U(1) gauged Q-balls and their properties

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Based on:

I. E. Gulamov, E. Y. Nugaev, A. G. Panin and M. N. Smolyakov,
"Some properties of U(1) gauged Q-balls",
Phys. Rev. D **92** (2015) 045011 [arXiv:1506.05786],

A. G. Panin and M. N. Smolyakov,
"Problem with classical stability of U(1) gauged Q-balls",
Phys. Rev. D **95** (2017) 065006 [arXiv:1612.00737].

Ordinary Q-balls

Proposed in G. Rosen, "Particlelike Solutions to Nonlinear Complex Scalar Field Theories with Positive-Definite Energy Densities", J. Math. Phys. **9** (1968) 996,

Became popular after S. R. Coleman, "Q-balls", Nucl. Phys. B **262** (1985) 263 [Erratum-ibid. B **269** (1986) 744].

$$S = \int d^4x \left(\partial^\mu \phi^* \partial_\mu \phi - V(\phi^*\phi)\right),$$

$$\phi(t,\vec{x}) = e^{i\omega t} f(r), \qquad f(r)|_{r\to\infty} \to 0, \qquad \partial_r f(r)|_{r=0} = 0,$$

$$r = \sqrt{\vec{x}^2}.$$

Without loss of generality $f(r) > 0$.

Q-ball charge:

$$Q = -i \int d^3x \left(\phi^* \dot{\phi} - \dot{\phi}^* \phi\right) = 8\pi\omega \int_0^\infty f^2\, r^2 dr$$

Q-ball energy:

$$E \quad = \quad \int d^3x \left(\dot{\phi}^* \dot{\phi} + \partial_i \phi^* \partial_i \phi + V(\phi^* \phi)\right)$$

It is easy to show that

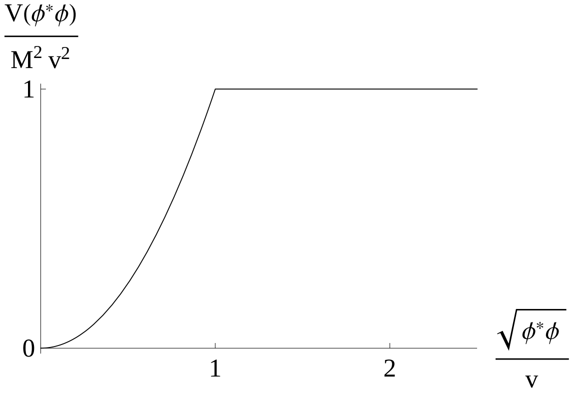
$$\frac{dE}{dQ} = \omega$$

$$\omega \to -\omega \quad \Rightarrow \quad Q \to -Q, \quad E \to E$$

A simple example

$$V(\phi^*\phi) = M^2\phi^*\phi\,\theta\left(1 - \frac{\phi^*\phi}{v^2}\right) + M^2 v^2\theta\left(\frac{\phi^*\phi}{v^2} - 1\right),$$

where θ is the Heaviside step function.

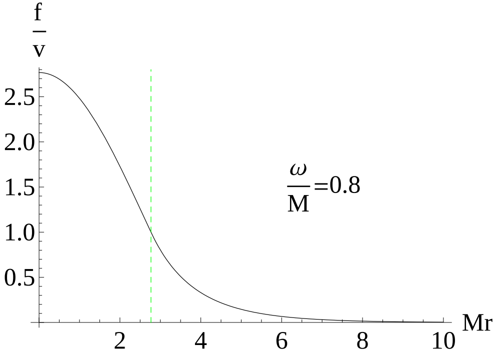


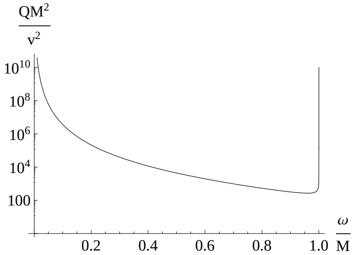
Solution:

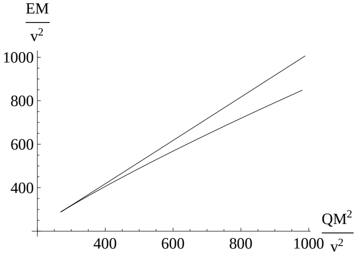
$$f = vR\frac{e^{-\sqrt{M^2-\omega^2}(r-R)}}{r}, \qquad \text{for} \qquad f < v \quad (r > R),$$

$$f = \frac{vR}{\sin(\omega R)}\frac{\sin(\omega r)}{r}, \qquad \text{for} \qquad f > v \quad (r < R).$$

$$R = \frac{1}{\omega}\left(\arctan\left(-\frac{\omega}{\sqrt{M^2-\omega^2}}\right)+\pi\right), \quad 0 < \omega < M.$$




