The XXII International Workshop
High Energy Physics and Quantum Field Theory

Hyperfine structure of P-states in muonic deuterium

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Muonic deuterium

- (μd) - bound state of negatively charged muon and deuteron. Lifetime of this simple atom is equal to muon's lifetime $\tau_\mu = 2.19703(4) * 10^{-6}$;
- $(m_\mu/m_e = 206.7682838(54))$ leads to a lower Bohr radius of the muon. Thus an influence of vacuum polarization and nuclear structure effects in energy structure increases;
- Muonic atoms play an important role in check of QED, theory of bound states and in precise measurement of fundamental constants;
- Measurement of the hyperfine structure in light muonic atoms allows us to obtain more precise values of charge radii and Zeemach radii of corresponding atoms.

A. Antognini et al., Science **339**, 417 (2013).

Lamb shift in (μp) for transition $(2P_{3/2}^{F=2} - 2S_{1/2}^{F=1})$ was meassured in PSI (Paul Scherrer Institute) with result 49881.88 (76) GHz (206.2949 (32) meV).

- New value of proton charge radius $r_p = 0.84087(39)$ fm;
- 7.0 σ discrepancy between new value and CODATA value $r_p = 0.8768(69) fm$.

Julian J. Krauth et al., arXiv:1506.01298v2 [physics.atom-ph]

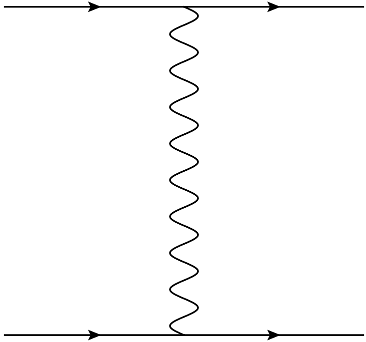
Experimental data on hyperfine structure in muonic deuterium has been already obtained and is being prepared for publication.

Purpose

The aim of this work is to calculate analytically and numerically corrections of order α^5 and α^6 in hyperfine structure of 2P state in muonic deuterium.

- We use quasipotential approach in QED;
- We include α^5 and α^6 vacuum polarization and nuclear structure corrections to achieve high accuracy;
- We improve previous calculations of hyperfine structure in muonic deuterium for 2P state.

Main contribution



Main contribution to hyperfine splitting of 2P state is given by hyperfine part of Breit hamiltonian:

$$\Delta V_B^{hfs}(r) = \frac{Z\alpha(1+\kappa_d)}{2m_1 m_2 r^3}\left[1+\frac{m_1\kappa_d}{m_2(1+\kappa_d)}\right](\boldsymbol{L}\cdot\boldsymbol{s}_2)-$$

$$-\frac{Z\alpha(1+\kappa_d)(1+a_\mu)}{2m_1 m_2 r^3}\left[(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)-3(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})\right].$$

This operator does not commute with muon's total angular momentum operator. Thus levels are mixed and off-diagonal matrix elements should be taken into account:

- diagonal elements $\left\langle 2P_{1/2}\,\middle|\,\Delta V_B^{hfs}\,\middle|\,2P_{1/2}\right\rangle$, $\left\langle 2P_{3/2}\,\middle|\,\Delta V_B^{hfs}\,\middle|\,2P_{3/2}\right\rangle$;
- off-diagonal elements $\left\langle 2P_{1/2}\,\middle|\,\Delta V_B^{hfs}\,\middle|\,2P_{3/2}\right\rangle^{F=1/2}$, $\left\langle 2P_{1/2}\,\middle|\,\Delta V_B^{hfs}\,\middle|\,2P_{3/2}\right\rangle^{F=3/2}$.

