

# Coulomb interaction in rare decays of B-Mesons

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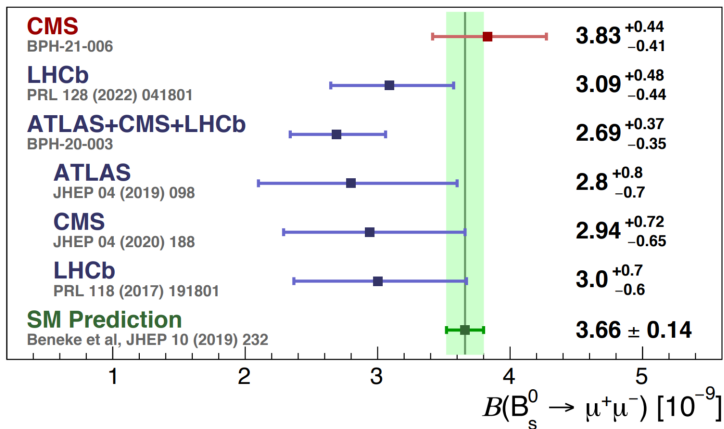
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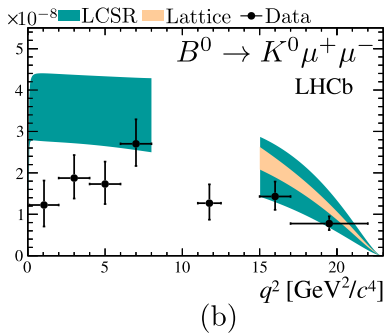
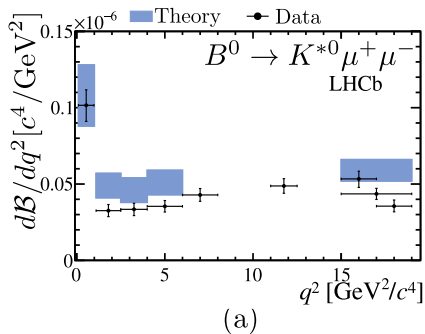
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# Problem statement and relevance of the work



The partial decay width of  $B_s^0 \rightarrow \mu^+\mu^-$ , measured in various experiments, as well as the averaged predictions of the Standard Model

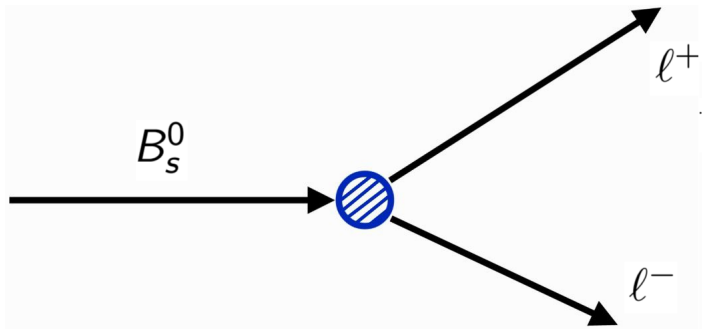
# Problem statement and relevance of the work



Differential decay width of  $B^0 \rightarrow K^{0(*)} \mu^+ \mu^-$  - predictions of the Standard Model, experimental values

# Part 1. Formulation and justification of method.

# Key idea - replacement of the secondary quantization procedure



Leptons cannot be considered free - in the final state, they interact with each other.

# Key idea - replacement of the secondary quantization procedure

The standard procedure of secondary quantization - expansion in plane waves:

$$\ell(x) = \sum_{s=1,2} \int \frac{d^3 p}{(2\pi)^3} (a_p u(p, s) e^{-ipx} + b_p^\dagger \bar{v}(p, s) e^{+ipx}) \quad (1)$$

Modified procedure of secondary quantization - expansion in exact solutions of the wave equation with an external potential (Furry's method):

$$\ell(x) = \sum_{s=1,2} \int \frac{d^3 p}{(2\pi)^3} (a_p \Psi_{\mathcal{E}\vec{p}}^{(+)}(x) e^{-i\mathcal{E}^{(+)}t} + b_p^\dagger \Psi_{-\mathcal{E}-\vec{p}}^{(-)}(x) e^{+i\mathcal{E}^{(-)}t}) \quad (2)$$

