Dispersion relations and Renormalization group:
QCD calculation of pion-photon transition form factor

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based on PRD 98 (2018) 096017

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Exclusive hard process

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Pion-photon transition form factor

at large standard QCD corrections

within Light Cone Sum Rules

M.S. &Pimikov A. & Stefanis N., PRD 93 (2016) 114018;
Ayala C. &M.S. &Stefanis N., PRD 98 (2018) 096017

OUTLINE

1. Intro: Experimental and Theoretical motivations to modify fixed order pQCD (FOPT) calculation of transition FF for $\gamma\gamma^*(Q^2) \to \pi^0$ at low Q^2.

2. Current status of Light Cone SR (LCSR) predictions in $\text{N}^2\text{LO}_{\beta_0}$ FOPT

3. Dispersive form for pion TFF $+$ RG generates
   a "New" perturbation theory - fractional APT.
   Behavior of FAPT couplings.

4. Light cone sum rules within FAPT:
   new prediction for the pion-photon TFF

5. Conclusions

Experimental status of pion transition FF

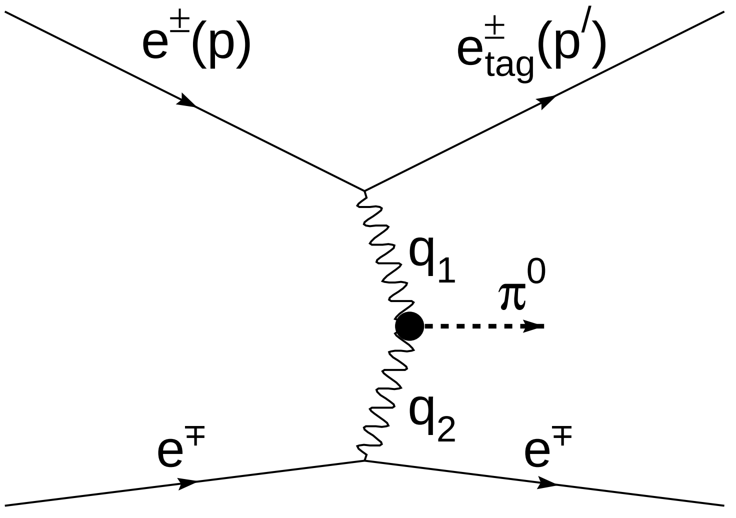
Why it is interesting for QCD?

The measurements of TFF is the clean experiment that has the best accuracy (BESIII!) among others exclusive hard reactions

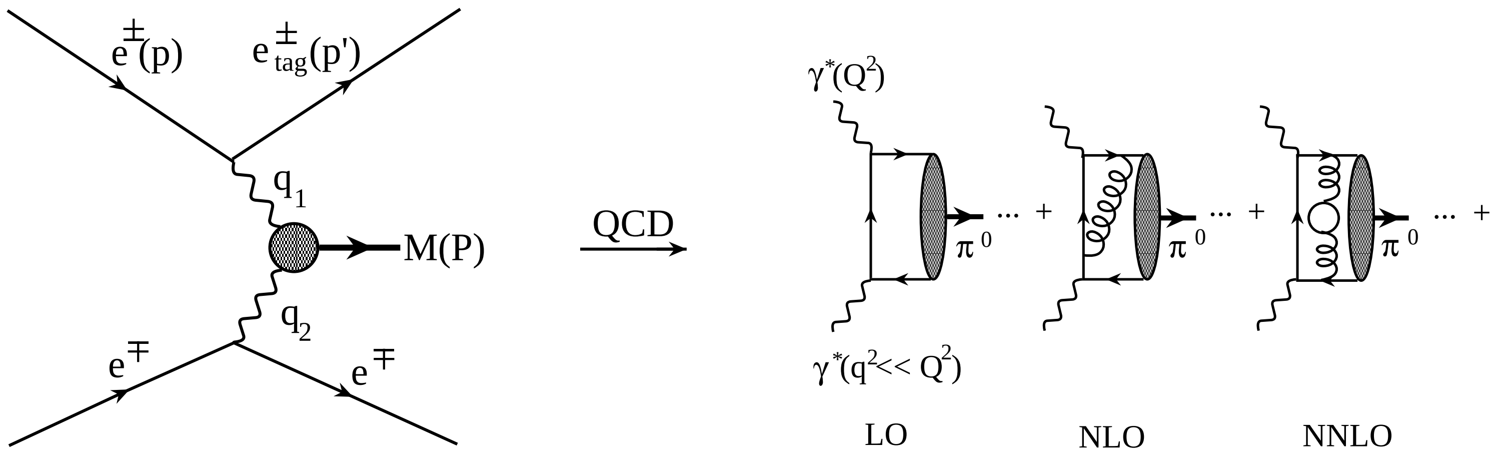
CELLO (1991) $Q^2 : 0.7 - 2.2$ **GeV**^{2}
CLEO (1998) $Q^2 : 1.6 - 8.0$ **GeV**^{2}
agrees with collinear QCD
BaBar (2009) $Q^2 : 4 - 40$ **GeV**^{2}
TFF has growing tendency with Q^2 **creating the "BaBar puzzle"**

