Some analytic results for the contribution to the anomalous magnetic moments of leptons due to the polarization of vacuum via lepton loops

Olga Solovtsova[1,2)]

in collaboration with **V.I. Lashkevich**[1)] and **A.V. Sidorov**[2)]

[1)]Gomel State Technical University, Gomel, Belarus

BLTP [2)] Joint Institute for Nuclear Research, Dubna, Russia

Outline

- Introduction & Motivation
- Theoretical overview
- New analytic result for the mass-dependent three-bubble diagram
- Summary

The anomalous magnetic moments

$a = (g-2)/2$

of the electron (a_e) and of the muon (a_μ) are the most precisely measured quantities in particle physics.

(The experimental study of the a_τ is difficult due to the short τ-life time ~ 2.9×10^{-13} s.)

There are numerous excellent reviews and in-depth articles on the subject.

F. Jegerlehner, The anomalous magnetic moment of the muon, Springer 2017, 693p.

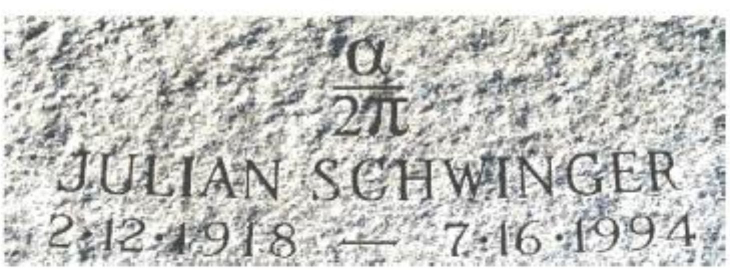
(the earlier references therein),

review **I. Logashenko and S. Eidelman, Anomalous magnetic moment of the muon, Phys. Usp. 61(2018)**

mini-review **H. Davoudiasl and W.J. Marciano, Tale of two anomalies, Phys. Rev. D 98 (2018)** and so on.

Historically

- ★ $g = 2$ (tree level, Dirac)
- ★ $a = \alpha/(2\pi)$ (1-loop QED, Schwinger)



A. Petermann
C.M. Sommerfield
B. Lautrup and E.de Rafael
M. Caffo, S. Turrini and E. Remiddi
S. Laporta, M. L.Laursen and M.A.Samuel
G. Li, R. Mendel and M. A. Samuel
T. Kinoshita et al., A. Kurz
T. Liu, P. Marquard and M. Steinhauser
P.A. Baikov, A. Maier and P. Marquard
and so on

Today

★ **One of the best measured quantities**

The latest available experimental results read

$$a_\mu^{exp} = 116\,592\,089(63)\ \times\ 10^{-11}\ \ (\text{Bennett } et\ al.[Muon\ g-2\ Collab.])$$

$$a_e^{exp} = 1\,159\,652\,180.73(.28)\ \times\ 10^{-12}\ \ (\text{Parker } et\ al.\ 2018)$$

★ There is a long-standing deviation between the experimental measurements and theoretical predictions

$$a_\mu = a_\mu^{exp} - a_\mu^{theor} \sim\ 270(85)\ \times 10^{-11} \qquad (3-4\ \sigma)$$

$$a_e = a_e^{exp} - a_e^{theor} \sim -0.88(36)\ \times 10^{-12} \qquad (\sim 2.5\ \sigma)$$

$(a\ rather\ unexpected\ fact\ that\ the\ sign\ of\ \Delta a_e\ is\ opposite\ to\ \Delta a_\mu)$

Extremely useful in probing/constraining physics beyond the SM

★ New exp. at Fermilab and J-PARC expected to reduce the uncertainty of $(g-2)_\mu$ by a factor of 4

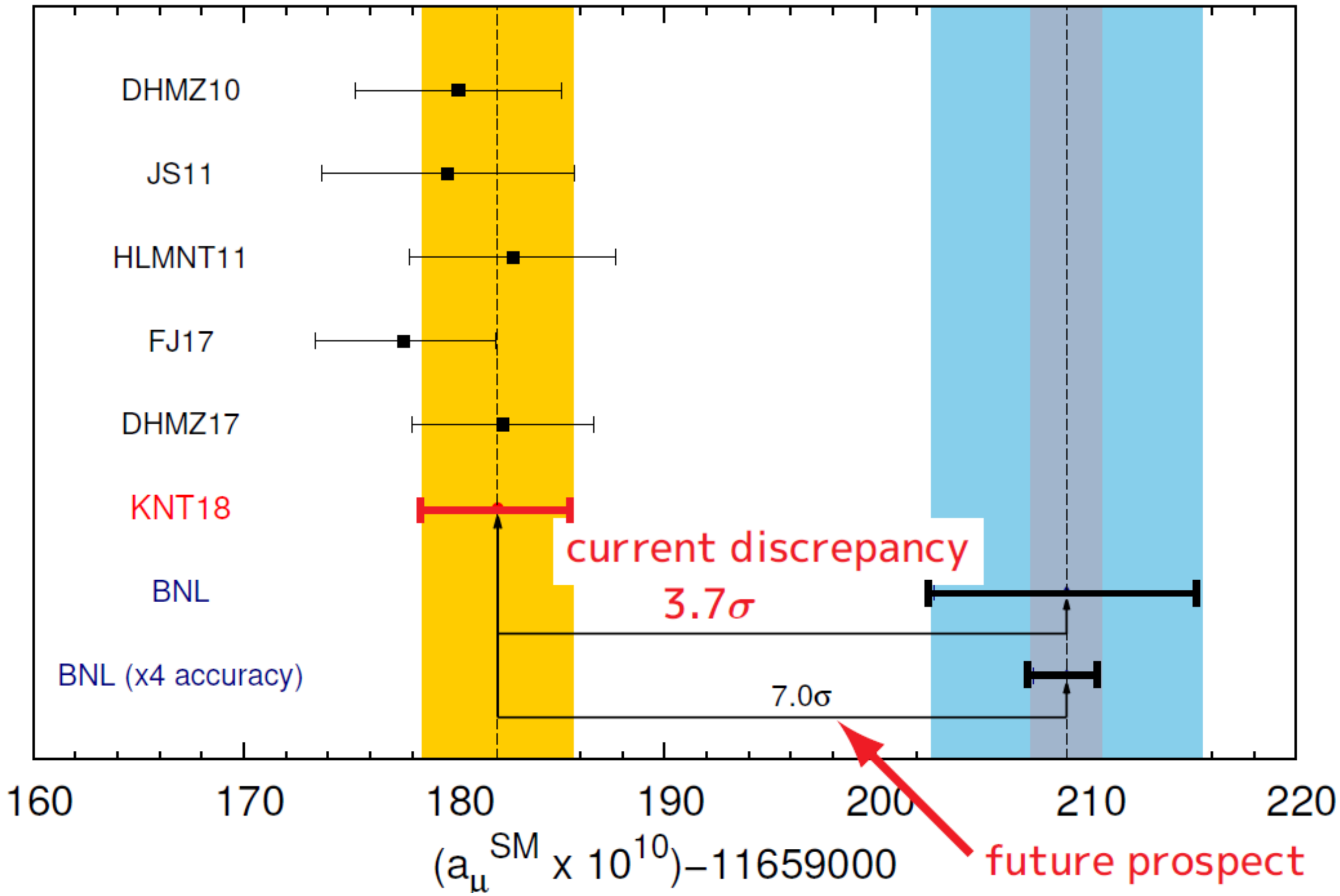


Figure from the paper A. Keshavarzi, D. Nomura and T. Teubner (KNT) Phys. Rev. D97 (2018)

Currently, the accuracy of theoretical calculations is high but the discrepancy stimulates a thorough theoretical re-examination.

The paper "Muon Anomaly from Lepton Vacuum Polarization and the Mellin-Barnes Representation" by J.-P. Aguilar, E. de Rafael and D. Greynat (ARG) Phys. Rev. D77 (2008)

It was presented a powerful technique to obtain asymptotic expansions. …
It would be reassuring to have, at least, two independent calculations of the various theoretical contributions, as well as of the higher order estimates.
We investigated the high-precision numerical estimation of the MB integrals based on the on the stationary phase contour [A. Sidorov, V. Lashkevich, OS, Phys. Rev. D97(2018).

N -the number terms in the quadrature formula.
In parentheses are presented the errors of numerical integration.