## Three methods for accounting for Coulomb interaction

- 1 The non-relativistic Gamow-Sommerfeld-Sakharov factor:

$$\mathcal{K}^{(GSS)} = \frac{2\pi\alpha/v}{1 - e^{-2\pi\alpha/v}}, \quad (3)$$

- 2 The exact relativistic method of Crater (arXiv:hep-ph/9912386) and Sazdjian (PhysRevD 33, 3401):

$$\mathcal{K}^{(CS)} = \left| \frac{\Gamma(\sqrt{\frac{1}{4} - \alpha^2} + \frac{1}{2} + i\frac{\alpha}{v})}{\Gamma(\sqrt{1 - 4\alpha^2} + 1)} \right|^2 \cdot e^{\pi\alpha/v}, \quad (4)$$

- 3 The approximate relativistic method of Furry (PhysRev 81,115):

$$\mathcal{K}^{(Furry)} = e^{\pi\alpha/v}. \quad (5)$$

# Justification of the approach using the decay $B \rightarrow S^+ S^-$

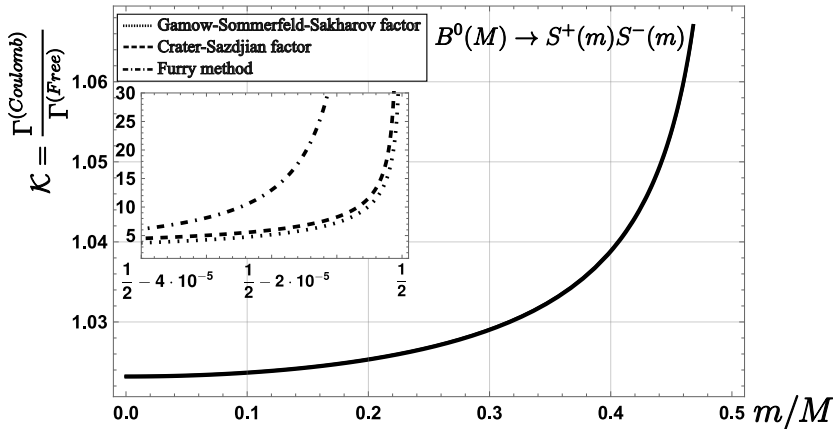


Figure: Comparison of three methods for accounting Coulomb interaction between particles.

Part 2. Coulomb interaction in  $B_{s,d}^0 \rightarrow l^+ l^-$ .

# Coulomb interaction in $B_{s,d}^0 \rightarrow l^+ l^-$

The decay width for  $B_s^0 \rightarrow l^+ l^-$  is given by:

$$\Gamma_{B_s^0 \rightarrow l^+ l^-} = \mathcal{K}_{B_s^0 \rightarrow \mu^+ \mu^-}^{(Furry)} \cdot |D|^2 \frac{\sqrt{M^2 - 4m^2}}{8\pi}, \text{ where}$$

$$D = \frac{iG_F}{\sqrt{2}} \frac{\alpha_{em}}{2\pi} \cdot V_{tb} V_{ts}^* f_{B_s^0} 2m C_{10A}$$

$$\mathcal{K}_{B_s^0 \rightarrow \mu^+ \mu^-}^{(Furry)} = \frac{\Gamma^{(Coulomb)}}{\Gamma^{(free)}} = e^{\pi\alpha\mathcal{E}/p}$$

here  $\mathcal{E}$  and  $p$  are the lepton's energy and momentum, respectively.