Belle (2012) $Q^2 : 4 - 40$ **GeV**^{2}
returns to collinear QCD

BESIII (2019) $Q^2 : 0.3 - 3.1$ **GeV**^{2}
Promising very precise data (preliminary, arXiv:1810.00654)



Why it is interesting for QCD?



Theoretical advances in both parts of QCD factorization:

- **high order NNLO$_\beta$ contribution $O(\alpha_s^2\beta_0)$ to the hard part;**
- **distrib. amplit. (BMS2001&QCDSR) of twist-2 for pion part;**
- **contributions from twist-4 and corrections a'la twist-6 .**

3 steps to TFF: Perturbative & twist expansion;
Dispersive representation in q^2 for LCSR ,
LCSR with duality interval s_0 for $q^2 \to 0$.

Theoretical status of pion TFF in QCD FOPT

Hard process at $-Q^2, -q^2 \gg m_\rho^2 \Rightarrow$ **collinear factorization**

$$F^{\text{(tw=2)}}_{\text{FOPT}}(Q^2, q^2) = N_{\text{T}} \left(T_{\text{LO}} + a_s T_{\text{NLO}} + a_s^2 T_{\text{NNLO}} + \ldots\right) \otimes \varphi_\pi^{(2)}$$

$$\begin{aligned}
T_{\text{LO}} &= a_s^0\, T_0(x) \equiv 1/\left(q^2\bar{y} + Q^2 y\right) \\
a_s T_{\text{NLO}} &= a_s^1\, T_0(y) \otimes \left[\mathcal{T}^{(1)} + \underline{L\, V_0}\right](y,x)\,, \\
a_s^2 T_{\text{NNLO}} &= a_s^2\, T_0(y) \otimes \left[\mathcal{T}^{(2)} - \underline{L\, \mathcal{T}^{(1)}\beta_0} + \underline{L\, \mathcal{T}^{(1)} \otimes V_0} - \underline{\frac{L^2}{2}\,\beta_0 V_0} \right. \\
&\qquad \left. + \underline{\frac{L^2}{2}\, V_0 \otimes V_0} + \underline{\underline{L\, V_1}}\right](y,x)\,,
\end{aligned}$$

$L = L(y) = \ln\left[(q^2\bar{y} + Q^2 y)/\mu_F^2\right]$ **Plain terms** $\mathcal{T}^{(1)}, \mathcal{T}^{(2)} (\mathcal{T}_\beta^{(2)})$ **- corrections to parton subprocess;**

Underlined terms due to $\bar{a}_s(y)$ **and ERBL,** V_0 **- kernel;**

underlined term - two loops ERBL, V_1 **- kernel.**

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Status of Light Cone Sum Rules at N^{2}LO