Coulomb wave function of (μd) 2P state:

$$\Psi_{2P}(\boldsymbol{r}) = \frac{1}{2\sqrt{6}} W^{\frac{5}{2}} r e^{-\frac{Wr}{2}} Y_{1m}(\theta, \phi),$$

$$W = \mu Z \alpha.$$

To make averaging over angles in $\Delta V_B^{hfs}(r)$ we use the following relations:

$$\boldsymbol{s}_1 \to \boldsymbol{J} \frac{\overline{(\boldsymbol{s}_1 \cdot \boldsymbol{J})}}{J^2},$$

$$\boldsymbol{L} \to \boldsymbol{J} \frac{\overline{(\boldsymbol{L} \cdot \boldsymbol{J})}}{J^2},$$

$$\overline{(\boldsymbol{s}_1 \cdot \boldsymbol{J})} = \frac{1}{2}\left[j(j+1) - l(l+1) + \frac{3}{4} \right],$$

$$\overline{(\boldsymbol{L} \cdot \boldsymbol{J})} = \frac{1}{2}\left[j(j+1) + l(l+1) - \frac{3}{4} \right],$$

$$\langle \delta_i j - 3 n_i n_j \rangle = -\frac{1}{5}(4\delta_{ij} - 3L_i L_j - 3L_j L_i).$$

After averaging over Coulomb wave functions we get the following general expression for diagonal matrix elements of hyperfine structure:

$$E_B^{hfs} = \frac{\alpha^4\mu^3(1+\kappa_d)}{48 m_1 m_2}\left[\overline{T_1} + \frac{m_1\kappa_d}{m_2(1+\kappa_d)}\overline{T_1} - (1+a_\mu)\overline{T_2}\right],$$

where

$$T_1 = (\boldsymbol{L}\cdot\boldsymbol{s}_2),\quad T_2 = [(\boldsymbol{s}_1\cdot\boldsymbol{s}_2) - 3(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})],$$
$$T_3 = [(\boldsymbol{s}_1\cdot\boldsymbol{s}_2) - (\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})].$$

After averaging over angles in off-diagonal elements we get:

$$\overline{T}_1 = 2\overline{T}_2 = -2\overline{T}_3 = \begin{cases} -\frac{\sqrt{2}}{3}, & F = 1/2, \\ -\frac{\sqrt{5}}{3}, & F = 3/2. \end{cases}$$

Off-diagonal matrix elements of hyperfine structure:

$$E_{F=1/2}^{hfs,off-diag} = \frac{\alpha^4\mu^3(1+\kappa_d)}{48 m_1 m_2}\left(-\frac{\sqrt{2}}{6}\right)\left[1 + \frac{2m_1\kappa_d}{m_2(1+\kappa_d)} - a_\mu\right],$$

$$E_{F=3/2}^{hfs,off-diag} = \frac{\alpha^4\mu^3(1+\kappa_d)}{48 m_1 m_2}\left(-\frac{\sqrt{5}}{6}\right)\left[1 + \frac{2m_1\kappa_d}{m_2(1+\kappa_d)} - a_\mu\right].$$

Relativistic correction of order α^6 is known in analytical form and can be evaluated withe the help of Dirac theory. Dirac hamiltonian of a particle in central field is:

$$H = c\boldsymbol{\alpha}(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A}) + \beta m_0 c^2 + e\Phi.$$

Hyperfine part is:

$$\Delta H^{hfs} = e_0 \boldsymbol{\alpha} \boldsymbol{A}.$$

To calculate the correction we perform averaging of $e_0 \boldsymbol{\alpha} \boldsymbol{A}$ over relativistic wave functions of muon-deuteron system. Finally we get the following expressions for corrections to diagonal matrix elements:

$$E^{hfs}_{rel}(2P_{1/2}) = \frac{\alpha^6 (1+\kappa_d)\mu^3}{48 m_1 m_2} \frac{m_1^3}{\mu^3} \frac{47}{9} \times \frac{1}{2}[F(F+1) - J(J+1) - I(I+1)],$$

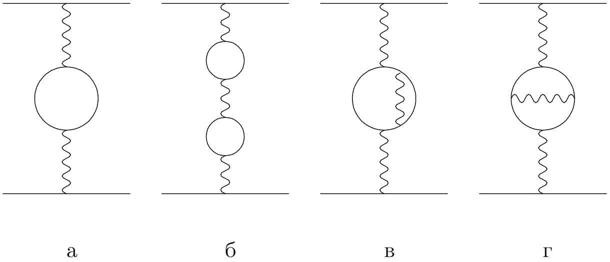
$$E^{hfs}_{rel}(2P_{3/2}) = \frac{\alpha^6 (1+\kappa_d)\mu^3}{48 m_1 m_2} \frac{m_1^3}{\mu^3} \frac{7}{45} \times \frac{1}{2}[F(F+1) - J(J+1) - I(I+1)].$$