N	$a_\mu^{(e)}\,(\alpha/\pi)^{-2}$	$a_\mu^{(\mu)}\,(\alpha/\pi)^{-2}$	$a_\mu^{(\tau)}\,(\alpha/\pi)^{-2}$
1	1.07(3)	0.014(2)	0.000 076 (3)
2	1.093(2)	0.015 4(3)	0.000 077 7(10)
3	1.094 21(7)	0.015 72(5)	0.000 077 99(9)
4	1.094 258 8(7)	0.015 72(5)	0.000 078 074 (3)
6	1.094 258 2(2)	0.015 681(8)	**0.000 078 075 0(10)**
8	1.094 258 303(9)	0.015 688 0(7)	0.000 078 075 6(3)
10	**1.094 258 308 8(5)**	0.015 687 47(6)	0.000 078 075 78(4)
16	1.094 258 309 216(2)	0.015 687 40(3)	0.000 078 075 806 0(2)
30	1.094 258 309 215 796 31(2)	0.015 687 421 7(2)	0.000 078 075 805 927 3(2)
35	1.094 258 309 215 796 321(1)	0.015 687 421 854(6)	0.000 078 075 805 927 46(3)
Exact	1.094 258 309 215 796 321	0.015 687 421 859 102	0.000 078 075 805 927 46

	$a_\mu^{(eee)}\,(\alpha/\pi)^{-4}$	$a_\mu^{(ee\mu)}\,(\alpha/\pi)^{-4}$	$a_\mu^{(e\mu\mu)}\,(\alpha/\pi)^{-4}$
our ARG	7.223 076 945 278 57 7.223 076 98(14)	0.494 072 045 042 0.494 072 046(5)	0.027 988 322 686 013 0.027 988 322 7(1)
	$a_\mu^{(eeee)}\,(\alpha/\pi)^{-5}$	$a_\mu^{(eee\mu)}\,(\alpha/\pi)^{-5}$	$a_\mu^{(ee\mu\mu)}\,(\alpha/\pi)^{-5}$
our ARG	20.142 813 095 88 20.142 813 2(5)	2.203 327 312 1947 2.203 327 32(3)	0.206 959 089 016 185 0.206 959 089 0(86)

Theoretical overview

According to the SM, the contributions to a lepton anomaly a_L can be classified into the quantum electromagnetic (QED), hadronic and electroweak type.
Currently, the calculations of the QED contributions of the fourth- and fifth-order, which are important in finding the theoretical error, are mainly determined by numerical integration.
An independent determination of the QED contributions may be important to improve the reliability of calculations.

$$a_L^{\mathrm{QED}} = A_1 + A_2(m_{\ell_1}/m_L) + A_2(m_{\ell_2}/m_L) + A_3(m_{\ell_1}/m_L, m_{\ell_2}/m_L),$$

m_{ℓ_1} and m_{ℓ_2} are the masses of the leptons ``inside'' and m_L is the mass of the external lepton.

A_1, A_2, A_3 can be expanded as power series in the fine-structure constant α:

$$A_i = A_i^{(2)}\left(\frac{\alpha}{\pi}\right) + A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + A_i^{(8)}\left(\frac{\alpha}{\pi}\right)^4 + A_i^{(10)}\left(\frac{\alpha}{\pi}\right)^5 + \ \ldots .$$

A recent improved determination of the fine structure constant
α^{-1} (Cs)= 137.035999046(27) [Parker et al., Science 360, 191–195 (2018)].
(The Era of precision uncertainty)

A_1 is mass independent. The first four coefficients of A_1 (e, μ, τ) are

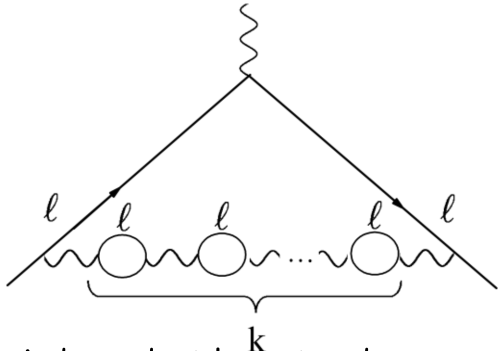
$$A_1^{(2)} = \frac{1}{2} \qquad [\text{Schwinger'1948}]$$

$$A_1^{(4)} = -0.328478965\ldots, \quad A_1^{(6)} = 1.181241456\ldots \ \text{(known analytically)}$$

$$A_1^{(8)} = -1.912245764926445574\ldots. \ \text{(1100-digits result [S. Laporta'2018])}$$

$$A_1^{(10)} = \ 6.675(192) \ [\text{Aoyama, Kinoshita, Nio, Phys.Rev.D 97 } (2018)]$$

The universal contribution formed by the same leptons as external leptons, or in the case where there are no lepton loops.



Mass-independent lowest order vacuum polarization diagram with k lepton loops

$$A_1^{(2)} = A_{1,v.p.}^{(2)} = 1/2$$

$$A_{1,v.p.}^{(4)},\ A_{1,v.p.}^{(6)},\ A_{1,v.p.}^{(8)},\ A_{1,v.p.}^{(10)}\ \ldots$$

where n $(k = n - 1)$ is the number of loops of the Feynman diagrams.

The Riemann ζ-zeta function

$[\zeta(2) = \pi^2/6,\ \zeta(4) = \pi^4/90,\ \zeta(6) = \pi^6/945\ \ldots]$

k	$A_1^{(2k+2)}$	Analytic expression	Numerical value $\times 10^4$
1	$A_{1\ v.p.}^{(4)}$	$\frac{119}{36} - \frac{\pi^2}{3}$	156.8742185910
2	$A_{1\ v.p.}^{(6)}$	$-\frac{943}{324} - \frac{4\ \pi^2}{135} + \frac{8\ \zeta(3)}{3}$	25.5852493652
3	$A_{1\ v.p.}^{(8)}$	$\frac{151849}{40824} - \frac{2\ \pi^4}{45} + \frac{32\ \zeta(3)}{63}$	8.7686585889
4	$A_{1\ v.p.}^{(10)}$	$-\frac{3689383}{656100} - \frac{21928\ \pi^4}{1403325} - \frac{128\ \zeta(3)}{675} + \frac{64\ \zeta(5)}{9}$	4.7090571603
5	$A_{1\ v.p.}^{(12)}$	$\frac{428632663}{42987672} + \frac{19672\ \pi^4}{1607445} - \frac{80\ \pi^6}{5103} + \frac{83360\ \zeta(5)}{22113}$	3.4468727205
6	$A_{1\ v.p.}^{(14)}$	$-\frac{23973913987}{1169170200} - \frac{256\ \pi^4}{637875} - \frac{3027328\ \pi^6}{260513253} - \frac{1119488\zeta(5)}{280665} + \frac{320\ \zeta(7}{9}$	3.1811933817

$(\times 10^{-4})$

7	$A_{1\ v.p.}^{(16)}$	$\frac{8807062662626447}{181934574822000} - \frac{448\,\pi^4}{4100625} + \frac{667648488\,\pi^6}{4433103675} - \frac{112\,\pi^8}{10935} - \frac{130048\,\zeta(5)}{382725} + \frac{68084800\,\zeta(7)}{1980693}$	3.5280404768
. . .			
10	$A_{1\ v.p.}^{(22)}$	$-\frac{50894130940213523554005 97}{391323858211704085668 0}$	

$$7 \quad A_{1\ v.p.}^{(16)} \quad \frac{8807062662626447}{181934574822000} - \frac{448\,\pi^4}{4100625} + \frac{667648488\,\pi^6}{4433103675} - \frac{112\,\pi^8}{10935} - \frac{130048\,\zeta(5)}{382725} + \frac{68084800\,\zeta(7)}{1980693} \qquad 3.5280404768$$

. . .

$$10 \quad A_{1\ v.p.}^{(22)} \quad -\frac{5089413094021352355400597}{39132385821170408566 80} + \frac{5761024\,\pi^6}{16275380625} + \frac{967395724391296\,\pi^8}{625149940910244 75} - \frac{70653378563072\,\pi^{10}}{3782086263828873} + \frac{249123798016\,\zeta(7)}{14481472005} - \frac{1291228747857920\,\zeta(9)}{1610083552077} + \frac{896000\,\zeta(11)}{243} \qquad 10.9948293424$$