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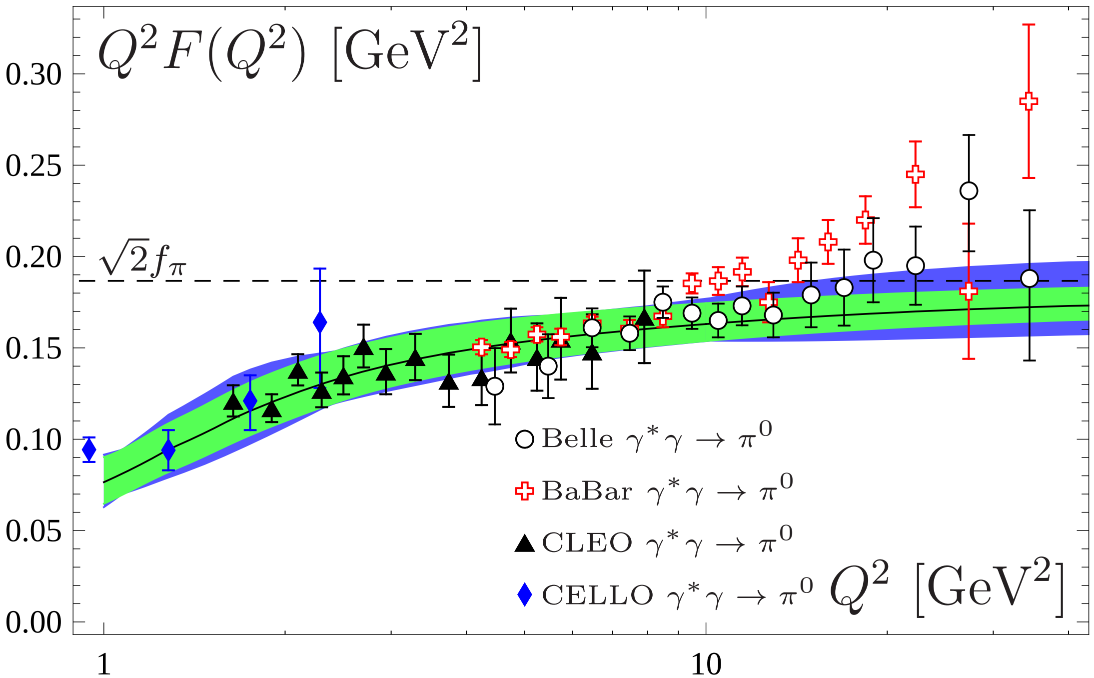
Pion TFF in LCSR in FOPT vs exp. data

Theor. predictions on $F_{\gamma\gamma^*\pi}$: LCSR $\oplus$ N^{2}LO $\oplus$DA BMS $\oplus$ tw4,6

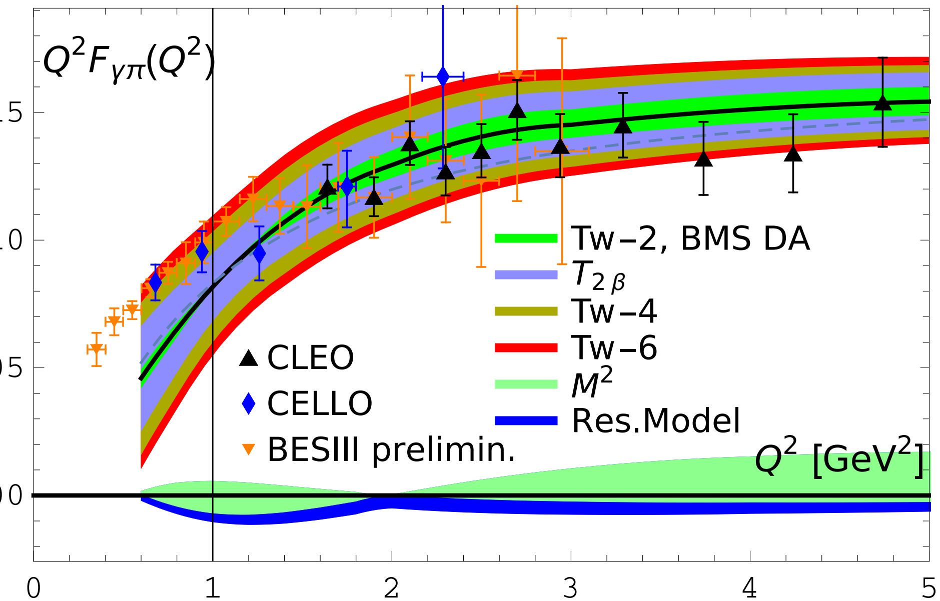


Data	Collab.
◆	CELLO (1991)
▲	CLEO (1998)
✚	BaBar (2009)
○	Belle (2012)
—	BMPS result(2013)

