R_{\mbox{PV}})$. The masses of these superfields are

$$M_\varphi = a_\varphi \Lambda; \qquad M = a\Lambda,$$

where the coefficients a_φ and a do not depend on couplings.

The β-function as an integral of double total derivatives

The NSVZ equation naturally appears with the higher derivative regularization because in this case the integrals giving the β-function defined in terms of the bare couplings

$$\beta(\alpha_0,\lambda_0) \equiv \frac{d\alpha_0}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}$$

are integrals of double total derivatives with respect to the loop momenta

A.V.Smilga, A.I.Vainshtein, Nucl.Phys. **B 704** (2005) 445;
K.S., Nucl.Phys. **B 852** (2011) 71; JHEP **10** (2019) 011.

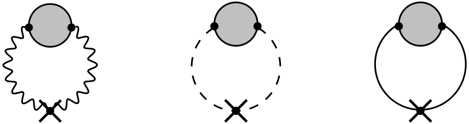
This can be seen even in the one-loop approximation

S. S. Aleshin, A. E. Kazantsev, M. B. Skoptsov, K.S., JHEP **05** (2016), 014.

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = \int \frac{d^4q}{(2\pi)^4}\frac{d}{d\ln\Lambda}\frac{\partial}{\partial q^\mu}\frac{\partial}{\partial q_\mu}\Big\{-\frac{\pi C_2}{q^2}\Big[\ln\Big(1+\frac{M_\varphi^2}{q^2R^2(q^2/\Lambda^2)}\Big)$$
$$+2\ln\Big(1+\frac{M_\varphi^2}{q^2}\Big)\Big] + \frac{\pi T(R)}{q^2}\ln\Big(1+\frac{M^2}{q^2F^2(q^2/\Lambda^2)}\Big)\Big\} + O(\alpha_0,\lambda_0^2),$$

where a small vicinity of the singular point $q^\mu = 0$ is excluded from the integration region.

The double total derivatives effectively cut internal lines in the supergraphs and reduce a number of loop integrations by 1



Then the NSVZ equation for the renormalization group functions (RGFs) defined in terms of the bare couplings is obtained by summing singular contributions in all orders

K.S., Eur. Phys. J. C **80** (2020) no.10, 911.

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{1}{2\pi}\Big(3C_2 - T(R) - 2C_2\gamma_c(\alpha_0,\lambda_0)$$
$$-2C_2\gamma_V(\alpha_0,\lambda_0) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r\Big).$$

Note that qualitatively this result can be obtained with the help of a certain modification of the one-loop calculation, namely, by replacing tree propagators with the exact propagators.

The nonrenormalization of the triple gauge-ghost vertices

The original NSVZ equation is reproduced after taking into account the non-renormalization of the triple gauge-ghost vertices

K.S., Nucl.Phys. **B909** (2016) 316.

This nonrenormalization theorem produces the equation

$$\frac{d}{d\ln\Lambda}(Z_\alpha^{-1/2} Z_c Z_V) = 0,$$

which allows to express the β-function in terms of the anomalous dimensions of quantum superfields,

$$\beta(\alpha_0,\lambda_0) = 2\alpha_0\Big(\gamma_c(\alpha_0,\lambda_0) + \gamma_V(\alpha_0,\lambda_0)\Big).$$

Using this relation we obtain the NSVZ β-function in the original form

$$\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0^2} = -\frac{3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha_0,\lambda_0)/r}{2\pi} + \frac{C_2}{2\pi}\cdot\frac{\beta(\alpha_0,\lambda_0)}{\alpha_0}.$$

The NSVZ scheme for the standard RGFs

For the standard RGFs

$$\widetilde{\beta}(\alpha,\lambda) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad \widetilde{\gamma}(\alpha,\lambda) \equiv \frac{d\ln Z}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}$$

the all-loop NSVZ scheme turns out to be the HD+MSL scheme, when a theory is regularized by Higher Derivatives, and divergences are removed by Minimal Subtractions of Logarithms, because in this case