✓ ***The growth of coefficients for n>6***

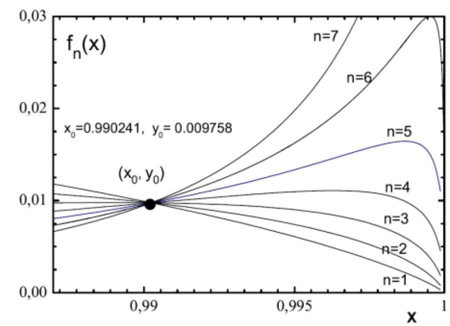
✓ ***Asymptotic formula***

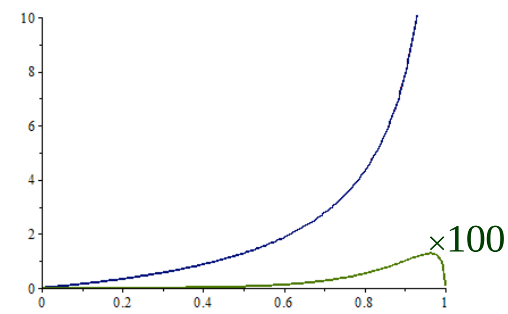
$$A_{1v.p.}^{(2k+2)} = \frac{1}{2}\frac{k!}{6^k} e^{-10/3}\left[1+O(1/k)\right] \quad \text{[Lautrup'1977]}$$

$$A_{1v.p.}^{(2n)} = \int_0^1 dx f_n(x), \quad f_n(x) = (1-x)\left[-\Pi(x)\right]^{n-1}\left(\frac{\alpha}{\pi}\right)^{-n+1}$$

✓ ***The cancellation of terms***

$$A_{1,v.p.}^{(14)} = -20.5067 - 0.039093 - 11.171947 - 4.135991 + 35.852418 \simeq 0.000318$$

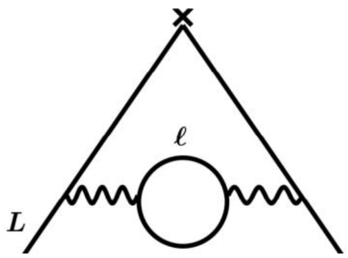




One closed loop

$$a_L^{\mathrm{QED}} = A_1 + A_2(m_{\ell_1}/m_L) + A_2(m_{\ell_2}/m_L) + A_3(m_{\ell_1}/m_L, m_{\ell_2}/m_L)$$

S.Friot, D. Greynat, E. De Rafael, "Asymptotics of Feynman diagrams and the Mellin-Barnes representation", Phys. Lett. B 628 (2005) 73;
"On convergent series representations of Mellin-Barnes integrals", J. Math. Phys 53 (2012) 023508



$$\tilde{A}_2^{(4)}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, t^{-s}\,\frac{(1-s)\,[\Gamma(s)\Gamma(1-s)]^2}{(2+s)(1+2s)(3+2s)}$$

where $t \equiv (m_\ell/m_L)^2$, and $c \in]0,1[$ is the fundamental strip. This integral may be performed using the Cauchy residue theorem through residues. Then, closing the contour of integration in the complex plane to the left, we get the following result:

$$\tilde{A}_2^{(4)}(t) \underset{t<1}{=} -\frac{25}{36} + \frac{1}{4}\pi^2\sqrt{t} + 3\,t - \frac{5}{4}\pi^2\, t^{3/2} + \left(\frac{44}{9} + \frac{\pi^2}{3}\right) t^2$$
$$-\frac{1}{6}\ \ln(t) + \frac{3}{2}\, t\ln(t) + \frac{1}{2}\sqrt{t}\ \mathrm{arctanh}(\sqrt{t})\ln(t)(1-5\,t) - t^2\ln(1-t)\ln(t) + \frac{1}{2}\ t^2\ln^2(t) - t^2\mathrm{Li}_2(t)\,,$$

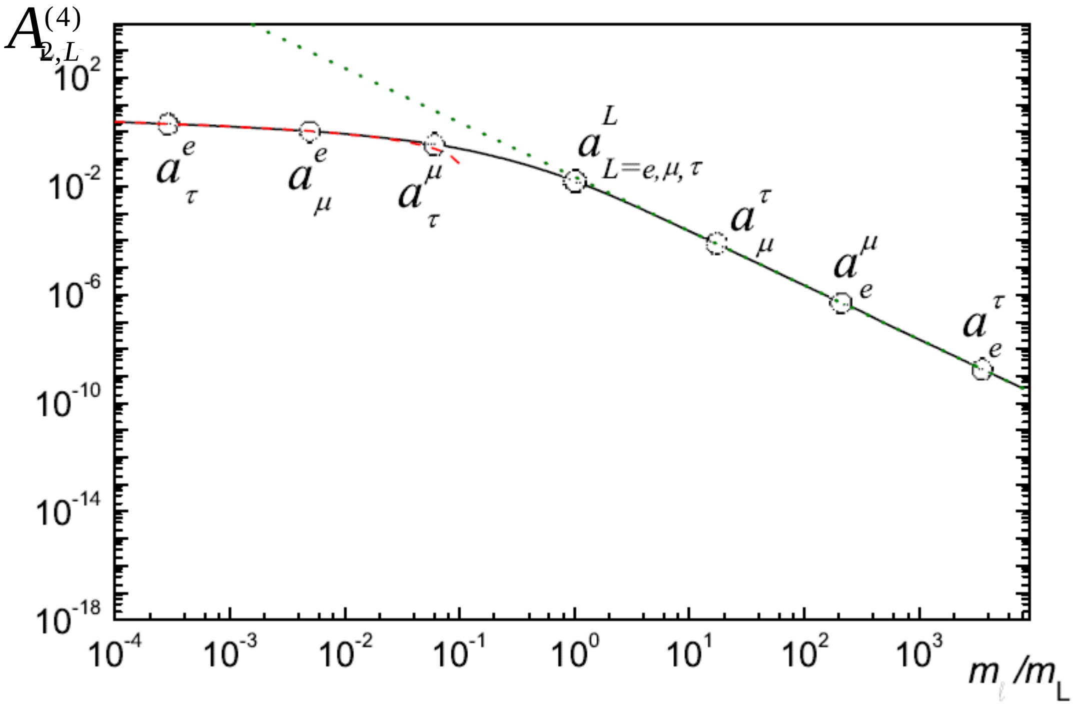
where $\Phi(z,s,a)$ denotes the Lerch function: $\Phi(z,s,a) \doteq \sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s}$ for $|z|<1$ and $a \neq 0,-1,-2,\ldots$. By closing the contour of integration to the right, we get another expression

$$\tilde{A}_2^{(4)}(t) \underset{t\geq 1}{=} -\frac{1}{4} - t + \frac{1}{4\,t}\left[\Phi\left(\frac{1}{t},2,\frac{3}{2}\right) - 5\,\Phi\left(\frac{1}{t},2,\frac{5}{2}\right)\right] - \frac{1}{6}\ \ln(t)$$
$$+\frac{3}{2}\,t\ln(t) + \frac{1}{2}\sqrt{t}\ \mathrm{arccoth}(\sqrt{t})\ln(t)(1-5\,t) - t^2\ln\left(1-\frac{1}{t}\right)\ln(t) + t^2\,\mathrm{Li}_2\left(\frac{1}{t}\right)\,,$$

$$t = \frac{m_l^2}{m_L^2}$$

where $\mathrm{Li}_n(z)$ is the polylogarithm function.

These expressions are analytic continuations of each other.



$$a^\ell_L = A^{(4),\ell}_{2,L}\left(\frac{\alpha}{\pi}\right)^2$$

$$A^{(4),L}_{2,L} = A^{(4)}_1 = \frac{119}{36} - \frac{\pi^2}{36}$$

Open circles correspond values for the physical masses of leptons.

$$A1a_{as} \underset{t<1}{=} -\frac{25}{36} + \frac{1}{4}\pi^2\sqrt{t} - \frac{1}{6}\ln(t) + [3 + 2\ln(t)]\,t \qquad t = \frac{m_l^2}{m_L^2}$$

$$- \frac{5}{4}\pi^2 t^{3/2} + \left[\frac{44}{9} + \frac{1}{3}\pi^2 - \frac{7}{3}\ln(t) + \frac{1}{2}\ln^2(t)\right] t^2 + O(t^3)$$

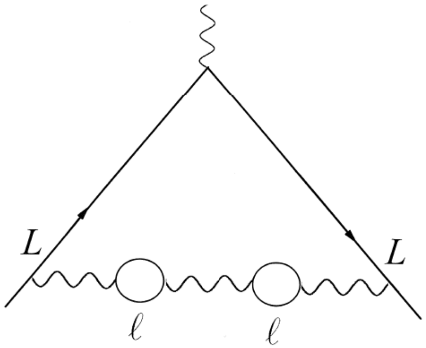
$$A1b_{as} \underset{t>1}{=} \frac{1}{45\,t} + \left[\frac{9}{19600} - \frac{1}{140}\ln(t)\right]\frac{1}{t^2}$$

$$- \left[\frac{2}{315}\ln(t) + \frac{131}{99225}\right]\frac{1}{t^3} + O\left(\frac{1}{t^4}\right)$$

well known result first obtained by Lautrup and Rafael' 1968

(rapid convergence for t>1)