BMPS:Bakulev,MS,Pimikov,Stefanis

The data points which agree well with BMSP LCSR predictions
CELLO, CLEO, BaBar$_{Q^2<9\ \mathrm{GeV}^2}$ (2009), BaBar$^{\eta}_{\eta'}$ (2011), the most of Belle (2012)
New development for analysis of all experimental data in [1904.02631, Stefanis].

Pion TFF in LCSR in FOPT vs exp. data



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Challenge !

Total rad. corrections	-18% at 3 GeV^2
Source	**Uncertainty (%)**
Unknown NNLO term $\mathcal{T}_c^{(2)}$	∓ 5
Range of Tw-2 BMS DAs	$-3.4 \div 4.1$
Tw-4 coupling $\delta^2 = [0.152 - 0.228]$ **GeV**^{2}	± 3.0
Tw-6 $\langle \bar{q}q \rangle^2 = (0.24 \pm 0.01)^6$ **GeV**^{2}	$-2.4 \div 3.0$
Total	$-13.6 \div 14.9$

Pion TFF in pQCD with RG improvement

Collecting all of the "underlined" terms of RG-evolution into $a_s(\mu^2) \to \bar{a}_s(y) \equiv \bar{a}_s\left(q^2\bar{y} + Q^2 y\right)$ **and ERBL–factor.**

$$F^{(\text{tw=2})}(Q^2, q^2) = N_{\mathsf{T}} T_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x) + \bar{a}_s^2(y)\mathcal{T}^{(2)}(y,x) + \ldots\right] \underset{x}{\otimes} \exp\left[-\int_{a_s}^{\bar{a}_s(y)} d\alpha \frac{V(\alpha; x, z)}{\beta(\alpha)}\right]\right\} \underset{z}{\otimes} \varphi_\pi^{(2)}(z, \mu^2)\,,$$

$$\varphi_\pi^{(2)}(x, \mu^2) = \psi_0(x) + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2)\, \psi_n(x) -$$ **Gegenbauer harmonics**

$$F^{(\text{tw=2})}(Q^2, q^2) \;=\; F_0^{\text{RG}}(Q^2, q^2) + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2) F_n^{\text{RG}}(Q^2, q^2)$$

$$F_n^{\text{RG}}(Q^2, q^2) = N_{\mathsf{T}} T_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{T}^{(1)}(y,x)\right] \left(\frac{\bar{a}_s(y)}{a_s(\mu^2)}\right)^{\nu_n} \right\} \underset{x}{\otimes} \psi_n(x)$$

One loop resumed result, $\nu_n = \gamma_n / 2\beta_0$

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Dispersive form for pion TFF $+$ RG

a “New” perturbation theory -

fractional APT.

Properties of FAPT couplings.

Dispersive form of TFF leads to fractional APT

$$\left[F_{(1l)n}(Q^2,q^2)\right]_{\text{an}} = \int_{m^2}^{\infty} \frac{\rho_F(Q^2,\sigma)}{\sigma+q^2-i\epsilon}\,d\sigma,\ \ \rho_F(\sigma) = \frac{\mathbf{Im}}{\pi}\left[F_{(1l)n}(Q^2,-\sigma)\right]$$

Appear the FAPT $\mathcal{A}_\nu, \mathfrak{A}_\nu$ couplings + a New one - $\mathcal{I}_\nu$!

$$\nu(0){=}0;\ F^{\text{FAPT}}_{(1l),0}(Q^2,q^2;\boldsymbol{m^2}){=}N_T T_0(Q^2,q^2;y)\underset{y}{\otimes}\left\{1+\mathbb{A}_1(\boldsymbol{m^2},y)\mathcal{T}^{(1)}(y,x)\right\}\underset{x}{\otimes}\psi_0(x)$$

$$\nu(n){\neq}0;\ F^{\text{FAPT}}_{(1l),n}(Q^2,q^2;\boldsymbol{m^2}){=}\frac{N_T}{a_s^{\nu_n}(\mu^2)}T_0(Q^2,q^2;y)\underset{y}{\otimes}$$
$$\left\{\mathbb{A}_{\nu_n}(\boldsymbol{m^2},y)1+\mathbb{A}_{1+\nu_n}(\boldsymbol{m^2},y)\mathcal{T}^{(1)}(y,x)\right\}\underset{x}{\otimes}\psi_n(x)$$