A.L.Kataev and K.S., Nucl.Phys. **B875** (2013) 459

$$\widetilde{\beta}(\alpha,\lambda)\Big|_{\text{HD+MSL}} = \beta(\alpha_0\to\alpha,\lambda_0\to\lambda); \quad \widetilde{\gamma}(\alpha,\lambda)\Big|_{\text{HD+MSL}} = \gamma(\alpha_0\to\alpha,\lambda_0\to\lambda).$$

Note that the NSVZ equation can be written in the form of the renormalization group invariant expression (RGI) that does not receive quantum corrections in all orders. For instance, for the pure $\mathcal{N}=1$ SYM theory

$$\Big(\frac{\mu^3}{\alpha}\Big)^{C_2}\exp\Big(-\frac{2\pi}{\alpha}\Big) = \text{RGI}.$$

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381.

Usual supersymmetric theories in higher dimensions are not renormalizable, because the degree of divergence increases with the number of loops. However, in this case it is possible to consider theories with higher derivatives. We will consider a model analogous to the $6D$, $\mathcal{N} = (1,0)$ higher derivative supersymmetric theory proposed in

> E. A. Ivanov, A. V. Smilga and B. M. Zupnik, Nucl. Phys. B **726** (2005), 131.

It is convenient to formulate it in $6D$, $\mathcal{N} = (1,0)$ harmonic superspace

> P. S. Howe, G. Sierra and P. K. Townsend, Nucl. Phys. B **221** (1983), 331;
> P. S. Howe, K. S. Stelle and P. C. West, Class. Quant. Grav. **2** (1985), 815;
> B. M. Zupnik, Sov. J. Nucl. Phys. **44** (1986), 512;
> E. A. Ivanov and A. V. Smilga, Phys. Lett. B **637** (2006), 374;
> I. L. Buchbinder and N. G. Pletnev, Nucl. Phys. B **892** (2015), 21,

which is similar to the usual $4D$, $\mathcal{N} = 2$ harmonic superspace

> A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1** (1984), 469;
> A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, "Harmonic superspace", Cambridge, UK: Univ. Pr. (2001) 306 p..

The harmonic superspace is especially convenient for formulating $6D$, $\mathcal{N} = (1,0)$ supersymmetric theories because it makes $\mathcal{N} = (1,0)$ supersymmetry manifest.

The $6D$, $\mathcal{N} = (1,0)$ harmonic superspace is parametrized by the coordinates $(x^\mu, \theta^a_i, u^{\pm i})$, where $\mu = 0, \ldots, 5$, θ^a_i (with $a = 1, \ldots, 4$ and $i = 1, 2$) are the anticommuting left-handed spinors, and the harmonic variables $u^{\pm i}$ satisfy the relations

$$u_i^- = (u^{+i})^*, \qquad u^{+i} u_i^- = 1, \qquad u_i^\pm = \varepsilon_{ij} u^{\pm j}.$$

It contains the analytic subspace closed under supersymmetry transformations with coordinates

$$x_A^\mu = x^\mu + \frac{i}{2}\theta^- \gamma^\mu \theta^+, \qquad \theta^{+a} = u_i^+ \theta^{ai}, \qquad u_i^\pm.$$

The gauge superfield and the hypermultiplet (in the adjoint representation) are described by the analytic superfields $V^{++} = e_0 V^{++A} t^A$ and $q^+ = e_0 q^{+A} t^A$, respectively,

$$D_a^+ V^{++} = 0; \qquad D_a^+ q^+ = 0,$$

where $D_a^+ = u_i^+ D_a^i$. Also we will need the harmonic derivatives

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}}, \qquad D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}}.$$

Following the paper

I. L. Buchbinder, A. S. Budekhina, E. A. Ivanov and K.S., Phys. Rev. D **111** (2025) no.12, 125014.

we consider the theory which in the $6D$, $\mathcal{N}=(1,0)$ harmonic superspace is described by the action

$$S=\pm\frac{1}{2e_0^2}\text{tr}\int d\zeta^{(-4)}(F^{++})^2-\frac{2}{e_0^2}\,\text{tr}\int d\zeta^{(-4)}\widetilde{q^+}\nabla^{++}q^+.$$