S. Laporta, 1993



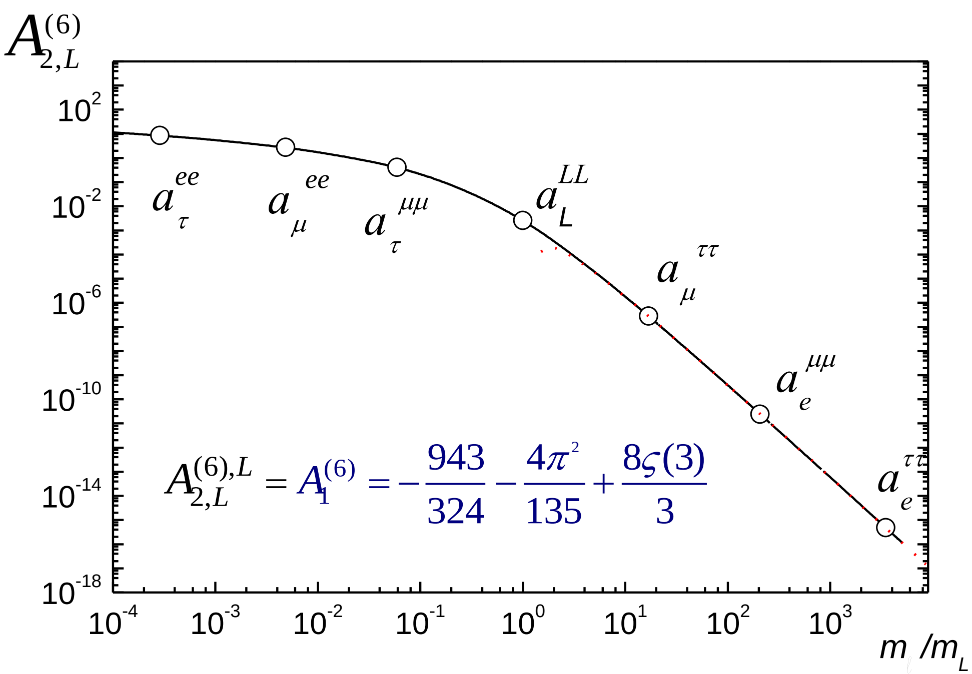
$$
\begin{aligned}
A2a \underset{t<1}{=} \; & \frac{317}{324} + \frac{1}{27}\,\pi^2 - \frac{4}{45}\,\pi^2\,\sqrt{t} - \left(\frac{35}{9} + \frac{4}{9}\,\pi^2\right) t + \left(\frac{32}{81} + \frac{5}{9}\,\pi^2\right) t^2 \\
& + \left(\frac{16}{1125} - \frac{16}{135}\pi^2\right) t^3 + \left[\frac{25}{54} - \frac{4}{9}\,\pi^2 t^2 - \frac{127}{45}\,t + \frac{16}{45}\,t^2 \right. \\
& \left. + \frac{16}{45}\,t^3 \ln(1-t)\right] \ln(t) - \frac{8}{45}\sqrt{t}\,\operatorname{arctanh}(\sqrt{t})\,\ln(t) \\
& + \left(\frac{1}{18} - \frac{2}{3}\,t + \frac{5}{6}\,t^2 - \frac{8}{45}\,t^3\right) \ln^2(t) - \frac{2}{9}\,t^2\,\ln^3(t) \\
& + \frac{8}{3}\,t^2\,\mathrm{Li}_3(t) + \left(\frac{4}{3} - \frac{5}{3}\,t\right) t\,\ln(1-t)\,\ln(t) - \left(\frac{1}{9} - \frac{4}{3}\,t + \frac{5}{3}\,t^2 \right. \\
& \left. - \frac{16}{45}\,t^3 + \frac{4}{3}\,t^2\,\ln(t)\right) \mathrm{Li}_2(t) + \frac{4}{45}\,t^4\,\Phi\left(t, 2, \frac{7}{2}\right) - \frac{1}{9}\ln(1-t)\,\ln(t).
\end{aligned}
$$

$$
t = \frac{m_l^2}{m_L^2}
$$

$$
\begin{aligned}
A2a_{as} \underset{t<1}{=} \; & \frac{317}{324} + \frac{\pi^2}{27} + \frac{25}{54}\ln(t) + \frac{1}{18}\ln^2(t) - \frac{4\pi^2\sqrt{t}}{45} \\
& - \left[4 + \frac{4}{9}\pi^2 + \frac{26}{9}\ln(t) + \frac{2}{3}\ln^2(t)\right] t + \left[\frac{551}{324} + \frac{5}{9}\pi^2 - \left(\frac{53}{54} - \frac{4}{9}\pi^2\right)\ln(t) + \frac{5}{6}\ln^2(t) - \frac{2}{9}\ln^3(t)\right] t^2 \\
& + \left[\frac{13519}{10125} - \frac{16\pi^2}{135} - \frac{224\ln(t)}{675} - \frac{8\ln^2(t)}{45}\right] t^3 + \left[\frac{8905}{21168} - \frac{25\ln(t)}{84}\right] t^4 + O\left(t^5\right).
\end{aligned}
$$

Two-loop result $A_2^{(6)}$

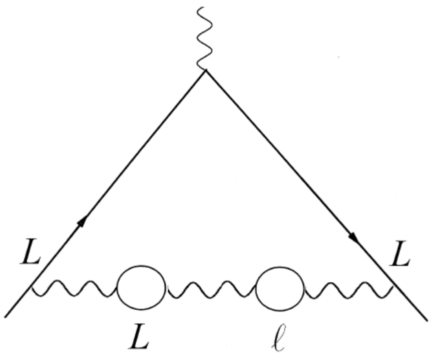
$$A2b \underset{t\geq 1}{=} -\frac{1}{1620\,t^2}\left\{64\,t - 1009\,t^2 + 6876\,t^3 - 576\,t^4 + 144\,\Phi\left(\frac{1}{t}, 2, \frac{5}{2}\right)\right.$$
$$-6\,t^2\ln(t)\left(125 - 762\,t + 96\,t^2\right) + 288\,t^{5/2}\operatorname{arcth}\left(\frac{1}{\sqrt{t}}\right)\ln(t)$$
$$+180\,t^2\left(1 - 12\,t + 15\,t^2 - \frac{16}{5}\,t^3\right)\left[\ln\left(\frac{t-1}{t}\right)\ln(t) - \mathrm{Li}_2\left(\frac{1}{t}\right)\right]$$
$$\left.-2160\,t^4\left[\mathrm{Li}_2\left(\frac{1}{t}\right)\ln(t) + 2\,\mathrm{Li}_3\left(\frac{1}{t}\right)\right]\right\}.$$



There is no such expression in the literature – only the analytic expansions:

$$A2b_{as} \underset{t>1}{=} \frac{1}{t^2}\left[-\frac{161}{54000} + \frac{1}{225}\ln(t)\right]$$
$$+\frac{1}{t^3}\left[-\frac{239}{1984500} + \frac{1}{175}\ln(t)\right]$$
$$+\frac{1}{t^4}\left[\frac{8677}{14288400} + \frac{61}{11340}\ln(t)\right] + O\left(\frac{1}{t^5}\right)$$

S. Laporta, 1993.



$$t = \frac{m_l^2}{m_L^2}$$

$$\begin{aligned}
\bar{A}2a \underset{t<1}{=} & -\frac{233}{810} + \frac{\pi^2}{27} + \frac{32t}{15} + \frac{1}{27}\left(-115 + 2\pi^2\right) t - \frac{4}{45}\pi^2 t^{3/2} + \frac{78512t^2}{10125} - \frac{13\pi^2 t^2}{27} \\
& + \left(\frac{2858}{10125} + \frac{22t}{375}\right) t^3 + \frac{4}{45} t^3 \Phi\left(t, 2, \frac{3}{2}\right) + t^3 \left(\frac{1}{4} + \frac{t}{6} + \frac{11}{12}t^2\right) \Phi\left(t, 3, \frac{7}{2}\right) \\
& + \Bigg[-\frac{1}{4} + \frac{t}{12} - \frac{13t^2}{18} + \frac{\mathrm{ArcTanh}\left(\sqrt{t}\right)}{4\sqrt{t}} + \frac{1}{6}\sqrt{t}\,\mathrm{ArcTanh}\left(\sqrt{t}\right) + \frac{11}{12}t^{3/2}\mathrm{ArcTanh}\left(\sqrt{t}\right) \\
& + \frac{1}{3}\ln(1-t) + \frac{1}{3}t^2\ln(1-t)\Bigg] \ln^2(t) + \left(-\frac{1}{9} - \frac{4}{45t} - \frac{4t}{3} + \frac{13t^2}{9}\right) \mathrm{Li}_2(t) \\
& + \ln(t) \Bigg[-\frac{4}{45} + \frac{1}{162}\left(-357 + 36\pi^2\right) - \frac{67t}{45} - \frac{85t^2}{54} + \frac{1}{450}\left(-993 + 100\pi^2\right) t^2 \\
& - \left(\frac{293}{675} + \frac{11t}{75}\right) t^3 - \frac{8}{45} t^{3/2} \mathrm{ArcTanh}\left(\sqrt{t}\right) - t^3 \left(\frac{1}{4} + \frac{t}{6} + \frac{11}{12}t^2\right) \Phi\left(t, 2, \frac{7}{2}\right) \\
& + \left(-\frac{1}{9} - \frac{4}{45t} - \frac{4t}{3} + \frac{13t^2}{9}\right) \ln(1-t) + \frac{4}{3}\left(1 + t^2\right) \mathrm{Li}_2(t)\Bigg] - 2\left(1 + t^2\right) \mathrm{Li}_3(t) \,.
\end{aligned}$$