QM^2 / v^2
10^10
10^8
10^6
10^4
100
0.2
0.4
0.6
0.8
1.0
ω / M


EM / v^2
1000
800
600
400
400
600
800
1000
QM^2 / v^2

U(1) gauged Q-balls

Proposed in G. Rosen, "Charged particlelike solutions to nonlinear complex scalar field theories", J. Math. Phys. **9** (1968) 999.,

Became popular after K.-M. Lee, J. A. Stein-Schabes, R. Watkins and L. M. Widrow, "Gauged Q-balls", Phys. Rev. D **39** (1989) 1665.

$$S = \int d^4x \left((\partial^\mu \phi^* - ieA^\mu \phi^*)(\partial_\mu \phi + ieA_\mu \phi) - V(\phi^*\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right),$$

$$\begin{aligned}
\phi(t,\vec{x}) &= \mathrm{e}^{i\omega t} f(r), \qquad f(r)|_{r\to\infty} \to 0, \qquad \partial_r f(r)|_{r=0} = 0, \\
A_0(t,\vec{x}) &= A_0(r), \qquad A_0(r)|_{r\to\infty} \to 0, \qquad \partial_r A_0(r)|_{r=0} = 0, \\
A_i(t,\vec{x}) &\equiv 0.
\end{aligned}$$

Without loss of generality $f(r) > 0$.

Q-ball charge:

$$Q = 8\pi \int (\omega + eA_0) f^2 r^2 dr.$$

Q-ball energy:

$$E = 4\pi \int_0^\infty \left((\omega + eA_0)^2 f^2 + \partial_r f \partial_r f + V(f) + \frac{1}{2} \partial_r A_0 \partial_r A_0 \right) r^2 dr.$$

It is possible to show (I. E. Gulamov, E. Y. Nugaev and M. N. Smolyakov, Phys. Rev. D **89** (2014) 085006) that the relation

$$\frac{dE}{dQ} = \omega$$

holds for U(1) gauged Q-balls.

- $\omega \to -\omega \quad \Rightarrow \quad A_0 \to -A_0.$
- The sign of $\omega + eA_0$ coincides with the sign of ω

  (K.-M. Lee, J. A. Stein-Schabes, R. Watkins and L. M. Widrow, Phys. Rev. D **39** (1989) 1665).
- If $A_0 \equiv 0$, then $\omega = 0$

  (G. Rosen, J. Math. Phys. **9** (1968) 999).

Thus

$$\omega \to -\omega \quad \Rightarrow \quad Q \to -Q, \quad E \to E$$

Q-balls at $r \to \infty$: ordinary Q-balls

Let us consider $V(\phi^*\phi)$ such that $V(\phi^*\phi) \to M^2\phi^*\phi$ for $\phi^*\phi \to 0$.

For $f(r) \to 0$ the equation of motion reduces to

$$(\omega^2 - M^2)f + \frac{1}{r}\frac{d^2}{dr^2}(rf) \approx 0, \quad \Rightarrow \quad f(r) \sim \frac{e^{-\sqrt{M^2-\omega^2}r}}{r}$$

There are no Q-balls with $\omega = M$, because $Q \to \infty$ (as well as $E \to \infty$) for $\omega \to M$.

For $f(r) \to 0$ the equation of motion reduces to

$$(\omega^2 - M^2)f - \frac{2\omega e^2 Q}{4\pi r} f + \frac{1}{r}\frac{d^2}{dr^2}(rf) \approx 0.$$

- For $\omega < M$

$$f(r) \sim \frac{\mathrm{e}^{-\sqrt{M^2-\omega^2}\,r}}{r^{1+\frac{\omega e^2 Q}{4\pi\sqrt{M^2-\omega^2}}}} \quad \text{for} \quad r \to \infty.$$

- For $\omega = M$

$$f(r) = C\frac{K_1\left(\sqrt{\frac{2Me^2Q}{\pi}\,r}\right)}{\sqrt{r}},$$

where C is a constant and $K_1(b,z)$ is the modified Bessel function of the second kind, leading to

$$f(r) \sim \frac{\mathrm{e}^{-\sqrt{\frac{2Me^2Q}{\pi}\,r}}}{r^{\frac{3}{4}}} \quad \text{for} \quad r \to \infty.$$