For off-diagonal matrix elements we get relativistic corrections as follows:

$$E^{hfs,off-diag}_{rel,F=1/2} = -\frac{\alpha^6(1+\kappa_d)\mu^3}{48 m_1 m_2}\frac{m_1^3}{\mu^3}\frac{3\sqrt{2}}{32} = -0.0043\ meV,$$

$$E^{hfs,off-diag}_{rel,F=3/2} = -\frac{\alpha^6(1+\kappa_d)\mu^3}{48 m_1 m_2}\frac{m_1^3}{\mu^3}\frac{3\sqrt{5}}{32} = -0.0067\ meV.$$

One- and two-loop vacuum polarization corrections in one photon interaction



One-loop vacuum polarization contribution to hyperfine splitting of 2P state in muonic deuterium in coordinate representation looks as follows:

$$\Delta V^{hfs}_{1\gamma,VP}(r)=\frac{Z\alpha(1+\kappa_d)}{2m_1m_2r^3}\frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi e^{-2m_e\xi r}\left\{\left(1+\frac{m_1\kappa_d}{m_2(1+\kappa_d)}\right)\times\right.$$
$$\times(\boldsymbol{L}\cdot\boldsymbol{s}_2)(1+2m_e\xi r)-(1+a_\mu)\left(4m_e^2\xi^2r^2[(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)-(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})]+\right.$$
$$\left.\left.+(1+2m_e\xi r)[(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)-3(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})]\right)\right\}.$$

where $\rho(\xi)=\sqrt{\xi^2-1}(2\xi^2+1)/\xi^4$. To obtain this potential we use the following substitution in photon propagator:

$$\frac{1}{k^2}\to\frac{\alpha}{3\pi}\int_1^\infty\rho(\xi)d\xi\frac{1}{k^2+4m_e^2\xi^2},$$

Averaging over wave functions we get the following expression:

$$E^{hfs}_{1\gamma,VP}(r)=\frac{\alpha^4\mu^3(1+\kappa_d)}{24m_1m_2r^3}\frac{\alpha}{6\pi}\int_1^\infty\rho(\xi)d\xi\int_0^\infty xdxe^{-x[1+\frac{2m_e\xi}{W}]}\times$$
$$\times\left[\left(1+\frac{m_1\kappa_d}{m_2(1+\kappa_d)}\right)\times\overline{T_1}(1+\frac{2m_e\xi}{W}x)-(1+a_\mu)\left(\frac{4m_e^2\xi^2x^2}{W^2}\overline{T_3}+\right.\right.$$
$$\left.\left.+(1+\frac{2m_e\xi}{W}x)\overline{T_2}\right)\right].$$

This expression allows us to get corrections to both diagonal and off-diagonal matrix elements. For 2 loops in series:

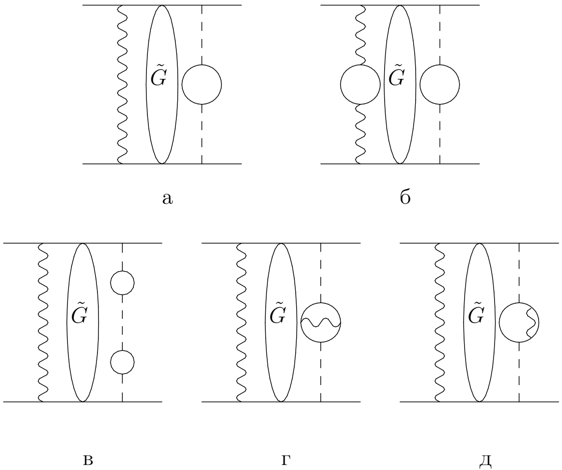
$$\Delta V^{hfs}_{1\gamma,VPVP}(r)=\frac{Z\alpha(1+\kappa_d)}{2m_1m_2r^3}\left(\frac{\alpha}{3\pi}\right)^2\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\frac{1}{\xi^2-\eta^2}\times$$
$$\times\left[\left(1+\frac{m_1\kappa_d}{m_2(1+\kappa_d)}\right)(\boldsymbol{L}\cdot\boldsymbol{s}_2)[\xi^2(1+2m_e\xi r)e^{-2m_e\xi r}-\eta^2(1+2m_e\eta r)e^{-2m_e\eta r}]-\right.$$
$$-(1+a_\mu)\left(4m_e^2r^2[\xi^4e^{-2m_e\xi r}-\eta^4e^{-2m_e\eta r}][(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)-(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})]+\right.$$
$$\left.\left.+[\xi^2(1+2m_e\xi r)e^{-2m_e\xi r}-\eta^2(1+2m_e\eta r)e^{-2m_e\eta r}][(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)-3(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})]\right)\right].$$