$$\begin{aligned}
A2b_{as} \underset{t<1}{=} & \left(\frac{2}{9}\pi^2 - \frac{119}{54}\right) \ln(t) + \frac{\pi^2}{27} - \frac{61}{162} + \left(\frac{4\pi^2}{9} - \frac{115}{27}\right) t - \frac{4}{45}\pi^2 t^{3/2} \\
& + t^2 \left(\frac{2}{15}\ln^2(t) + \left(\frac{2}{9}\pi^2 - \frac{331}{150}\right) \ln(t) - \frac{13}{27}\pi^2 + \frac{124199}{20250}\right) \\
& + \left(-\frac{22}{315}\ln^2(t) + \frac{8243}{33075}\ln(t) - \frac{9074699}{13891500}\right) t^3 \,.
\end{aligned}$$

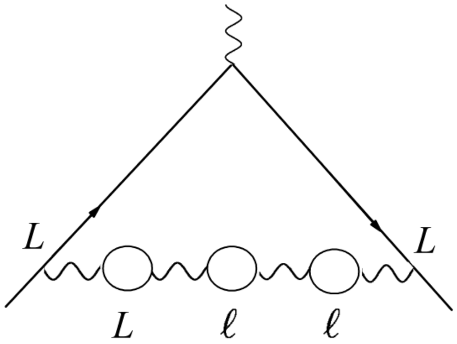
Two-loop result: t>1

$$A2b \underset{t>1}{=} \frac{1}{8100t^2}\left[-360t - 2475t^2 - 4725t^3 + 2025t^{3/2}\mathrm{ArcTanh}\left(\frac{1}{\sqrt{t}}\right)\right.$$
$$+1350t^{5/2}\mathrm{ArcTanh}\left(\frac{1}{\sqrt{t}}\right) + 7425t^{7/2}\mathrm{ArcTanh}\left(\frac{1}{\sqrt{t}}\right) + 2700t^2\ln\left(\frac{-1+t}{t}\right)$$
$$\left.+2700t^4\ln\left(\frac{-1+t}{t}\right)\right]\ln^2(t) + \frac{1}{8100t^2}\ln(t)\left[1116 + \frac{324}{t} + 9888t - 1770t^2\right.$$
$$+23940t^3 - 1440t^{7/2}\mathrm{ArcTanh}\left(\frac{1}{\sqrt{t}}\right) + \left(7425 + \frac{2025}{t^2} + \frac{1350}{t}\right)\Phi\left(t, 2, \frac{7}{2}\right)$$
$$+\left(-720t - 900t^2 - 10800t^3 + 11700t^4\right)\ln\left(\frac{-1+t}{t}\right)$$
$$\left.+\left(-10800t^2 - 10800t^4\right)\mathrm{Li}_2\left(\frac{1}{t}\right)\right] + \frac{1}{8100t^2}\left[\frac{3432}{5} + \frac{648}{5t} + 16960t - 240\pi^2 t\right.$$
$$-5850t^2 + 27900t^3 - 720\Phi\left(\frac{1}{t}, 2, \frac{7}{2}\right) + \left(7425 + \frac{2025}{t^2} + \frac{1350}{t}\right)\Phi\left(\frac{1}{t}, 3, \frac{7}{2}\right)$$
$$\left.+\left(720t + 900t^2 + 10800t^3 - 11700t^4\right)\mathrm{Li}_2\left(\frac{1}{t}\right) - 16200t^2\left(1+t^2\right)\mathrm{Li}_3\left(\frac{1}{t}\right)\right].$$

$$A2b_{as} \underset{t>1}{=} \frac{1}{t}\left(\frac{41}{135} - \frac{4\pi^2}{135}\right) + \frac{1}{t^2}\left(-\frac{96913}{12348000} - \frac{37}{22050}\ln(t) - \frac{1}{420}\ln^2(t)\right)$$
$$+\frac{1}{t^3}\left(-\frac{329573}{75014100} + \frac{1061}{297675}\ln(t) - \frac{2}{945}\ln^2(t)\right).$$

3-bubble result (eighth order): t<1

$$\tilde{A}_2^{(8)}(t) \underset{t<1}{=} \frac{7627}{1944} + \frac{13\pi^2}{27} - \frac{4\,\pi^4}{45} + \frac{175\,t}{18} - \frac{4\pi^2 t}{3} - \frac{54346t^2}{151875} + \frac{67\pi^2 t^2}{81} - \frac{8\pi^4 t^2}{45} + \frac{31168t^3}{13505625}$$
$$+\frac{2\pi^2 t^3}{81} - \frac{32t^4}{15435} - \frac{4}{45}t^5\Phi\left(t,3,\frac{9}{2}\right) + \frac{12}{35}t^4\Phi\left(t,3,\frac{9}{2}\right) + \frac{4}{945}t^2(-21+11t)\ln^3(t) \qquad (7.11)$$
$$+\frac{1}{27}\left(-39+108t-67t^2-2t^3\right)\mathrm{Li}_2(t) - \frac{1}{3780\sqrt{t}}\ln^2(t)\Big\{48(-27+7t)\mathrm{ArcTanh}\left(\sqrt{t}\right)$$
$$+\sqrt{t}\left[-2869+1566t-4162t^2-140t^3+420\pi^2\left(1+2t^2\right)\right.$$
$$\left.+6\left(35+420t-623t^2+88t^3\right)\ln(1-t)\right] - 1260\sqrt{t}\left(1+2t^2\right)\mathrm{Li}_2(t)\Big\} +$$
$$\frac{1}{27}\left(9+108t-\frac{657t^2}{5}+\frac{264t^3}{35}\right)\mathrm{Li}_3(t) + \frac{1}{27}\ln(t)$$
$$\left[\frac{61}{6} - \pi^2 + 136t - 12\pi^2 t - \frac{3734t^2}{375} + 13\pi^2 t^2 - \frac{15936t^3}{42875} + \frac{48t^4}{245}\right.$$
$$\frac{12}{5}t^5\Phi\left(t,2,\frac{9}{2}\right) - \frac{324}{35}t^4\Phi\left(t,2,\frac{9}{2}\right) + \left(-39+108t-67t^2-2t^3\right)\ln(1-t) +$$
$$\left.\left(-6-72t+\frac{462}{5}t^2-\frac{264}{35}t^3\right)\mathrm{Li}_2(t) - 54\left(1+2t^2\right)\mathrm{Li}_3(t)\right] + 4\left(1+2t^2\right)\mathrm{Li}_4(t)\,.$$



3-bubble result

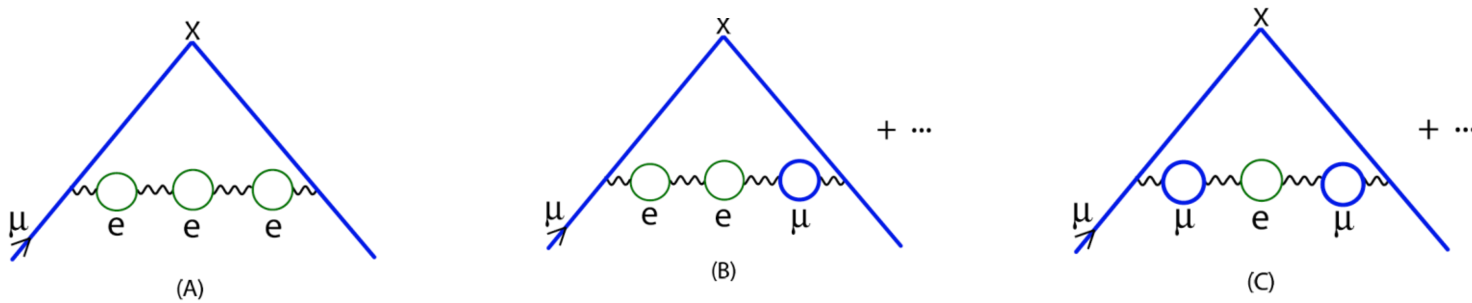
$$\tilde{A}^{(8)}_{2,\ as.}(t) \underset{t<1}{=} \left(\frac{119}{8} - \frac{\pi^2}{9}\right)\ln^2(t) + \left(\frac{61}{162} - \frac{\pi^2}{27}\right)\ln(t) + \frac{7627}{1944} + \frac{13}{27}\pi^2$$
$$-\frac{4}{45}\pi^4 + \left[\left(\frac{115}{27} - \frac{4}{9}\pi^2\right)\ln(t) - \frac{227}{18} + \frac{4}{3}\pi^2\right] t$$
$$+ \left[-\frac{1}{45}\ln^3(t) + \left(\frac{863}{540} - \frac{2}{9}\pi^2\right)\ln^2(t) + \left(-\frac{268061}{40500} + \frac{13}{27}\pi^2\right)\ln(t)\right.$$
$$\left. + \frac{9200857}{1215000} + \frac{67}{81}\pi^2 - \frac{8}{45}\pi^4\right] t^2 + O\left(t^3\right) ,$$

