Cosmological models with non-minimal coupling and bounce solutions

Sergey Yu. Vernov

Skobeltsyn Institute of Nuclear Physics,
Lomonosov Moscow State University

based on the work in collaboration with

E.O. Pozdeeva, M.A. Skugoreva, and A.V. Toporensky,
JCAP **1612** (2016) 006 (arXiv:1608.08214)

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ACTION AND EQUATIONS

Let us consider

$$S=\int d^4x\sqrt{-g}\left[U(\varphi)R-\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}-V(\varphi)\right], \qquad (1)$$

where $U(\varphi)$ and $V(\varphi)$ are differentiable functions of the scalar field φ.

Models with scalar fields are very useful to describe the observable global evolution of the Universe as the dynamics of the spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) background with the interval

$$ds^2 = -\,dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right).$$

In the spatially flat FLRW metric the equations are

$$6UH^2 + 6\dot{U}H - \frac{1}{2}\dot{\varphi}^2 - V = 0, \qquad (2)$$

$$2U\left[2\dot{H} + 3H^2\right] + 2U'\left[\ddot{\varphi} + 2H\dot{\varphi}\right] = V - \left[2U'' + \frac{1}{2}\right]\dot{\varphi}^2, \qquad (3)$$

$$\ddot{\varphi} + 3H\dot{\varphi} - 6U'\left[\dot{H} + 2H^2\right] + V' = 0, \qquad (4)$$

where a "dot" means a derivative with respect to the cosmic time t and a "prime" means a derivative with respect to the scalar field φ.
The function $H = \dot{a}/a$ is the Hubble parameter.
If a solution of Eqs. (2)–(4) has such a point t_b that

$$H(t_b) = 0, \qquad \dot{H}(t_b) > 0,$$

then it is a bounce solution.
Let us find conditions that are necessary for the existence of a bounce solution.
Using Eq. (2), we get that from $H(t_b) = 0$ it follows $V(\varphi(t_b)) \leqslant 0$.

Subtracting equation (2) from equation (3), we obtain

$$4U\dot{H} = -\dot{\varphi}^2 - 2\ddot{U} + 2H\dot{U}. \tag{5}$$

Therefore, if $U = const > 0$ a bounce solution **does not exist.**
At the bounce point we get

$$2\left(U + 3U'^2\right)\dot{H}(t_b) = U'V' + [2U'' + 1]\,V. \tag{6}$$

Functions U and V and their derivatives are taken at the point $\varphi(t_b)$.
The condition $\dot{H}(t_b) > 0$ gives the restriction on functions U and V at the bounce point.

MINIMAL AND NON-MINIMAL COUPLING

An effective gravitation constant in the model considered is

$$G_{eff} = \frac{1}{16\pi U}.$$

The dynamics of the FLRW Universe can be prolonged smoothly into the region of $G_{eff} < 0$, however, any anisotropic or inhomogeneous corrections are expected to diverge while G_{eff} tends to infinity.
We analyze such bounce solutions that $U(\varphi(t)) > 0$ for any $t \geqslant t_b$ and conditions of their existence.
If $U = const$ the coupling is minimal. It is the Einstein frame.
Let us introduce new variables to get equations that look similar in both in the Einstein frame and in the Jordan frame.
The effective potential is

$$V_{eff}(\varphi) = \frac{V(\varphi)}{4K^2 U(\varphi)^2}. \quad (7)$$

If $K = 8\pi G$, then V_{eff} coincides with the potential of the corresponding model in the Einstein frame.

We also introduce the functions

$$P \equiv \frac{H}{\sqrt{U}} + \frac{U'\dot{\varphi}}{2U\sqrt{U}}, \qquad A \equiv \frac{U + 3{U'}^2}{4U^3}$$

and get equations:

$$3P^2 = A\dot{\varphi}^2 + 2K^2 V_{eff}\,, \qquad (8)$$

$$\dot{P} = \, -A\sqrt{U}\,\dot{\varphi}^2. \qquad (9)$$

If $U(\varphi) > 0$, then $A(\varphi) > 0$ as well.
The Hubble parameter $H(t)$ is a monotonically decreasing function in a model with $U = const > 0$.
The function $P(t)$ is a monotonically decreasing function in a model with an arbitrary positive $U(\varphi)$.
The functions P and V_{eff} have been introduce in
M.A. Skugoreva, A.V. Toporensky, S.Yu. Vernov, Phys. Rev. D **90** (2014) 064044 (arXiv:1404.6226)