For diagrams with one loop inside another:

$$\Delta V^{hfs}_{2-loop}(r) = \frac{Z\alpha(1+\kappa_d)}{2m_1 m_2 r^3}\frac{2}{3}\left(\frac{\alpha}{\pi}\right)^2\int_0^1 \frac{f(v)dv}{1-v^2}e^{-\frac{2m_e r}{\sqrt{1-v^2}}}\times$$
$$\times\Bigg[\left(1+\frac{m_1\kappa_d}{m_2(1+\kappa_d)}\right)\left[1+\frac{2m_e r}{\sqrt{1-v^2}}\right](\boldsymbol{L}\cdot\boldsymbol{s}_2)-$$
$$-(1+a_\mu)\Bigg(\frac{4m_e^2 r^2}{1-v^2}[(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)-(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})]+$$
$$+\left(1+\frac{2m_e r}{\sqrt{1-v^2}}\right)[(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)-3(\boldsymbol{s}_1\cdot\boldsymbol{n})(\boldsymbol{s}_2\cdot\boldsymbol{n})]\Bigg)\Bigg].$$

Muonic vacuum polarization correction of order α^6 is also included by simple replacement $m_e \to m_1$.

Vacuum polarization effects in second and third order perturbation theory



Green's function

Green's function is a solution of the following equation:

$$(\hat{H} - E)G_E(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

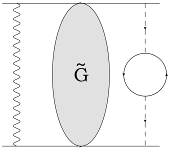
Spectral decomposition looks as follows:

$$G_E(\boldsymbol{r}, \boldsymbol{r'}) = \sum_i \frac{\psi_i^*(\boldsymbol{r})\psi_i(\boldsymbol{r'})}{E_i - E}.$$

Second and third order PT corrections are defined by reduced Green's function with the following partial decomposition:

$$\tilde{G}_n(\mathbf{r}, \mathbf{r}') = \sum_{l,m} \tilde{g}_{nl}(r, r') Y_{lm}(\mathbf{n}) Y_{lm}^*(\mathbf{n}').$$

Second order perturbation theory



Main contribution of vacuum polarization to HFS in second order PT has the following general structure:

$$\Delta E^{hfs}_{SOPT\ VP\ 1} = 2 < \psi | \Delta V^{C}_{VP} \cdot \tilde{G} \cdot \Delta V^{hfs}_{B} | \psi >,$$

where modified Coulomb potential

$$\Delta V^{C}_{VP}(r) = \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \left(-\frac{Z\alpha}{r} \right) e^{-2m_e \xi r}.$$

Reduced Green's function in 2P state takes form:

$$G_{2P}(\mathbf{r},\mathbf{r}') = -\frac{\mu^2(Z\alpha)}{36z^2z'^2}\left(\frac{3}{4\pi}\mathbf{n}\mathbf{n}'\right)e^{-(z+z')/2}g(z,z'),$$

$$g(z,z') = 24z_<^3 + 36z_<^3z_> + 36z_<^3z_>^2 + 24z_>^3 + 36z_<z_>^3 + 36z_<^2z_>^3 + 49z_<^3z_>^3 - 3z_<^4z_>^3 -$$
$$-12e^{z_<}(2 + z_< + z_<^2)z_>^3 - 3z_<^3z_>^4 + 12z_<^3z_>^3[-2C + Ei(z_<) - \ln z_< - \ln z_>],$$

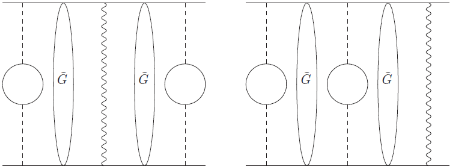
where $C = 0.5772...$ - Euler constant,
$z = Wr,\ z_< = min(z,z'),\ z_> = max(z,z')$.