The asymptotic result ($t<1$) is the same as in the paper Phys. Rev. D77 (2008) 093010 (E. De Rafael *et al.*)

Eighth Order MB Integrals

$t = \frac{m_l^2}{m_L^2} \qquad (t < 1)$

ARG [Phys.Rev. D77 (2008) 093010]



The Mellin-Barnes representation (in the notation of Ref. ARG)

$$a_\mu^{(eee)} = \left(\frac{\alpha}{\pi}\right)^4 \frac{\mathbf{1}}{2\pi i} \int\limits_{c_s - i\infty}^{c_s + i\infty} ds \left(\frac{4m_e^2}{m_\mu^2}\right)^{-s} \Gamma(s)\Gamma(1-s)\ \Omega_0(s)\ R_3(s)$$

$$a_\mu^{(ee\mu)} = \left(\frac{\alpha}{\pi}\right)^4 \frac{\mathbf{3}}{2\pi i} \int\limits_{c_s - i\infty}^{c_s + i\infty} ds \left(\frac{4m_e^2}{m_\mu^2}\right)^{-s} \Gamma(s)\Gamma(1-s)\ \Omega_1(s)\ R_2(s)$$

$$a_\mu^{(e\mu\mu)} = \left(\frac{\alpha}{\pi}\right)^4 \frac{\mathbf{3}}{2\pi i} \int\limits_{c_s - i\infty}^{c_s + i\infty} ds \left(\frac{4m_e^2}{m_\mu^2}\right)^{-s} \Gamma(s)\Gamma(1-s)\ \Omega_2(s)\ R_1(s)$$

$$\Omega_0(s) = \frac{\Gamma(1+2s)\Gamma(2-s)}{\Gamma(3+s)}$$

$$\begin{aligned}\Omega_1(s) = \Gamma(2-s)\Bigg\{ & -\frac{4}{3}\frac{\Gamma(-1+2s)}{\Gamma(1+s)} + \frac{4}{3}\frac{\Gamma(2s)}{\Gamma(2+s)} + \frac{5}{9}\frac{\Gamma(1+2s)}{\Gamma(3+s)} \\ & + \left[-\frac{4}{3}\frac{\Gamma(-2+2s)}{\Gamma(s)} + 2\frac{\Gamma(-1+2s)}{\Gamma(1+s)} - \frac{1}{3}\frac{\Gamma(1+2s)}{\Gamma(3+s)}\right]\mathrm{H}_{1-s} + \\ & \frac{4}{3}\frac{\Gamma(-2+2s)}{\Gamma(s)}\mathrm{H}_{-1+s} - 2\frac{\Gamma(-1+2s)}{\Gamma(1+s)}\mathrm{H}_s + \frac{1}{3}\frac{\Gamma(1+2s)}{\Gamma(3+s)}\mathrm{H}_{2+s}\Bigg\}\end{aligned}$$

$$\mathrm{H}_s = \psi(1+s) + \gamma_{\mathrm{E}}$$

$$R_1(s) = \frac{\sqrt{\pi}}{4}\,\frac{1}{s}\,\frac{\Gamma(2+s)}{\Gamma\left(\frac{5}{2}+s\right)}$$

$$R_2(s) = \frac{\sqrt{\pi}}{9}\,\frac{(-1+s)(6+13s+4s^2)}{s^2(2+s)(3+s)}\,\frac{\Gamma(1+s)}{\Gamma\left(\frac{3}{2}+s\right)}$$

$$R_3(s) = \frac{\sqrt{\pi}}{864}\frac{\Gamma(s)}{\Gamma\left(\frac{11}{3}+s\right)}\left[\frac{P_7(s)}{s(1+s)(2+s)} - (1+s)(35+21s+3s^2)\left(27\pi^2 - 162\,\psi^{(1)}(s)\right)\right]$$

$$P_7(s) = 3492 - 8748s - 26575s^2 - 9214s^3 + 18395s^4 + 17018s^5 + 5120s^6 + 512s^7$$

3-bubble result: t>1

$$\tilde{A}_2^{(8)}(t) \underset{t>1}{=} \frac{31937}{68040} + \frac{32}{315t^2} + \frac{23104}{8505t} + \frac{6509t}{630} - \frac{334t^2}{945} + \frac{12\Phi\left(\frac{1}{t},3,\frac{5}{2}\right)}{35t^3} - \frac{4\Phi\left(\frac{1}{t},3,\frac{5}{2}\right)}{45t^2} -$$

$$\frac{1}{27}\left(-39+108t-67t^2-2t^3\right)\mathrm{Li}_2\left(\frac{1}{t}\right) - \frac{1}{3780\sqrt{t}}\ln^2(t)\times$$

$$\Bigg\{48(-27+7t)\mathrm{ArcTanh}\left(\frac{1}{\sqrt{t}}\right) + \sqrt{t}\,\Big[-139-5994t+528t^2+$$

$$6\left(35+420t-623t^2+88t^3\right)\ln\left(\frac{-1+t}{t}\right)\Big] + 1260\sqrt{t}\left(1+2t^2\right)\mathrm{Li}_2\left(\frac{1}{t}\right)\Bigg\}$$

$$-\frac{1}{27}\left(-9-108t+\frac{657t^2}{5}-\frac{264t^3}{35}\right)\mathrm{Li}_3\left(\frac{1}{t}\right) - \frac{1}{5670t^2}\ln(t)\times$$

$$\Bigg[-864-\frac{1944}{t}\Phi\left(\frac{1}{t},2,\frac{5}{2}\right) + 504\Phi\left(\frac{1}{t},2,\frac{5}{2}\right) + t\,\Big(-7552-7895t-27408t^2+2004t^3+$$

$$8190t\ln\left(\frac{-1+t}{t}\right) - 22680t^2\ln\left(\frac{-1+t}{t}\right) + 14070t^3\ln\left(\frac{-1+t}{t}\right) +$$

$$420t^4\ln\left(\frac{-1+t}{t}\right) - 36t\left(35+420t-539t^2+44t^3\right)\mathrm{Li}_2\left(\frac{1}{t}\right) +$$

$$11340\left(t+2t^3\right)\mathrm{Li}_3\left(\frac{1}{t}\right)\Big)\Bigg] - 4\left(1+2t^2\right)\mathrm{Li}_4\left(\frac{1}{t}\right).$$

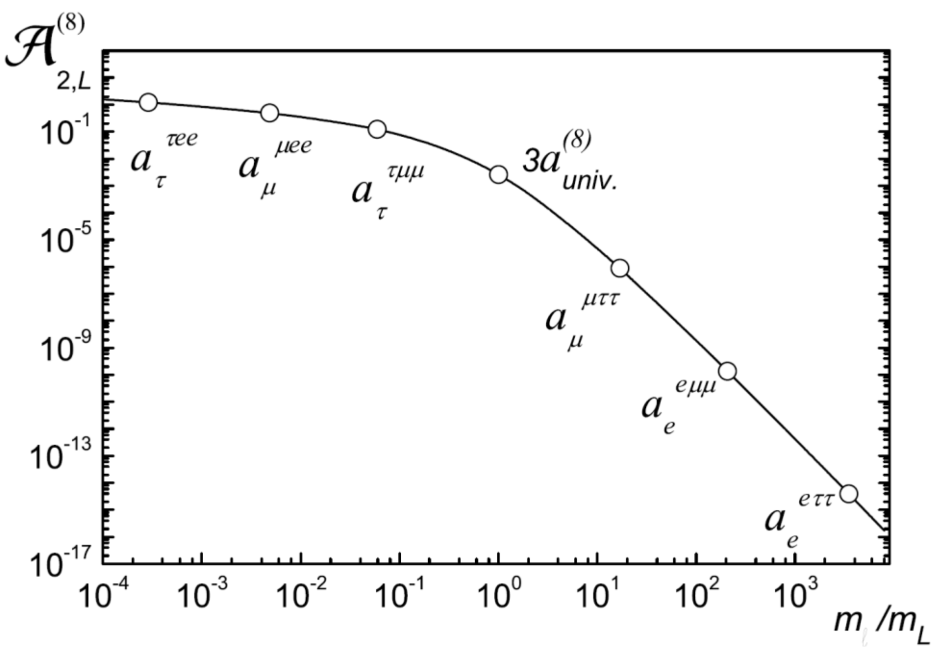
Asymptotic: t>1

$$\tilde{A}^{(8)}_{2,as}(t) \underset{t>1}{=} \left[\frac{5809}{1080000} + \frac{61}{54000}\ln\left(\frac{1}{t}\right) + \frac{1}{450}\ln^2\left(\frac{1}{t}\right)\right]\frac{1}{t^2}$$