# Coulomb interaction in $B_{s,d}^0 \rightarrow \ell^+ \ell^-$

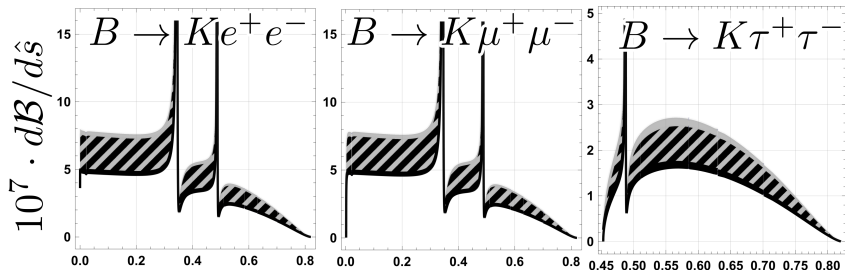
	$\mathcal{B}^{(exp)}$	$\mathcal{B}^{(free)}$	$\mathcal{B}^{(Coulomb)}$
$B_s^0 \rightarrow \mu^+ \mu^- [10^{-9}]$	$3.83_{-0.41}^{+0.44}$	$3.66 \pm 0.14$	$3.75 \pm 0.14$
$B^0 \rightarrow \mu^+ \mu^- [10^{-11}]$	$< 19$	$1.03 \pm 0.05$	$1.05 \pm 0.05$
$B_s^0 \rightarrow e^+ e^- [10^{-11}]$	$< 940$	$1.77 \pm 0.08$	$1.81 \pm 0.09$
$B^0 \rightarrow e^+ e^- [10^{-13}]$	$< 25000$	$4.99 \pm 0.25$	$5.10 \pm 0.26$
$B_s^0 \rightarrow \tau^+ \tau^- [10^{-8}]$	$< 6.8 \cdot 10^5$	$4.61 \pm 0.22$	$4.75 \pm 0.23$
$B^0 \rightarrow \tau^+ \tau^- [10^{-9}]$	$< 2.1 \cdot 10^6$	$1.28 \pm 0.07$	$1.32 \pm 0.07$

Table: Branching ratio  $\mathcal{B} = \Gamma_{B_{d,s}^0 \rightarrow \ell^+ \ell^-} / \Gamma_{B_{d,s}^0}^{(total)}$

### Part 3. Coulomb interaction in

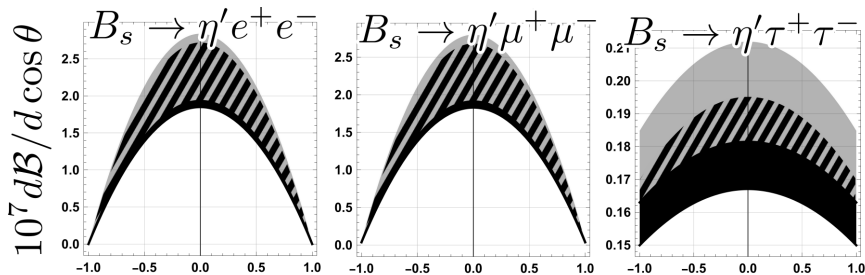
$$B_{s,d}^0 \rightarrow h^0 l^+ l^-.$$

# Coulomb interaction in $B_{s,d}^0 \rightarrow h^0 \ell^+ \ell^-$



The differential decay width  $10^7 \cdot \frac{1}{\Gamma^{(total)}} \frac{d\Gamma}{ds}$  as a function of  $s/M_B^2$  where  $s = (p_{B^0} - p_{K^0})^2$ .

# Coulomb interaction in $B_{s,d}^0 \rightarrow \ell^+ \ell^-$



$10^7 \cdot dB/d \cos \theta$  as a function of  $\cos \theta$ , where  $\theta = \angle(\mathbf{p}_{\ell^+}, \mathbf{p}_K)$  in the rest frame of the lepton pair