Corresponding contribution to HFS (μd):

$$E^{hfs}_{VP,SOPT} = \frac{\alpha^4\mu^3(1+\kappa_d)}{24m_1m_2}\frac{\alpha}{54\pi}\int_1^\infty \rho(\xi)d\xi\int_0^\infty dx\int_0^\infty \frac{e^{-x'}}{x'^2}dx'e^{-x\left(1+\frac{2m_e\xi}{W}\right)}\times$$
$$\times\left[\overline{T_1} + \frac{m_1\kappa_d}{m_2(1+\kappa_d)}\overline{T_1} - (1+a_\mu)\overline{T_2}\right].$$

H.F. Hameka, Jour. Chem. Phys. 47, 2728 (1967).

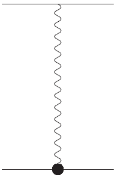
Third order perturbation theory



Formula for vacuum polarization correction of order α^6 int third order PT can be written as follows:

$$
\begin{aligned}
\Delta E^{\mathrm{HFS}}_{\mathrm{TOPT}} = & \langle\psi_n|\Delta V^C_{\mathrm{VP}}\cdot\tilde{G}\cdot\Delta V^{\mathrm{HFS}}\cdot\tilde{G}\cdot\Delta V^C_{\mathrm{VP}}|\psi_n\rangle \\
& +2\langle\psi_n|\Delta V^C_{\mathrm{VP}}\cdot\tilde{G}\cdot\Delta V^C_{\mathrm{VP}}\cdot\tilde{G}\cdot\Delta V^{\mathrm{HFS}}|\psi_n\rangle \\
& -\langle\psi_n|\Delta V^{\mathrm{HFS}}|\psi_n\rangle\langle\psi_n|\Delta V^C_{\mathrm{VP}}\cdot\tilde{G}\cdot\tilde{G}\cdot\Delta V^C_{VP}|\psi_n\rangle \\
& -2\langle\psi_n|\Delta V^C_{\mathrm{VP}}|\psi_n\rangle\langle\psi_n|\Delta V^C_{\mathrm{VP}}\cdot\tilde{G}\cdot\tilde{G}\cdot\Delta V^{\mathrm{HFS}}|\psi_n\rangle.
\end{aligned}
$$

S. G. Karshenboim, E. Yu. Korzinin, and V. G. Ivanov, JETP Lett. 88, 641(2008).



Nuclear structure correction of order α^6 for 2P state can be found by means of magnetic form-factor decomposition. In this case potential has the form:

$$V_{str}^{hfs}(k) = -\frac{4\pi}{2m_1m_2}\frac{Z\alpha(1+a_\mu)r_M^2}{6}\left[(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)k^2-(\boldsymbol{s}_1\cdot\boldsymbol{k})(\boldsymbol{s}_2\cdot\boldsymbol{k})\right],$$

where $r_M = 2.1424(21)$ fm^2 - deuteron magnetic radius. After averaging over wae functions we get the final expression for nuclear structure correction:

$$E_{str}^{hfs} = \frac{\alpha^6\mu^5(1+a_\mu)r_M^2}{72m_1m_2}\overline{(\boldsymbol{s}_1\cdot\boldsymbol{s}_2)},$$