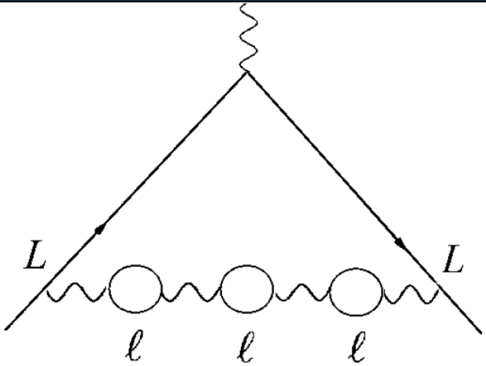
$$+ \left[\frac{1862387}{277830000} + \frac{6073}{992250}\ln\left(\frac{1}{t}\right) + \frac{1}{350}\ln^2\left(\frac{1}{t}\right)\right]\frac{1}{t^3}$$

$$+ \left[\frac{12916049}{9001692000} + \frac{1940611}{200037600}\ln\left(\frac{1}{t}\right) + \frac{671}{317520}\ln^2\left(\frac{1}{t}\right)\right]\frac{1}{t^4} +$$

$$\left[-\frac{15372207553}{31062505320000} + \frac{3811267}{373527000}\ln\left(\frac{1}{t}\right) + \frac{361}{242550}\ln^2\left(\frac{1}{t}\right)\right]\frac{1}{t^5} + O\left[\left(\frac{1}{t^6}\right)\right]$$



3-loop result: t>1 and t<1



$$\bar{A}^{(8)}_{2,\,as.}(t) \underset{t>1}{\cong} \frac{1}{27t^2}\left\{\frac{87709}{360360} - \frac{801}{5005}\zeta(3) + \frac{1}{t}\left[\frac{12204667}{67567500} - \frac{6}{125}\ln(t) - \frac{120}{1001}\zeta(3)\right] + \frac{1}{t^2}\left[\frac{73879547}{656370000} - \frac{9}{125}\ln(t) - \frac{1002}{12155}\zeta(3)\right] + \frac{1}{t^3}\left[\frac{671765975}{9505696200} - \frac{193}{2450}\ln(t) - \frac{18576}{323323}\zeta(3)\right]\right\} + O\left(\frac{1}{t^6}\right).$$

$$\tilde{A}^{(8)}_2(t) \underset{t<1}{\cong} -\frac{1}{54}\ln^3(t) - \frac{25}{108}\ln^2(t) - \left(\frac{317}{324} + \frac{\pi^2}{27}\right)\ln(t) - \frac{8609}{5832} - \frac{25}{162}\pi^2 - \frac{2}{9}\zeta(3) + \frac{101}{1536}\pi^4\sqrt{t}$$

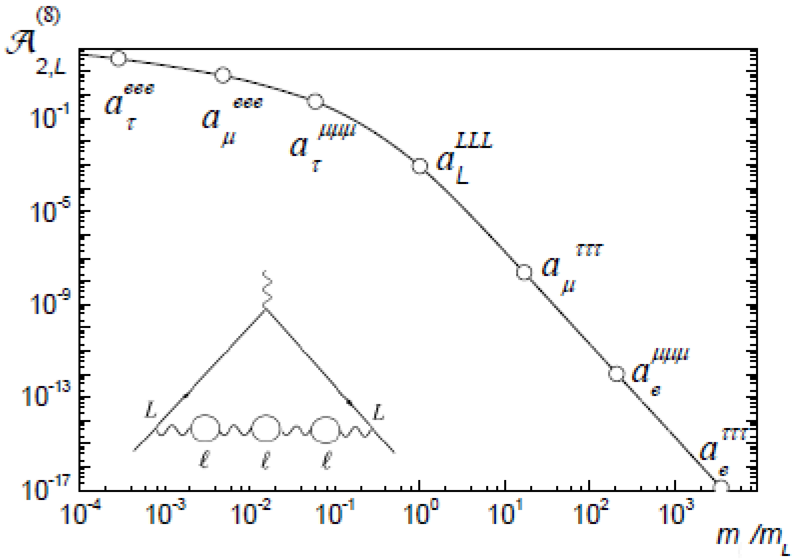
$$\left[-\frac{1}{54}\ln^3(t) - \frac{25}{108}\ln^2(t) - \left(\frac{317}{324} + \frac{\pi^2}{27}\right)\ln(t) - \frac{8609}{5832} - \frac{25}{162}\pi^2 - \frac{2}{9}\zeta(3)\right]$$

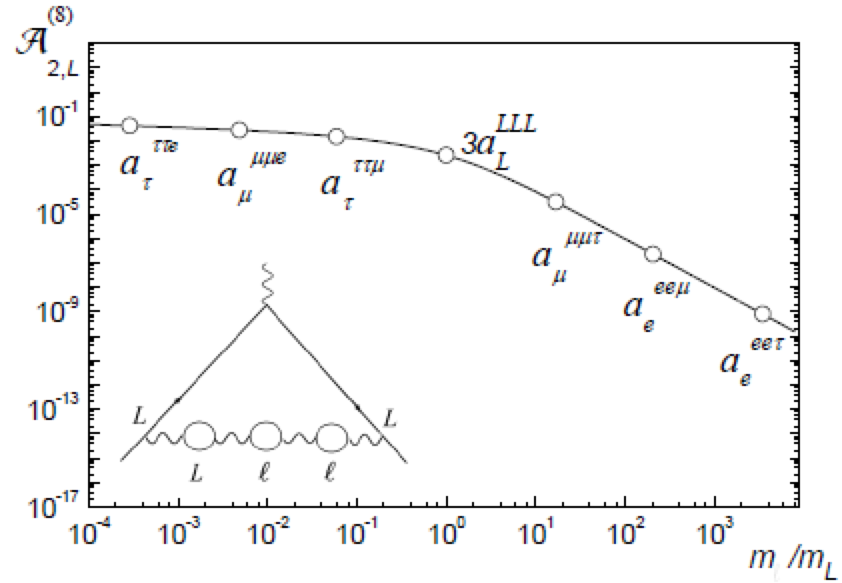
$$+ \left[\frac{2}{9}\ln^3(t) + \frac{13}{9}\ln^2(t) + \left(\frac{152}{27} + \frac{4}{9}\pi^2\right)\ln(t) + \frac{967}{315} + \frac{26}{27}\pi^2 + \frac{136}{35}\zeta(3)\right] t$$

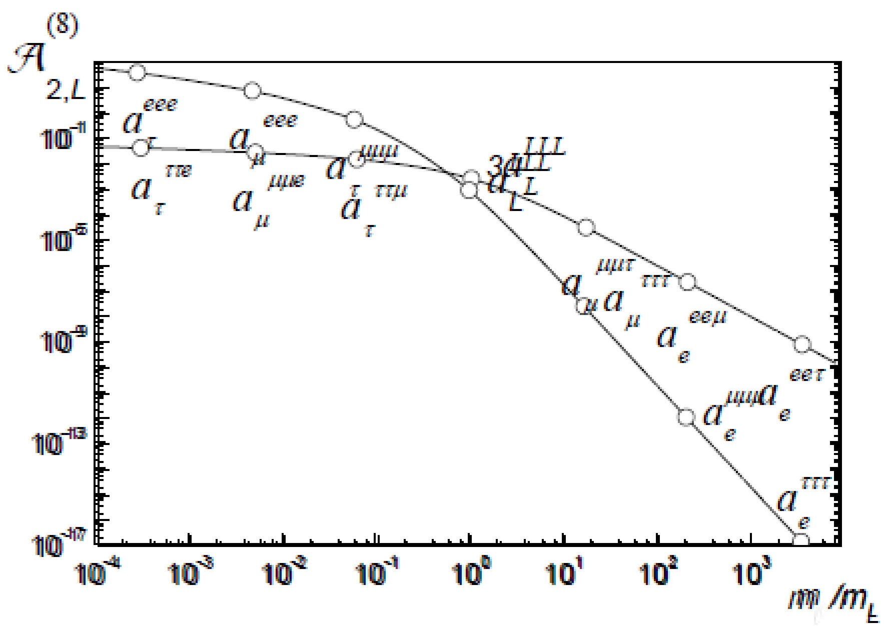
$$+ \left\{\frac{1}{12}\ln^4(t) - \frac{8}{27}\ln^3(t) + \left(\frac{127}{108} + \frac{\pi^2}{3}\right)\ln^2(t) - \left[\frac{236}{27} + \frac{16}{27}\pi^2 - 4\zeta(3)\right]\ln(t) + \frac{63233}{3240} + \frac{127}{162}\pi^2 + \frac{1}{5}\pi^4 - \frac{64}{15}\zeta(3)\right\} t^2 + O(t^3)$$