From Eqs. (8) and (9) it is easy to get the following system:

$$\dot{\varphi} = \psi\,, \qquad \dot{\psi} = \, -3P\sqrt{U}\psi - \frac{A'}{2A}\psi^2 - \frac{K^2 V'_{eff}}{A}. \qquad (10)$$

De Sitter solutions corresponds to $\psi = 0$, and hence,

$$V'_{eff}(\varphi_{dS}) = 0.$$

The corresponding Hubble parameter is

$$H_{dS} = P_{dS}\sqrt{U(\varphi_{dS})} = \pm\sqrt{\frac{V(\varphi_{dS})}{6U(\varphi_{dS})}}\,.$$

We analyze the stability of de Sitter solutions with $H_{dS} > 0$ and $U(\varphi_{dS}) > 0$ only.
For arbitrary differentiable functions V and $U > 0$, the model has a stable de Sitter solution with $H_{dS} > 0$ only if

$$V''_{eff}(\varphi_{dS}) > 0, \qquad V_{eff}(\varphi_{dS}) > 0.$$

The de Sitter point is a stable node (the scalar field decreases monotonically) at

$$\frac{3\left(U + 3U'^2\right)}{8U^2} \geqslant \frac{V''_{eff}}{V_{eff}}, \tag{11}$$

and a stable focus (the scalar field oscillations exist) at

$$\frac{3\left(U + 3U'^2\right)}{8U^2} < \frac{V''_{eff}}{V_{eff}}. \tag{12}$$

- In the spatially flat FLRW metric $R = 6(\dot{H} + 2H^2)$.
- From (2)–(4) we get

$$2R\left(U + 3{U'}^2\right) + (6U'' + 1)\,\dot{\varphi}^2 = 4V + 6V'U'\,. \qquad (13)$$

- From the structure of Eq. (13) it is easy to see that the simplest way to get a constant R is to choose such $U(\varphi)$ that

$$U + 3U'^2 = U_0, \qquad 6U'' + 1 = 0, \qquad U_0R = 2V + 3V'U'.$$

- The solution to the first two equations is

$$U_c(\varphi) = U_0 - \frac{\varphi^2}{12} \qquad (14)$$

For $U = U_c$ Eq. (13) can be simplified:

$$2U_0 R = 4V(\varphi) - \varphi V'(\varphi). \tag{15}$$

and has the following solution:

$$V_{int} = 2U_0\Lambda + C_4\varphi^4, \qquad R = 4\Lambda. \tag{16}$$

where C_4 is an integration constant.
Thus, requiring that the Ricci scalar is a constant one can define both functions $U(\varphi) = U_c$ and $V(\varphi) = V_{int}$. To get a positive G_{eff} for some values of φ we choose $U_0 > 0$.
This integrable cosmological model has been considered in
B. Boisseau, H. Giacomini, D. Polarski, and A.A. Starobinsky, *Bouncing Universes in Scalar-Tensor Gravity Models admitting Negative Potentials* J. Cosmol. Astropart. Phys. **1507** (2015) 002 (arXiv:1504.07927), where the behavior of bounce solutions has been studied in detail.

- Using the explicit form of functions U_c and V_{int} we get that the condition $\dot{H}_b > 0$ is equivalent to $\Lambda > 0$, hence, from $V(\varphi_b) < 0$ it follows $C_4 < 0$.
- Considering the equation

$$R = 6(\dot{H} + 2H^2) = 4\Lambda,$$

such as a differential equation for the Hubble parameter, we obtain two possible real solutions in dependence of the initial conditions:

$$H_1 = \sqrt{\frac{\Lambda}{3}}\tanh\left(\frac{2\sqrt{\Lambda}(t-t_0)}{\sqrt{3}}\right), \quad H_2 = \sqrt{\frac{\Lambda}{3}}\coth\left(\frac{2\sqrt{\Lambda}(t-t_0)}{\sqrt{3}}\right),$$

where t_0 is an integration constant.