The same expression as for RG-case, $\mathbb{A}_\nu(m^2,y) \Leftrightarrow \bar{a}_s^\nu(y)$

$$\mathbb{A}_\nu(\boldsymbol{m^2},y) = \mathcal{I}_\nu(\boldsymbol{m^2},Q(y)) - \mathfrak{A}_\nu(\boldsymbol{m^2});\ \ \mathbb{A}_\nu(0,y) = \mathcal{A}_\nu(Q(y)) - \mathfrak{A}_\nu(0)$$

the certain kinematics enters by means of $Q(y) \equiv q^2\bar{y}+Q^2 y$

Dispersive "Källen–Lehmann" representation

Different effective couplings in Euclidean, $\mathcal{A}_n$, and Minkowsk., $\mathfrak{A}_n$, regions $\overline{\alpha}_s^n \to \{\mathcal{A}_n, \mathfrak{A}_n\}$ [Shirkov&Solovtsov1997-07]

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma, \quad \rho_n(\sigma) = \frac{\text{Im}}{\pi}\left[\overline{a}_s^n(-\sigma)\right]\beta_0$$

For 1 loop run (here pole remover), $L = \ln\left(Q^2/\Lambda^2\right)$:

$$\rho_1(\sigma) \stackrel{1l}{=} \frac{1}{L_\sigma^2 + \pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2}\, d\sigma \stackrel{1l}{=} \frac{1}{L} - \frac{1}{e^L - 1}$$

$$\mathfrak{A}_1[L_s] = \int_s^\infty \frac{\rho_1(\sigma)}{\sigma}\, d\sigma \stackrel{1l}{=} \frac{1}{\pi}\,\text{arccos}\frac{L_s}{\sqrt{\pi^2 + L_s^2}}$$