This_result is the same as in the paper **AGR**

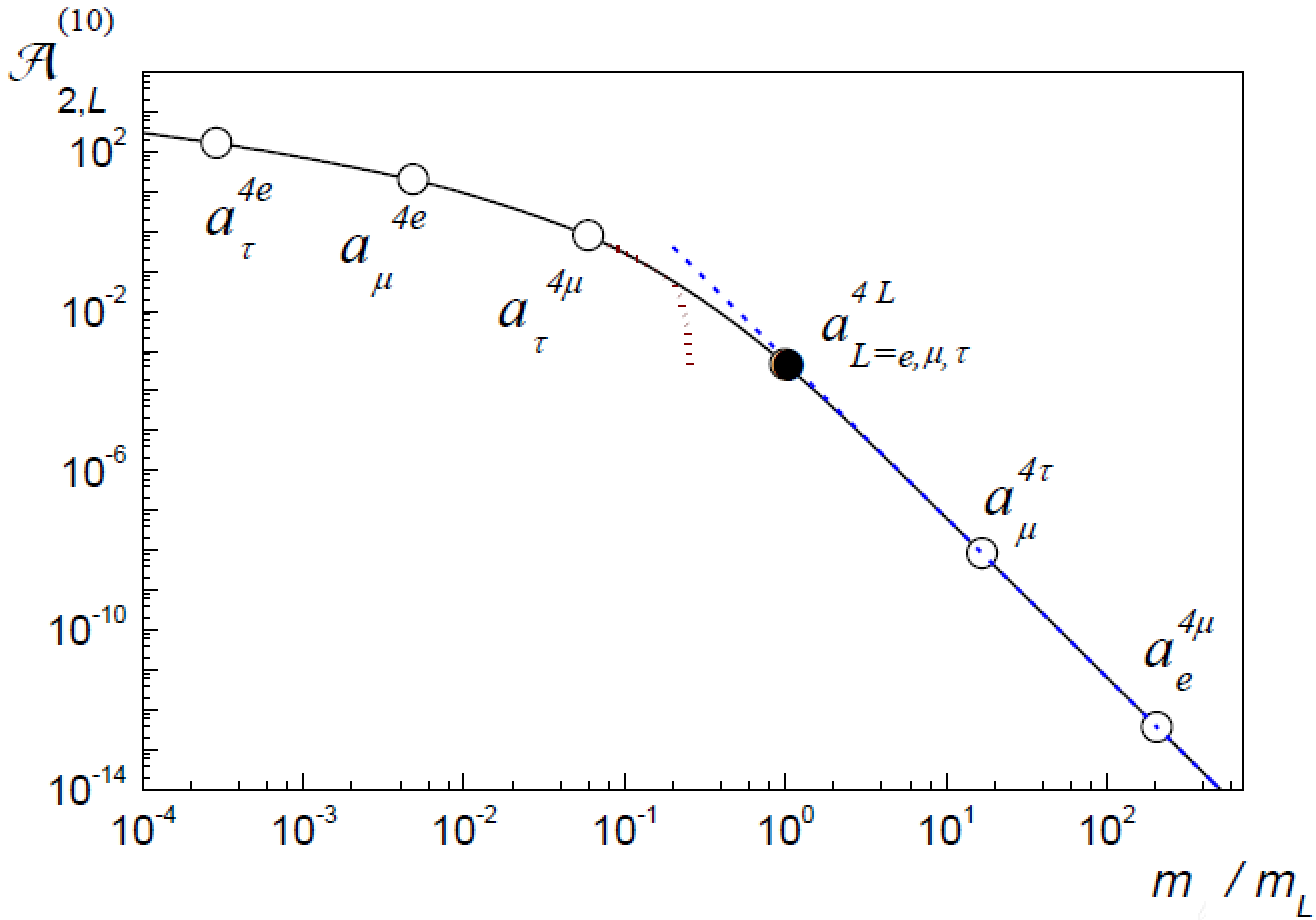
3-loop result







4-loop result: t>1 and t<1



Summary

We have obtained exact expressions for certain class of mass-dependent eighth-order coefficients determining the contributions of the three-bubble diagrams to the anomalous magnetic moment of all three charged leptons for all possible values of the mass ratio (only approximate formulas of the form of expansions were known). We find a good agreement with the known analytical expansions given in terms of the mass ratio.

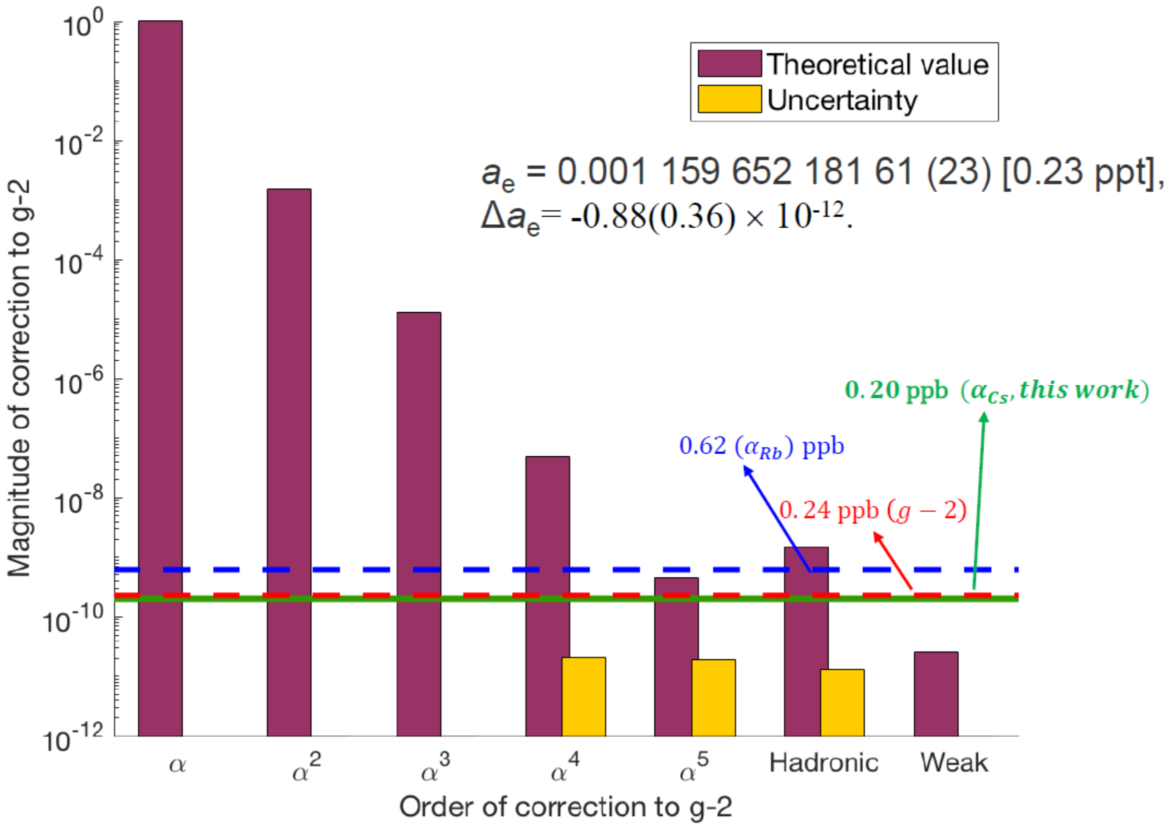
We demonstrated that exact expressions correspond to a common unified analytical function. Exact expressions allow us to calculate the coefficients with any accuracy and can be used to have an independent way to check the precision of numerical estimations.

Thanks for your attention !

Thanks for your attention !

The Era of precision uncertainty

Results



Parker et al., Science 360, 191–195 (2018)