Note that C_4 defines the initial condition of the Hubble parameter via Eq. (2):

$$6UH^2 + 6\dot{U}H - \frac{1}{2}\dot{\varphi}^2 - V = 0,$$

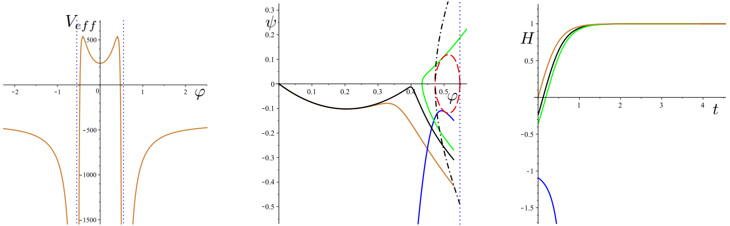

Figure: The effective potential (left picture), phase trajectories (middle picture) and the Hubble parameter $H(t)$ (right picture) for $V = C_4\varphi^4 + C_0$ and $U = U_0 - \varphi^2/12$. The parameters are $U_0 = 1/40$, $C_4 = -3$, $C_0 = 0.15$. The initial values are $\varphi_i = 0.53$, and $\psi_i = -0.4164479079$ (gold line), $\psi_i = -0.31$ (black line), $\psi_i = -0.27$ (green line), $\psi_i = -0.15$ (blue line). The black dash curve corresponds to $H = 0$. The blue point lines correspond to $U = 0$. The red dash curve corresponds to $P = 0$.

The red dash curve, defined by the equation $P = 0$, is the boundary of unreachable domain. Any point inside this curve corresponds to a non-real value of the Hubble parameter.
Such domain exists at any model with action (1) that has a bounce solution, because $V(\varphi_b) < 0$.
Solutions can touch the boundary of unreachable domain.

MODEL WITH A HIGGS-LIKE POTENTIAL

There are two way of modification: to modify U or to modify V.

- Let us modify V and consider the model with

$$U_c(\varphi) = \frac{1}{2K} - \frac{\varphi^2}{12} \tag{17}$$

and

$$V_c = C_4\varphi^4 + C_2\varphi^2 + C_0. \tag{18}$$

- Such as we consider only gravity regime ($G_{eff} > 0$) $U_c > 0$, we use the restriction $\varphi_b^2 < 6/K$.
- $$V_{eff} = \frac{36(C_4\varphi^4 + C_2\varphi^2 + C_0)}{(K\varphi^2 - 6)^2}. \tag{19}$$

  The even potential V_{eff} has an extremum at $\varphi = 0$ and at points

$$\varphi_m = \pm\sqrt{\frac{-2(3C_2 + KC_0)}{12C_4 + KC_2}}. \tag{20}$$

- The model with $V_c = C_4\varphi^4 + C_2\varphi^2 + C_0$ has a bounce solution only if

$$C_4\varphi_b^4 + C_2\varphi_b^2 + C_0 < 0, \quad C_2\varphi_b^2 + 2C_0 > 0, \quad C_2 + 2C_4\varphi_b^2 < 0.$$

- We specify the case $C_4 < 0$. Supposing that φ_m are real we get

$$0 > C_2 + 2\varphi_b^2 C_4 > C_2 + \frac{12}{K} C_4.$$

- So, the model with a bounce solution has real φ_m only at

$$3C_2 + KC_0 > 0 \quad \text{and} \quad KC_2 + 12C_4 < 0. \tag{21}$$

Using conditions to the model constants, we get

$$V''_{eff}(0) = \frac{C_0 K + 3C_2}{6} > 0, \quad V''_{eff}(\varphi_m) = -\frac{36(C_2 K + 12C_4)^3 (C_0 K + 3C_2)}{(C_0 K^2 + 6C_2 K + 36C_4)^3} < 0.$$

The effective potential has a minimum at $\varphi = 0$ and maxima at $\varphi = \varphi_m$.

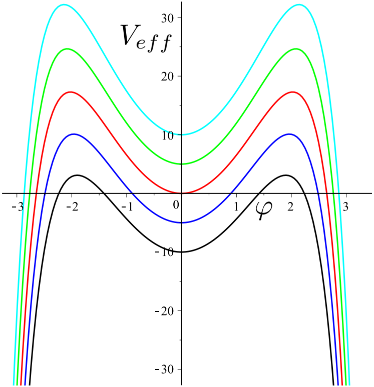

Figure: The effective potential V_{eff} at different values of parameters. In the picture we choose $K = 1/4$. The values of parameters are $C_4 = -1$, $C_2 = 7$ (left picture). The parameter $C_0 = -10$ (black curve), -5 (blue curve), 0 (red curve), 5 (green curve), and 10 (cyan curve).