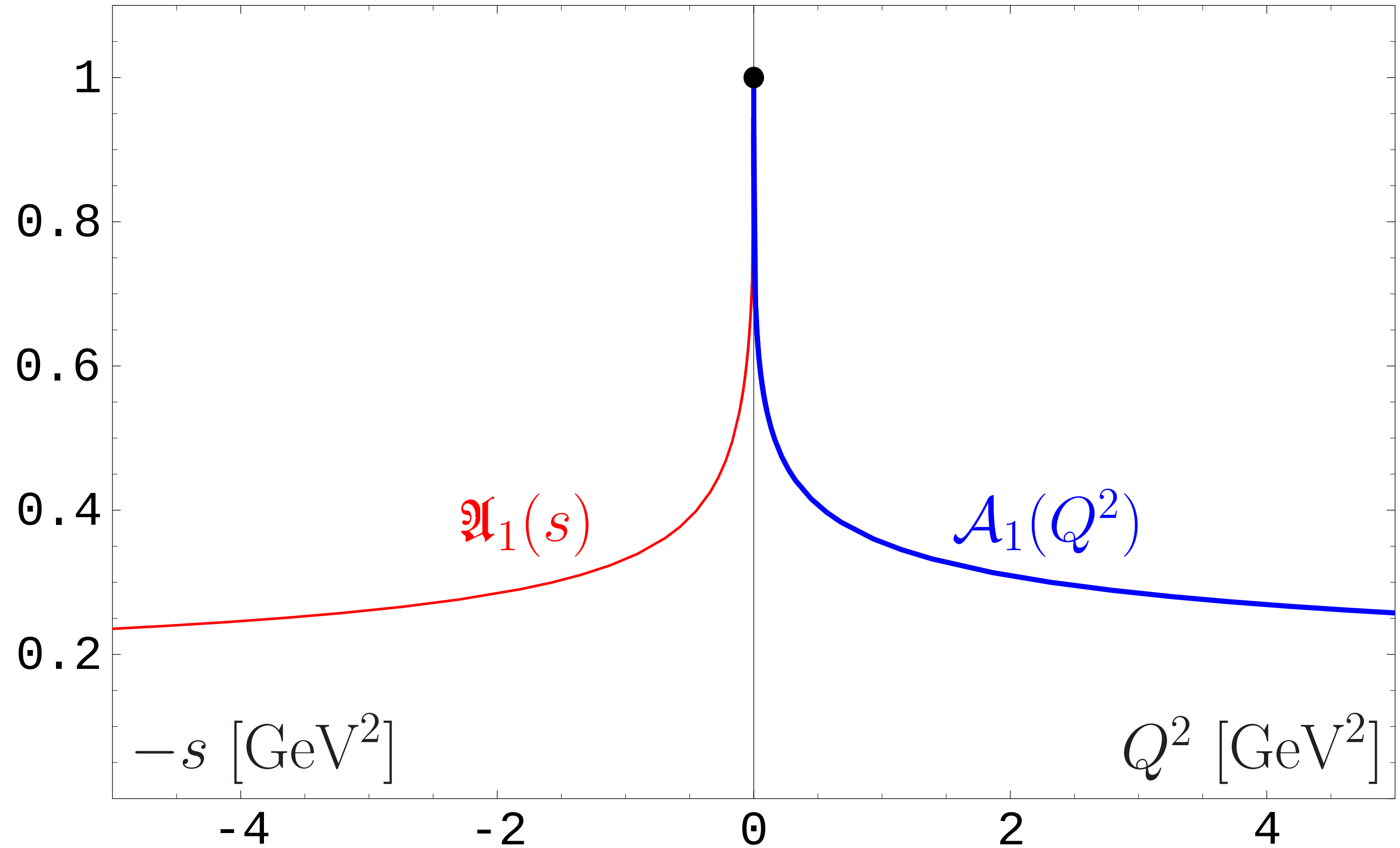
Inequality:

$$a_s^n[L] > \left(\mathcal{A}_n[L],\, \mathfrak{A}_n[L]\right) \stackrel{L\to\infty}{\longrightarrow} a_s^n[L]$$

APT: Distorting mirror

[Shirkov&Solovtsov1997-2007]