# Coulomb interaction in $B_{s,d}^0 \rightarrow h^0 \ell^+ \ell^-$

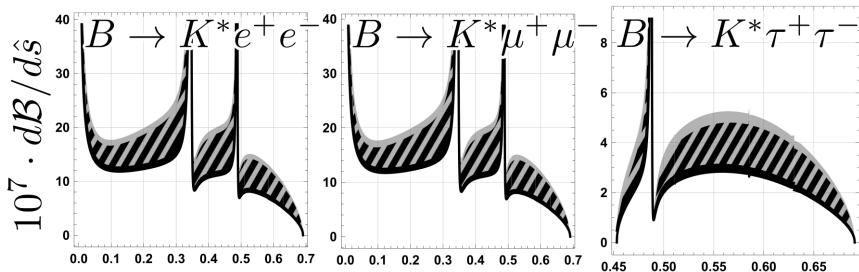
	$\mathcal{B}^{(exp)}$	$\mathcal{B}^{(th,free)}$	$\mathcal{B}^{(th,coulomb)}$	Corr.
$B^0 \rightarrow K^0 e^+ e^- [10^{-7}]$	$2.5^{+1.1}_{-0.9}$	$3.34 \pm 0.38$	$3.42 \pm 0.39$	2.33%
$B^0 \rightarrow K^0 \mu^+ \mu^- [10^{-7}]$	$3.39 \pm 0.35$	$3.33 \pm 0.38$	$3.41 \pm 0.38$	2.35%
$B^0 \rightarrow K^0 \tau^+ \tau^- [10^{-8}]$	-	$5.0 \pm 2.1$	$5.3 \pm 2.2$	5.75%
$B^0 \rightarrow \pi^0 e^+ e^- [10^{-8}]$	$< 8.4$	$1.17 \pm 0.3$	$1.19 \pm 0.3$	2.32%
$B^0 \rightarrow \pi^0 \mu^+ \mu^- [10^{-8}]$	$< 6.9$	$1.16 \pm 0.3$	$1.19 \pm 0.3$	2.34%
$B^0 \rightarrow \pi^0 \tau^+ \tau^- [10^{-9}]$	-	$3.05 \pm 0.73$	$3.21 \pm 0.76$	4.85%

	$\mathcal{B}^{(exp)}$	$\mathcal{B}^{(th,free)}$	$\mathcal{B}^{(th,coulomb)}$	Corr.
$B_s^0 \rightarrow \eta e^+ e^- [10^{-7}]$	—	$4.12 \pm 0.79$	$4.22 \pm 0.80$	2.33%
$B_s^0 \rightarrow \eta \mu^+ \mu^- [10^{-7}]$	—	$4.11 \pm 0.79$	$4.21 \pm 0.80$	2.35%
$B_s^0 \rightarrow \eta \tau^+ \tau^- [10^{-8}]$	—	$7.04 \pm 1.2$	$7.43 \pm 1.3$	5.55%
$B_s^0 \rightarrow \eta' e^+ e^- [10^{-7}]$	—	$3.04 \pm 0.58$	$3.11 \pm 0.59$	2.31%
$B_s^0 \rightarrow \eta' \mu^+ \mu^- [10^{-7}]$	—	$3.03 \pm 0.58$	$3.10 \pm 0.59$	2.34%
$B_s^0 \rightarrow \eta' \tau^+ \tau^- [10^{-8}]$	—	$2.37 \pm 0.36$	$2.54 \pm 0.38$	<b>6.9%</b>
$B_s^0 \rightarrow K^0 e^+ e^- [10^{-8}]$	—	$1.35 \pm 0.34$	$1.38 \pm 0.35$	2.33%
$B_s^0 \rightarrow K^0 \mu^+ \mu^- [10^{-8}]$	—	$1.34 \pm 0.34$	$1.37 \pm 0.35$	2.35%
$B_s^0 \rightarrow K^0 \tau^+ \tau^- [10^{-9}]$	—	$2.55 \pm 0.59$	$2.69 \pm 0.62$	5.42%

## Part 4. Coulomb interaction in

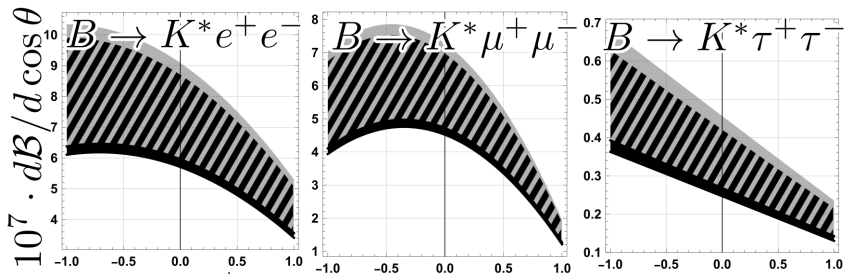
$$B_{s,d}^0 \rightarrow V^0 \ell^+ \ell^-.$$

# Coulomb interaction in $B_{s,d}^0 \rightarrow V^0 \ell^+ \ell^-$



The differential decay width  $10^7 \cdot \frac{1}{\Gamma^{(total)}} \frac{d\Gamma}{ds}$  as a function of  $s/M_B^2$  where  $s = (p_{B^0} - p_{K^0})^2$ .