ANALYSIS OF NUMERIC SOLUTIONS

Equations (2)–(4) can be transformed into the system of the first order equations which is useful for numerical calculations.
For $U = U_c$ and an arbitrary potential, this system has the following form:

$$\begin{cases} \dot{\varphi} = \psi, \\ \dot{\psi} = -3H\psi - \frac{1}{6}\left(6 - K\varphi^2\right)V' + \frac{2}{3}K\varphi V, \\ \dot{H} = -\frac{K}{6}\left[2\varphi^2 H^2 + (4H\psi + V')\varphi + 2\psi^2\right]. \end{cases} \tag{22}$$

If Eq. (2) is satisfied in the initial moment of time, then from system (22) it follows that Eq. (2) is satisfied at any moment of time.
We integrate this system with $V = V_c$ numerically.
The evolution of the scalar field starts at the bounce point $\varphi_b > 0$ with a negative velocity, defined by relation

$$\psi_i = \dot{\varphi}_b = -\sqrt{-2V(\varphi_b)}.$$

In the case $C_0 > 0$ there are three possible evolutions of the bounce solutions.

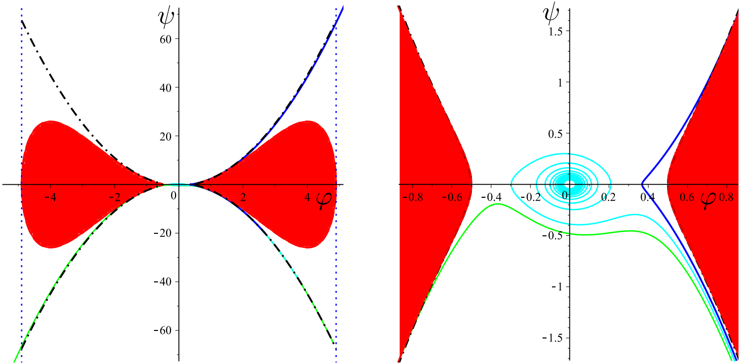

Figure: A phase trajectories for the models with U_c and V_c. The values of constants are $K = 1/4$, $C_4 = -4$, $C_2 = 1$, and $C_0 = 0$. The initial conditions are $\varphi_i = 2.7$, $\psi_i = -20.26259608$ (blue line), $\varphi_i = 3.7$, $\psi_i = -38.36598493$ (cyan line), and $\varphi_i = 4.8$, $\psi_i = -64.81244325$ (green line). The black curves are the lines of the points that correspond to $H = 0$. The unreachable domain, defined by $P < 0$, is in red. The blue point lines are $U = 0$.

For $C_0 > 0$ there exists the stable de Sitter solution $\varphi_{dS} = 0$ and $H_{dS} = \sqrt{\frac{C_0 K}{3}}$. It is a stable node at $KC_0 - 24C_2 \geqslant 0$ and a stable focus in the opposite case $C_0 K - 24C_2 < 0$.

The example of a stable node at $\varphi = 0$.

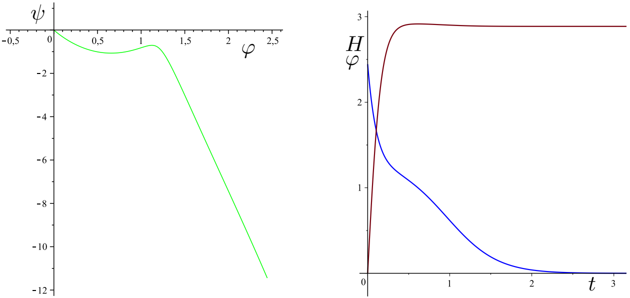

Figure: The field φ (blue line) and the Hubble parameter (red line) as functions of the cosmic time are presented in the right picture. The values of parameters are $K = 1$, $C_4 = -2.7$, $C_2 = 1$ and $C_0 = 25$. The initial conditions of the bounce solution are $\varphi_i = 2.445$, $\psi_i = -11.44650941$, and $H_i = 0$.

The example of a stable focus at $\varphi = 0$.