Here $\nabla^{++}q^+\equiv D^{++}q^++i[V^{++},q^+]$, and the integration measure is given by the expression

$$\int d\zeta^{(-4)}\equiv\int d^6x\,d^4\theta^+\,du.$$

In particular, this measure contains the integration over harmonics. The harmonic superspace analog of the gauge field strength is defined by the equations

$$F^{++}\equiv(D^+)^4V^{--},\qquad \text{where}$$

$$V^{--}(z,u)\equiv\sum_{n=1}^{\infty}(-i)^{n+1}\int du_1\dots du_n\frac{V^{++}(z,u_1)V^{++}(z,u_2)\dots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\dots(u_n^+u^+)}.$$

In components this action contains the term with higher derivatives of the gauge field (and its superpartners)

$$S = \text{tr} \int d^6x \left\{ \pm \frac{1}{e_0^2} (\mathcal{D}_\mu F^{\mu\nu})^2 + \dots \right\}.$$

Due to the higher derivatives the degree of divergence does not increase with a number of loops.

The theory could contain quadratic and logarithmical divergences, but the quadratic divergences cancel each other in the one-loop approximation (and presumably in all loops) due to the presence of the hypermultiplet in the adjoint representation. Moreover, due to the presence of the hypermultiplet, the theory is not anomalous

A. V. Smilga, Phys. Lett. B **647** (2007), 298

and seems to be renormalizable.

The hypermultiplet and ghosts do not receive divergent quantum corrections because the corresponding parts of the action are not given by the integrals over the total harmonic superspace. (This is analogous to the nonrenormalization of the superpotential in the $4D$ case.)

The higher covariant derivative regularization

Let us regularize the theory under consideration by higher covariant derivatives. The higher derivative term is constructed with the help of the operator

$$\Box \equiv \frac{1}{2}(D^+)^4(\nabla^{--})^2,$$

(where $\nabla^{--} = D^{--} + iV^{--}$), which is analogous to the Laplace operator when acting on analytic superfields. The sum of the original action and the higher derivative term can be written in the form

$$S_{\text{reg}} = \pm\frac{1}{2e_0^2}\text{tr}\int d\zeta^{(-4)} F^{++} R\Big(\frac{\Box}{\Lambda^2}\Big) F^{++} - \frac{2}{e_0^2}\,\text{tr}\int d\zeta^{(-4)} \widetilde{q^+}\nabla^{++}q^+,$$

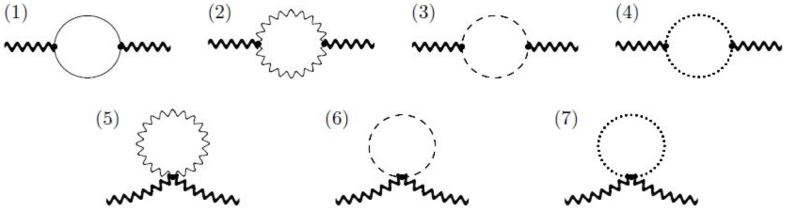
where $R(0) = 1$ and $R(x) \to \infty$ at $x \to \infty$. For regularizing the one-loop divergences it is also necessary to add the Pauli–Villars superfields with the mass $M = a\Lambda$. (For simplicity, we do not present the explicit expression for their action.) Then the generating functional of the regularized theory takes the form

$$Z[\text{sources}] = \int Dv^{++} D\widetilde{q^+} Dq^+ Db\, Dc\, D\varphi\, \text{Det}^{1/2}\Big[\Box^2 R\Big(\frac{\Box}{\Lambda^2}\Big)\Big]$$

$$\times\, \text{Det}(PV, M)\, \exp\Big(iS_{\text{reg}} + iS_{\text{gf}} + iS_{\text{FP}} + iS_{\text{NK}} + iS_{\text{sources}}\Big).$$

The divergent supergraphs giving the β-function

The renormalization of the coupling constant is determined by the following harmonic supergraphs:



The solid lines correspond to the hypemultiplet;
the wavy lines correspond to the gauge superfield;
the dashed lines denote propagators of the Faddeev–Popov ghosts;
the dotted lines denote propagators of the Nielsen–Kallosh ghosts.