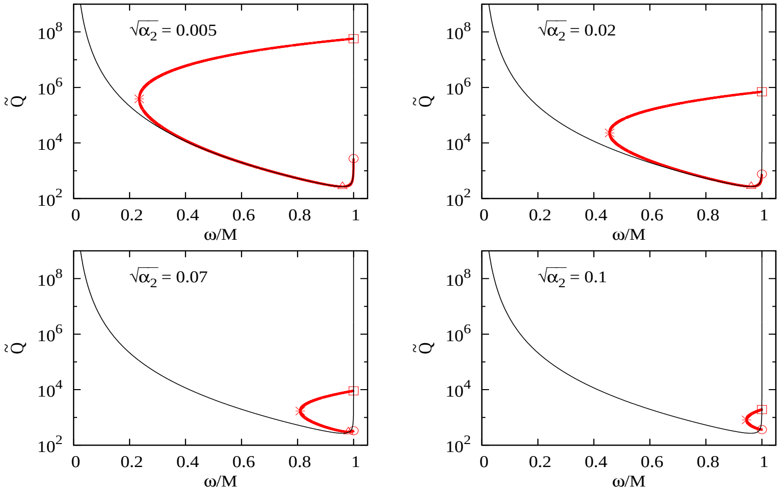


Figure 1: $V(f) = M^2 f^2\,\theta\left(1 - \frac{f^2}{v^2}\right) + M^2 v^2 \theta\left(\frac{f^2}{v^2} - 1\right)$.
$\tilde{Q} = \frac{M^2}{v^2} Q$ for different values of the parameter $\alpha_2 = \frac{e^2 v^2}{M^2}$ (thick lines). The thin lines stand for the nongauged case.

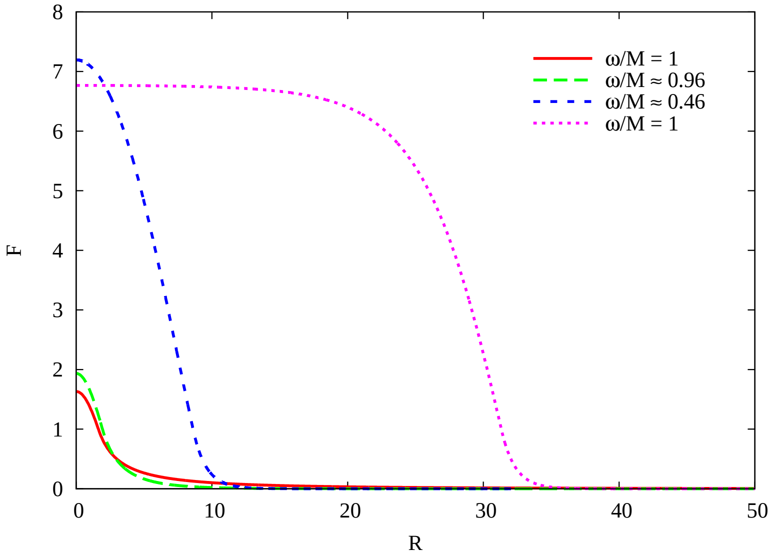


Figure 2: Profiles of the scalar field for different values of $\frac{\omega}{M}$. Here $\sqrt{\alpha_2} = 0.02$, $R = Mr$, $F(R) = \frac{1}{v} f(r)$.

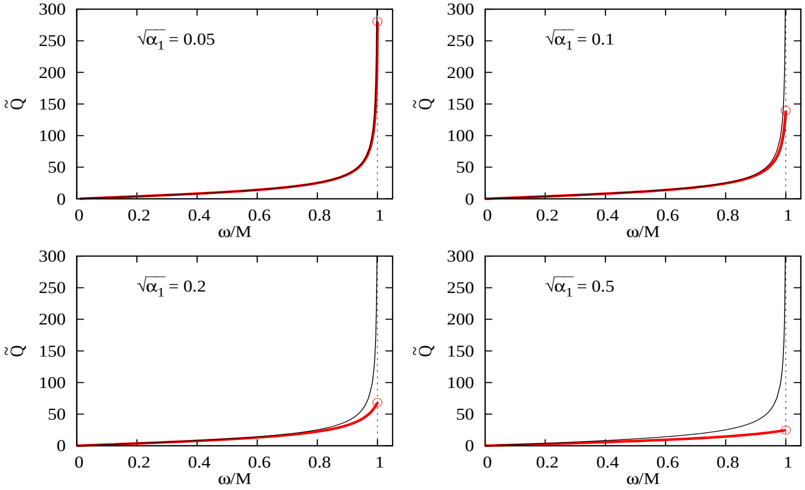


Figure 3: $V(f) = M^2 f^2 - \lambda f^4$.
$\tilde{Q} = \lambda Q$ for different values of the parameter $\alpha_1 = \frac{e^2}{\lambda}$ (thick lines). The thin lines stand for the nongauged case.