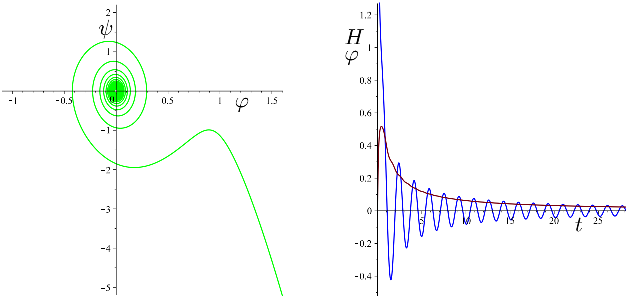

Figure: The values of constants are $K = 1/4$, $C_4 = -4$, $C_2 = 7$, $C_0 = 0$. The initial conditions are $\varphi_i = 3.4$ and $\psi_i = -30.12023904$. A zoom of the central part of phase plane is presented in the middle picture. The Hubble parameter (red) and the scalar field (blue) of functions of cosmic time are presented in the right picture.

We have presented solutions for models with positive value of C_0.
Let us consider now the phase trajectory at $C_0 = -0.1$.
We see that trajectories are similar at the beginning only. The scalar field tends to infinity and the system comes to antigravity domain with $U_c < 0$.

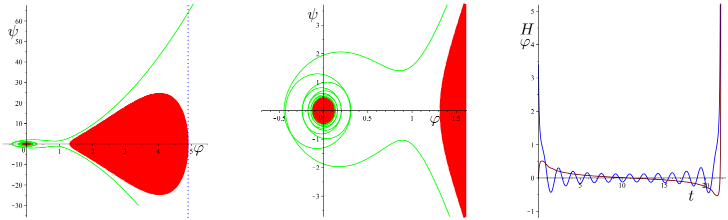

Figure: A phase trajectory for the models with U_c and V_c is presented in the left picture. The values of constants are $K = 1/4$, $C_4 = -4$, $C_2 = 7$, $C_0 = -0.1$. The initial conditions are $\varphi_i = 3.4$ and $\psi_i = -30.12355889$. A zoom of the central part of phase plane is presented in the middle picture. The Hubble parameter (red) and the scalar field (blue) of functions of cosmic time are presented in the right picture.

The difference between the solutions of system (22) with a positive and a negative C_0 is demonstrated. The cyan curves correspond to $C_0 = 10$, whereas the red curves correspond to $C_0 = -10$.
We see that the phase trajectories and the Hubble parameter are similar in the beginning, but stand essentially different in the future.

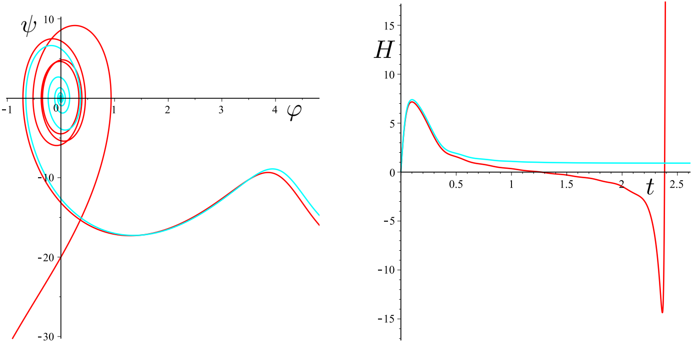

Figure: The phase trajectories (right picture) and the corresponding Hubble parameters (left picture) are presented. The values of parameters are $K = 1/4$, $C_4 = -4$, $C_2 = 90$. The initial conditions of bounce solution are $\varphi_i = 4.88$, $\psi_i = -15.17936692$ (cyan curve) and $\psi_i = -16.44424459$ (red curve).

We see that the behavior of solutions essentially depends on the sign of C_0. Let us understand the reason of this dependence.
Equation (2) is a quadratic equation for the Hubble parameter:

$$6UH^2 + 6\dot{U}H - \frac{1}{2}\dot{\varphi}^2 - V = 0, \tag{23}$$

This equation has the following solutions:

$$H_{\pm} = -\frac{\dot{U}}{2U} \pm \frac{1}{6U}\sqrt{9U'^2\dot{\varphi}^2 + 3U\dot{\varphi}^2 + 6UV}. \tag{24}$$

The values of the function $P(\varphi)$ that correspond to $H_{\pm}$ are

$$P = \pm\frac{1}{6U}\sqrt{3\left[\frac{3U'^2}{U}\dot{\varphi}^2 + \dot{\varphi}^2 + 2V\right]} = \pm\sqrt{\frac{A}{3}\dot{\varphi}^2 + \frac{2}{3}K^2 V_{eff}}\,. \tag{25}$$

In the domain with $V > 0$ the Hubble parameter is uniquely defined as a function φ and ψ by (24).
If $C_0 \geqslant 0$, then whole evolution of bounce solutions is evolution a solution of the second order system.
For $C_0 < 0$ the third order system (22) with the additional condition (2) is not equivalent to any second order system.