Quadratic divergences in these superdiagrams cancel each other, while the logarithmical divergences determine the β-function $\beta(\alpha_0)$ (defined in terms of the bare coupling constant), where $\alpha_0 \equiv e_0^2/4\pi$.

The one-loop β-function

The result for the β-function obtained after calculating the above superdiagrams can be written in the form

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \mp 2\pi C_2 \int \frac{d^6 q}{(2\pi)^6} \frac{d}{d\ln\Lambda} \frac{\partial}{\partial q_\mu} \frac{\partial}{\partial q^\mu} \Big[\frac{1}{q^4} \ln\Big(1 + \frac{M^4}{q^4 R(q^2/\Lambda^2)}\Big)\Big] + O(\alpha_0).$$

We see that exactly as in the $4D$ case the β-function is given by integrals of double total derivatives with respect to the loop momentum. Note that, due to the presence of an arbitrary regulator function $R(x)$, this fact is highly nontrivial. Calculating the integrals we obtain the one-loop result

$$\beta(\alpha_0) = \mp \frac{\alpha_0^2 C_2}{2\pi^2} + O(\alpha_0^3).$$

This expression agrees with the results of the calculations made with dimensional reduction in

E. A. Ivanov, A. V. Smilga and B. M. Zupnik, Nucl. Phys. B **726** (2005), 131;
L. Casarin and A. A. Tseytlin, JHEP **08** (2019), 159;
I. L. Buchbinder, E. A. Ivanov, B. S. Merzlikin and K.S., JHEP **08** (2020), 169; Nucl. Phys. B **961** (2020), 115249

by various methods if one takes into account the contribution of the hypermultiplet in the adjoint representation.

The resemblance in the structure of the one-loop results for $4D$, $\mathcal{N} = 1$ supersymmetric gauge theory and for the considered $6D$, $\mathcal{N} = (1,0)$ higher derivative theory allows to suggest that it may be possible to write down the all-loop exact expression for the β-function. It can be constructed by replacing the tree propagators by the exact ones in the one-loop singular contributions. (For $4D$, $\mathcal{N} = 1$ theories the similar procedure gives the NSVZ expression.) In the $6D$ case the result has the form

$$\beta(\alpha_0) = \mp \frac{\alpha_0^2 C_2}{2\pi^2\Big(1 \mp \alpha_0 C_2/8\pi^2\Big)}.$$

Certainly, this derivation is not rigourous and should be verified by explicit multiloop calculations. (If possible), it would be also expedient to construct its rigorous all-order proof analogous to the one for the $4D$, $\mathcal{N} = 1$ case.

Similarly to the pure $4D$, $\mathcal{N} = 1$ SYM theory, it is possible to integrate the renormalization group equation and obtain the expression that does not receive quantum corrections in any order of the perturbation theory,

$$\Big(\frac{\alpha}{\mu^4}\Big)^{C_2} \exp\Big(\pm\frac{8\pi^2}{\alpha}\Big) = \text{RGI}.$$

Conclusion

- With the help of the higher covariant derivative regularization it is possible to construct a simple all-loop NSVZ renormalization prescription given by the HD+MSL scheme.
- With the higher derivative regularization the NSVZ equation appears because the integrals giving the β-function are integrals of double total derivatives.
- The $6D$, $\mathcal{N}=(1,0)$ higher derivative theory interacting with the hypermultiplet in the adjoint representation seems to be renormalizable and free from the quadratic divergences.
- With the higher derivative regularization the β-function of this $(6D)$ theory is also given by integrals of double total derivatives, similarly to the $4D$ case.
- It is possible to suggest the existence of the all-loop NSVZ-like expression for the β-function of this theory analogous to the NSVZ equation for the pure $4D$, $\mathcal{N}=1$ SYM theory.
- The proposed expression should be verified in higher orders and proved (or rejected) by rigorous methods.

Thank you for the attention!