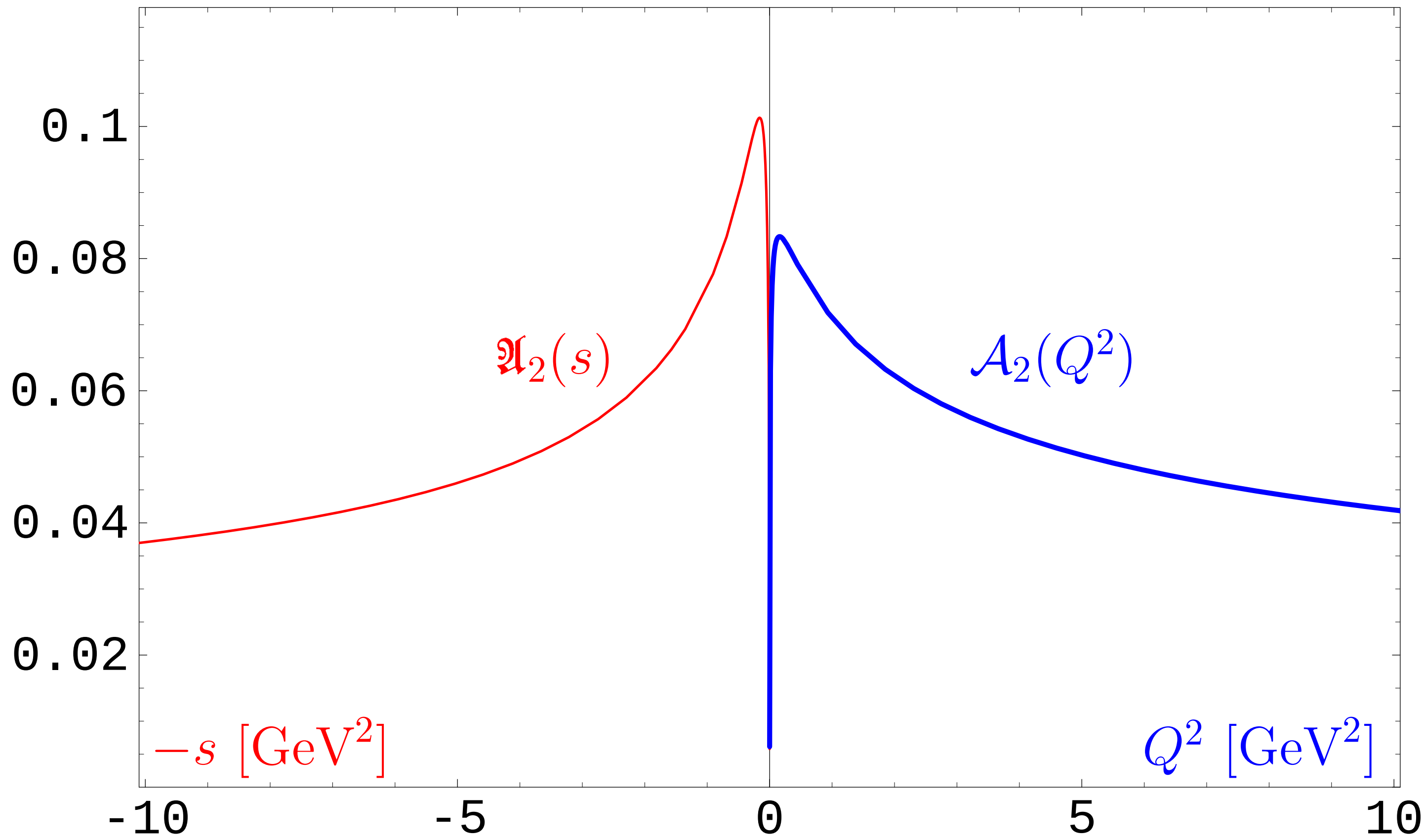
First, coupling images: $\mathfrak{A}_1(s)$ **and** $\mathcal{A}_1(Q^2)$



APT: Distorting mirror

[Shirkov&Solovtsov1997-2007]

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



FAPT(Eucl): Properties of $\mathcal{A}_\nu[L]$

Euclidean coupling (pole remover) [Bakulev,MS,Stefanis 2005-07]:

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Here $F(z,\nu)$ is reduced Lerch transcendental function.
It is analytic function in ν.

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It is analytic function in ν. Properties:
The charge $\mathcal{A}_\nu(Q^2)$ is Bounded for $\nu \geqslant 1$,

- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ **for** $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ **for** $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathcal{A}_\nu[\pm\infty] = 0$ **for** $\nu > 1$;

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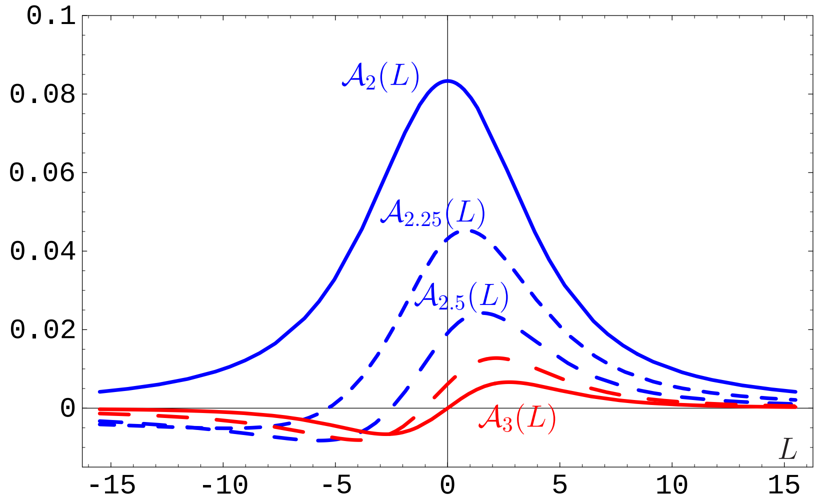
- $\mathcal{A}_\nu[\pm\infty] = 0$ for $\nu > 1$;