Let us consider the domain $|\varphi| < \sqrt{6/K}$.
From (24) it follows that

$$\dot{\varphi}^2 \geqslant -\frac{2UV}{U + 3U'^2} = -4KUV. \tag{26}$$

If $V \geqslant 0$ for all $-\varphi_m < \varphi < \varphi_m$, then this condition is always satisfied and φ tends to a minimum of V_{eff}.
For suitable initial $\dot{\varphi}$ the evolution is finished at $\varphi = 0$.
If $P > 0$ in the moment when potential stands positive and the function φ tends to zero, then $P > 0$ at any moment in future.

If the constants C_i are such that the potential change the sign and $V(0) < 0$ then the evolution is different.
If $C_0 < 0$, then there is a restricted domain in the neighborhood of $(0, 0)$ point on phase plane such that the values φ and ψ correspond to non-real value of the Hubble parameter.
The boundary of this domain is defines by equation $P = 0$.
We see that the phase trajectory rotates around this domain.
The trajectory can not cross the boundary, but all such trajectories touch the boundary at some finite moment of time.
Let for some moments of time t_1 and $t_2 > t_1$ we have $\varphi(t_2) = \varphi(t_1)$, then, using $U + 3U'^2 = \frac{1}{2K}$ and formula (9), we get

$$P(t_2) - P(t_1) = -\int_{t_1}^{t_2} \frac{U + 3U'^2}{4U\sqrt{U}} \psi^2 \, dt = -\int_{t_1}^{t_2} \frac{1}{8KU\sqrt{U}} \psi^2 \, dt \leqslant \tilde{C} < 0.$$

where $\tilde{C}$ is a negative number. Therefore, this integral has a finite negative value.

For any circle value of P decreases on some positive value, which doesn't tend to zero during evolution, when number of circles increase.
We come to conclusion that only a finite number of circles is necessary to get the value $P = 0$.
At this point $\dot{P} < 0$ as well, so the function P changes the sign. When $P = 0$ two possible values of the Hubble parameters: H_+ and H_- coincide. At this moment the value of the Hubble parameter changes from H_+ to H_-.
The similar result has been obtained in
I.Ya. Aref'eva, N.V. Bulatov, R.V. Gorbachev, S.Yu. Vernov, Class. Quantum Grav. **31** (2014) 065007 (arXiv:1206.2801).

INFLATIONARY MODELS

Using conformal transformation of the metric, one can get from the model with a constant R the corresponding model with a minimally coupled scalar field $\tilde{\varphi}$ has the potential

$$W(\varphi) = C_1 \cosh^4\left(\frac{\tilde{\varphi}}{2\sqrt{3U_0}}\right) - C_2 \sinh^4\left(\frac{\tilde{\varphi}}{2\sqrt{3U_0}}\right), \qquad (27)$$

C_1 and C_2 are constants.
The integrability of this system has been proved in
I. Bars and S.H. Chen,, 2011, *Phys. Rev.* D **83** 043522 (arXiv:1004.0752)
I. Bars and S.H. Chen, N. Turok, 2011, *Phys. Rev.* D **84** 083513 (arXiv:1105.3606)
B. Boisseau, H. Giacomini and D. Polarski, JCAP **1510** (2015) 033 (arXiv:1507.00792).
In I. Bars and S.H. Chen,
The Big Bang and Inflation United by an Analytic Solution, Phys. Rev. D **83** (2011) 043522 (arXiv:1004.0752)
the inflationary scenario has been constructed.

Conclusions

- The bounce solution with a non-monotonic Hubble parameter has been obtained.
- We show that the generalization of bounce potentials leads to different behaviors of the Hubble parameter.
- It would be very interesting to construct cosmological model with a non-minimal coupling standard scalar field, a bounce solution of which is suitable for inflationary scenario.