# Coulomb interaction in $B_{s,d}^0 \rightarrow V^0 \ell^+ \ell^-$



$10^7 \cdot dB/d \cos \theta$  as a function of  $\cos \theta$ , where  $\theta = \angle(\mathbf{p}_{\ell^+}, \mathbf{p}_K)$  in the rest frame of the lepton pair

Coulomb interaction in  $B_{s,d}^0 \rightarrow V^0 \ell^+ \ell^-$ 

	$\mathcal{B}^{(exp)}$	$\mathcal{B}^{(th,free)}$	$\mathcal{B}^{(th,coulomb)}$	Corr.
$B \rightarrow K^* e^+ e^- [10^{-6}]$	1.19 $\pm 0.20$	$1.36 \pm 0.15$	$1.40 \pm 0.15$	2.37%
$B \rightarrow K^* \mu^+ \mu^- [10^{-6}]$	1.06 $\pm 0.09$	$0.94 \pm 0.11$	$1.02 \pm 0.11$	2.40%
$B \rightarrow K^* \tau^+ \tau^- [10^{-8}]$	—	$6.67 \pm 0.87$	$7.10 \pm 0.93$	6.5%
$B \rightarrow \rho e^+ e^- [10^{-6}]$	—	$1.07 \pm 0.18$	$1.10 \pm 0.19$	2.36%
$B \rightarrow \rho \mu^+ \mu^- [10^{-7}]$	—	$8.9 \pm 1.6$	$9.1 \pm 1.7$	2.38%
$B \rightarrow \rho \tau^+ \tau^- [10^{-7}]$	—	$1.31 \pm 0.23$	$1.38 \pm 0.25$	5.81%

	$\mathcal{B}^{(exp)}$	$\mathcal{B}^{(th,free)}$	$\mathcal{B}^{(th,coulomb)}$	Corr.
$B_s^0 \rightarrow \phi e^+ e^- [10^{-6}]$	—	$1.52 \pm 0.17$	$1.56 \pm 0.17$	2.37%
$B_s^0 \rightarrow \phi \mu^+ \mu^- [10^{-6}]$	$0.84 \pm 0.4$	$1.15 \pm 0.14$	$1.18 \pm 0.15$	2.39%
$B_s^0 \rightarrow \phi \tau^+ \tau^- [10^{-7}]$	—	$1.05 \pm 0.13$	$1.12 \pm 0.14$	6.60%
$B_s^0 \rightarrow K^* e^+ e^- [10^{-8}]$	—	$5.56 \pm 0.65$	$5.69 \pm 0.67$	2.36%
$B_s^0 \rightarrow K^* \mu^+ \mu^- [10^{-8}]$	$2.9 \pm 1.1$	$4.42 \pm 0.56$	$4.53 \pm 0.57$	2.38%
$B_s^0 \rightarrow K^* \tau^+ \tau^- [10^{-9}]$	—	$5.33 \pm 0.62$	$5.65 \pm 0.65$	6.1%

- A method to account for the Coulomb interaction in lepton and semilepton decays of B-mesons was developed.
- Corrections to the decays were calculated  $B_{s,d}^0 \rightarrow l^+l^-$ ,  $B^0 \rightarrow \{K^0, \pi^0\}l^+l^-$  and  $B_s^0 \rightarrow \{\eta, \eta', K^0\}l^+l^-$ ,  $B^0 \rightarrow \{K^{*0}, \rho\}l^+l^-$  and  $B_s^0 \rightarrow \{\phi, K^{*0}\}l^+l^-$  (21 decays in total)
- For the  $B_s^0 \rightarrow \mu^+\mu^-$ ,  $B^0 \rightarrow K^0\mu^+\mu^-$ ,  $B^0 \rightarrow K^{*0}\mu^+\mu^-$  decays, accounting for the Coulomb interaction improves the agreement between the SM predictions and the experimental data
- The Coulomb correction can be up to 7%, which is similar to the error of the form factors

Thank You for Your Attention!