General structure of a potential of quadrupole correction is:

$$H^{quad}_{\mu d} = \sum_q (-1)^q T^2_q(d) \cdot T^2_{-q}(\mu),$$

where $T^2(d)$, $T^2(\mu)$ - irreducible tensor operators of rank 2, that describe quadrupole moments of nucleus and muon respectively.
Matrix elements have the following form:

$$\left\langle j' I F \,\middle|\, H^{quad}_{\mu d} \,\middle|\, j I F \right\rangle = (-1)^{J'+1/2-F-J} \left\{ \begin{array}{ccc} J & I & F \\ I & J' & 2 \end{array} \right\} \frac{Q}{2} \times$$
$$\times \left[\left(\begin{array}{ccc} I & 2 & I \\ -I & 0 & I \end{array} \right) \right]^{-1} \sqrt{2J+1}\sqrt{2J'+1} \left(\begin{array}{ccc} J' & 2 & J \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{array} \right) < \frac{\alpha}{r^3} > .$$

For diagonal elements for all levels of 2P hyperfine structure we have:

$$E^{quad}_{\mu d}(j = 1/2) = 0,$$

$$E^{quad}_{\mu d}(j = 3/2) = \frac{\alpha Q}{2} < \frac{1}{r^3} > (\delta_{F,1/2} - 4/5\delta_{F,3/2} + 1/5\delta_{F,5/2}).$$

For off-diagonal elements:

$$E^{quad}_{\mu d}(j = 3/2, j' = 1/2) = \frac{\alpha Q}{2} < \frac{1}{r^3} > (\sqrt{2}\delta_{F,1/2} - 1/\sqrt{5}\delta_{F,3/2}).$$

Averaging over coordinates has the form:

$$< \frac{1}{r^3} >= \int_0^\infty \frac{1}{r}(ff' + gg')dr.$$

In non-relativistic approximation we obtain:

$$< \frac{1}{r^3} >= \frac{\alpha^3\mu^3}{24}.$$

Results and discussion

E. Borie, Phys. Rev. A 72, 052511(2005).

E. Borie, Ann. Phys. 327, 733(2012).

In the Borie's paper hyperfine structure of 2P state in muonic deuterium was obtained. Besides main contribution and quadrupole interaction, only one loop vacuum polarization correction in first order PT was taken into account. Main contribution and quadrupole interaction terms agree with our results. Vacuum polarization correction differs because we include additional corrections of order α^5 and α^6.

Table: Diagonal matrix elements

Contribution	$2P^2_{1/2}$, μeV	$2P^4_{1/2}$, μeV	$2P^2_{3/2}$, μeV	$2P^4_{3/2}$, μeV	$2P^6_{3/2}$, μeV
α^4	-1380.3359	690.1679	8162.2889	8583.2316	9284.8027
rel α^6	-0.1676	0.0838	-0.0125	-0.0050	0.0075
VP α^5	-1.0706	0.5353	-0.2802	-0.1121	0.1681
VP α^6	-0.0011	0.0005	-0.0014	-0.0006	0.0008
str α^6	0.0042	-0.0021	-0.0104	-0.0042	0.0063
quad α^4	0	0	434.2329	-347.3863	86.8466
$\sum$	-1381.5710	690.7855	8596.2173	8235.7235	9371.8319

Table: Off-diagonal matrix elements

Contribution	$2P^2_{1/2}$, μeV	$2P^4_{1/2}$, μeV
α^4	-126.0372	-199.2824
rel α^6	-0.0043	-0.0067
VP α^5	-0.1437	-0.2271
VP α^6	0.00005	0.0001
quad α^4	614.0980	-194.1948
$\sum$	487.9129	-393.7110

E. Borie, Phys. Rev. A 72, 052511(2005).

Energy matrix before diagonalization, meV

$$\begin{pmatrix} -1.38157 & 0 & 0.487913 & 0 & 0 \\ 0 & 0.690785 & 0 & -0.393711 & 0 \\ 0.487913 & 0 & 8.59622 & 0 & 0 \\ 0 & -0.393711 & 0 & 8.23572 & 0 \\ 0 & 0 & 0 & 0 & 9.37183 \end{pmatrix}$$

Table: Hyperfine structure of 2P state in muonic deuterium, final results

State	Energy, meV	Borie, meV
$2^2P_{1/2}$	-1.4054	-1.4056
$4^2P_{1/2}$	0.6703	0.6703
$2^2P_{3/2}$	8.6200	8.6194
$4^2P_{3/2}$	8.2562	8.2560
$6^2P_{3/2}$	9.3718	9.3729

Thanks for your attention