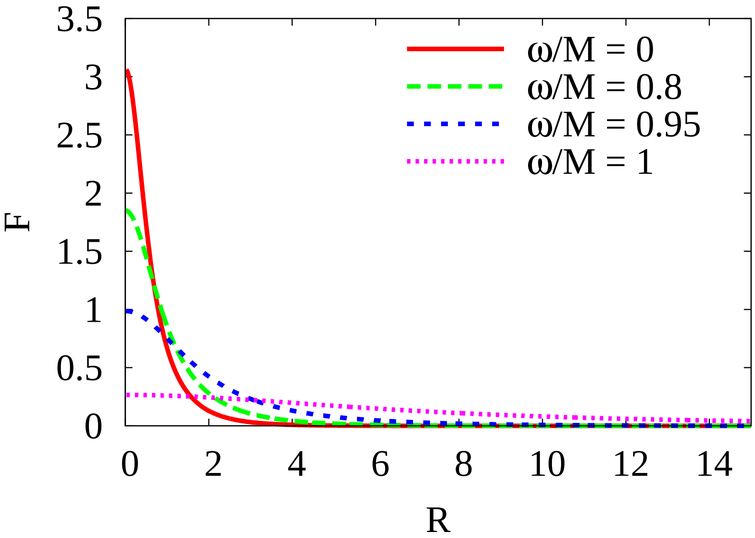


Figure 4: Profiles of the scalar field for different values of $\frac{\omega}{M}$. Here $\sqrt{\alpha_1} = 0.05$, $R = Mr$, $F(R) = \frac{\sqrt{\lambda}}{M} f(r)$.

Classical stability of Q-balls: ordinary Q-balls

Q-balls are stable with respect to small perturbations if the following conditions hold:

1. $$\frac{dQ}{d\omega} < 0$$

2. The operator

$$\hat{L} = -\Delta + \left.\frac{dV}{d(\phi^*\phi)}\right|_{\phi^*\phi=f^2(r)} + 2\left.\frac{d^2V}{d(\phi^*\phi)^2}\right|_{\phi^*\phi=f^2(r)} f^2(r) - \omega^2$$

   has *only one* negative eigenvalue. Here $\Delta = \partial_i\partial_i$.

T. D. Lee and Y. Pang, Phys. Rept. **221** (1992) 251
(based on the use of the energy functional of the system),
A. G. Panin and M. N. Smolyakov, Phys. Rev. D **95** (2017) 065006
(based on the use of the linearized equations of motion along the lines of the Vakhitov-Kolokolov method proposed in N. G. Vakhitov and A. A. Kolokolov, Radiophys. Quantum Electron. **16** (1973) 783).

Classical stability of Q-balls: U(1) gauged Q-balls

A. G. Panin and M. N. Smolyakov, Phys. Rev. D **95** (2017) 065006

Perturbations above the U(1) gauged Q-ball solution:

$$\phi(t,\vec{x}) = e^{i\omega t} f(r) + e^{i\omega t} e^{\gamma t}\left(u(\vec{x}) + iv(\vec{x})\right),$$
$$A_0(t,\vec{x}) = A_0(r) + e^{\gamma t} a_0(\vec{x}),$$
$$A_i(t,\vec{x}) = e^{\gamma t} a_i(\vec{x})$$

It is possible to show that U(1) gauged Q-balls are classically stable if the following conditions hold:

1. $$\frac{dQ}{d\omega} < 0$$
2. The corresponding operator $\hat{L}$ has *only one* negative eigenvalue.

$$\hat{L} = \begin{pmatrix} -\Delta + W(r) & -2e(\omega + eA_0)f & 0 \\ -2e(\omega + eA_0)f & \frac{\Delta}{2} - e^2 f^2 & 0 \\ 0 & 0 & \left(\frac{\Delta}{2} - e^2 f^2 - \frac{\gamma^2}{2}\right) I_{3\times 3} \end{pmatrix},$$

where $I_{3\times 3}$ is the 3×3 unit matrix and

$$W(r) = \left.\frac{dV}{d(\phi^*\phi)}\right|_{\phi^*\phi = f^2(r)} + 2\left.\frac{d^2 V}{d(\phi^*\phi)^2}\right|_{\phi^*\phi = f^2(r)} f^2(r) - (\omega + eA_0)^2.$$

There always exist negative eigenvalues corresponding to the perturbations a_i!