# Coulomb interaction in the decays $B_s^0 \rightarrow \ell^+ \ell^-$

The decay width for  $B_s^0 \rightarrow \ell^+ \ell^-$  is given by:

$$\Gamma_{B_s^0 \rightarrow \ell^+ \ell^-} = \mathcal{K}_{B_s^0 \rightarrow \mu^+ \mu^-}^{(Furry)} \cdot |D|^2 \frac{\sqrt{M^2 - 4m^2}}{8\pi}, \text{ where} \quad (6)$$

$$D = \frac{iG_F \alpha_{em}}{\sqrt{2} 2\pi} \cdot V_{tb} V_{ts}^* f_{B_s^0} 2m C_{10A}$$

$$\mathcal{K}_{B_s^0 \rightarrow \ell^+ \ell^-}^{(Furry)} = \frac{\Gamma(\text{Coulomb})}{\Gamma(\text{free})} = e^{\pi\alpha\mathcal{E}/p}$$

here  $\mathcal{E}$  and  $p$  are the lepton's energy and momentum, respectively.

# Coulomb interaction in the decay $B_s^0 \rightarrow K^0 l^+ l^-$

The differential decay width for  $B_s^0 \rightarrow K^0 l^+ l^-$  is given by:

$$\frac{d\Gamma}{d\hat{t}d\hat{s}} = \frac{G_F^2 \alpha_{em}^2 |V_{tb} V_{ts}^*|^2 M^5}{256\pi^5} (-\hat{\Pi} \beta_p + 2\hat{m} |C_{10A}|^2 \delta_p) \cdot \mathcal{K}^{(Coulomb)}, \quad (7)$$

$$\beta_p = |C_{9V} f_+(q^2) + 2MC_{7\gamma} s(q^2)|^2 + |C_{10A} f_+(q^2)|^2$$

$$\hat{\Pi} = (\hat{t} - 1)(\hat{t} - \hat{r}) + \hat{s}\hat{t} + \hat{m}(1 + \hat{r} + \hat{m} - \hat{s} - 2\hat{t})$$

$$\delta_p = \left(1 + \hat{r} - \frac{\hat{s}}{2}\right) |f_+(q^2)|^2 + (1 - \hat{r}) \text{Re}[f_+(q^2) f_-^*(q^2)] + \frac{\hat{s}}{2} |f_-(q^2)|^2$$

$$\mathcal{K}^{(Coulomb)} = \exp\left(\frac{\pi\alpha}{\sqrt{1 - 4\hat{m}/\hat{s}}}\right)$$

# General approach to accounting for Coulomb interaction

A general recipe that allows for the accounting of Coulomb interaction in decays, in which the final state contains an  $l^+l^-$  pair (and no more charged particles):

$$\langle l^+l^- H_2 | O | H_1 \rangle \rightarrow \langle l^+l^- H_2 | O | H_1 \rangle \cdot \frac{\Gamma(1 - i\frac{\alpha\mathcal{E}_l}{2p_l})}{\Gamma(1 + i\frac{\alpha\mathcal{E}_l}{2p_l})} \exp\left(\frac{\pi\alpha\mathcal{E}_l}{2p_l}\right)$$

$O$  - operator representing an arbitrary combination of  $\gamma$ -matrices, momenta, as well as quark and lepton fields,  $\mathcal{E}_l, p_l$  - energy and momentum of the charged lepton in the rest frame of the  $l^+l^-$  pair,  $\Gamma(x)$  - Euler's gamma function,  $H_1, H_2$  - neutral hadrons in the initial and final states respectively,  $\alpha = \alpha_{em} \approx 1/137$ .