In the spherically-symmetric case, $a_i \equiv 0$. Then $\hat{L}$ reduces to

$$\begin{pmatrix} -\Delta + W(r) & -2e(\omega + eA_0)f \\ -2e(\omega + eA_0)f & \frac{\Delta}{2} - e^2 f^2 \end{pmatrix}.$$

Numerical simulations

$$V(\phi^*\phi) = M^2\phi^*\phi\,\theta\left(1-\frac{\phi^*\phi}{v^2}\right) + M^2v^2\theta\left(\frac{\phi^*\phi}{v^2}-1\right)$$

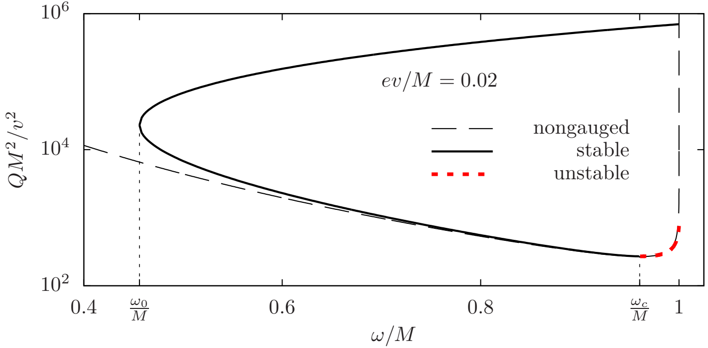


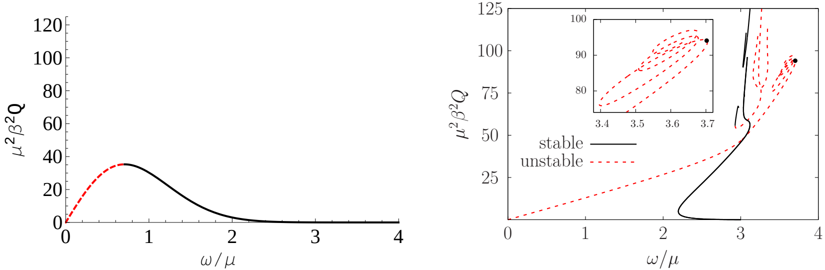
Figure 5: The long-dashed line stands for the nongauged Q-balls. The solid line stands for the classically stable gauged Q-balls. The short-dashed line stands for the classically unstable gauged Q-balls.

$$V(\phi^*\phi) = -\mu^2\phi^*\phi \ln\left(\beta^2\phi^*\phi\right)$$

In the nongauged case this scalar field potential was proposed in G. Rosen, "Dilatation covariance and exact solutions in local relativistic field theories", Phys. Rev. **183** (1969) 1186.

The linearized equations of motion for perturbations can be solved exactly in this model, see
G.C. Marques and I.Ventura, "Resonances within nonperturbative methods in field theories", Phys. Rev. D **14** (1976) 1056.

In the nongauged case, the $\frac{dQ}{d\omega} < 0$ stability criterion is valid in this model – Q-balls with $\frac{dQ}{d\omega} < 0$ are classically stable (and Q-balls with $\frac{dQ}{d\omega} > 0$ are classically unstable).



ordinary Q-balls U(1) gauged Q-balls

Figure 6: Ordinary (left plot) and U(1) gauged Q-balls for $e/\beta\mu = 1.1$ (right plot) for the logarithmic scalar field potential.

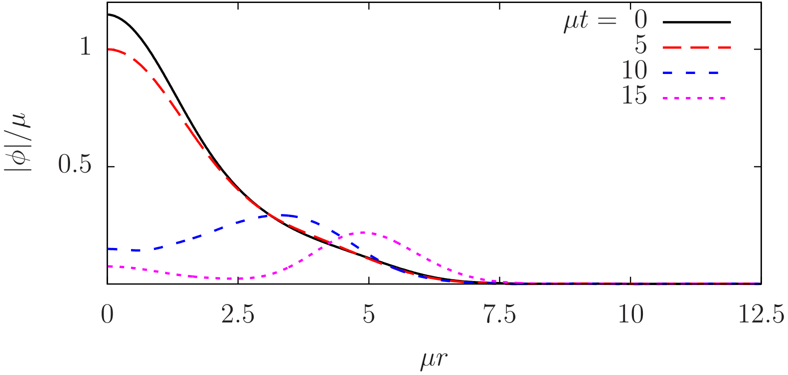


Figure 7: The scalar field profile of the classically unstable gauged Q-ball at different moments of time. The initial solution (at $\mu t = 0$) is marked by the dot in Fig. 6.

Thank you for your attention!