$\mathcal{A}_\nu[-\infty] = (\infty)^{1-\nu}$ for $\nu < 1$ *i.e.*,
$\mathcal{A}_\nu(Q^2 \to 0)$ becomes Unbounded for $\nu < 1$

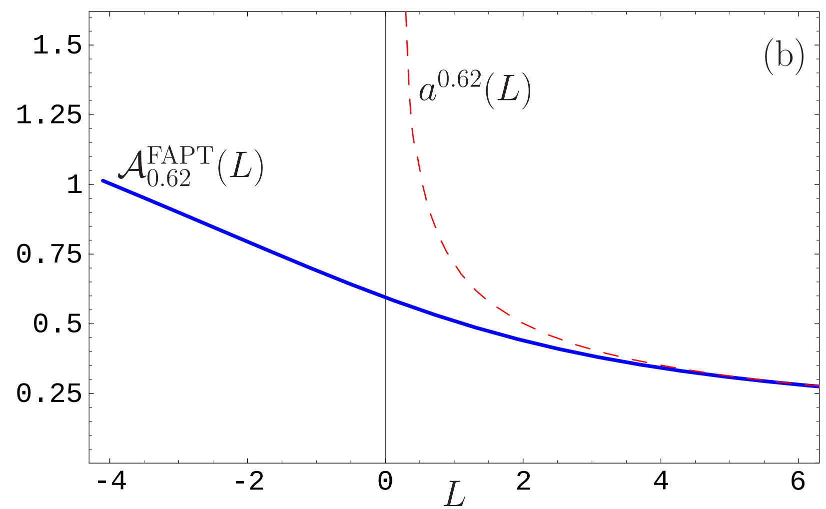
FAPT(Eucl): $\mathcal{A}_\nu[L]$ versus L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}$$

Fractional $\nu \in [2, 3]$:



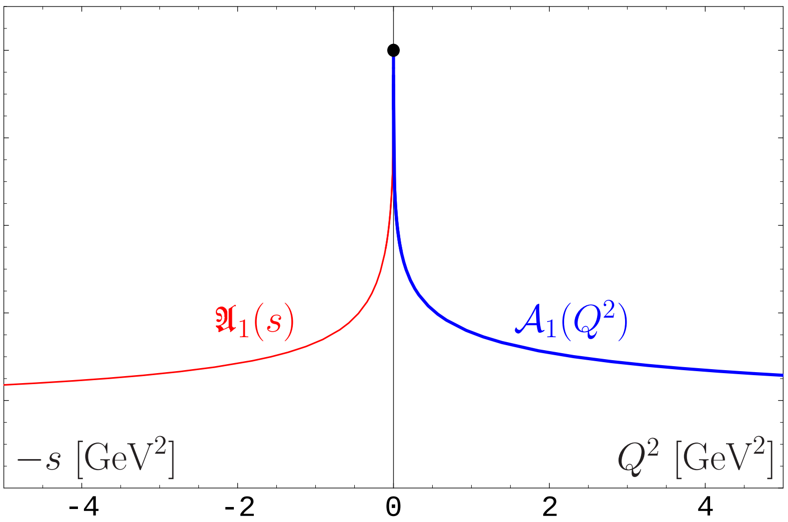
Comparison with $\bar{a}_s^\nu[L]$:



where $\nu = 0.62 = \gamma_2/2\beta_0$

$\bar{a}_s^{1+\nu}[L] \gg \mathcal{A}_{1+\nu}[L] \gg \mathcal{A}_{2+\nu}[L]$ **at $L \sim 1$**

PT vs FAPT for partial TFF. Conclusion.



This behavior in the vicinity of $Q^2 = 0$ is not appropriate!
$\mathfrak{A}_1(0), \mathcal{A}_1(0)$ should be equal to 0

To hold the correspondence with PT asymptotics we put "calibrated FAPT" condition:

$$\mathcal{A}_\nu(0) = \mathfrak{A}_\nu(0) = 0 \text{ for } 0 < \nu \leqslant 1$$

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

Light Cone Sum Rules within FAPT,

New prediction for the pion TFF

Ayala C. &M.S. &Stefanis N., PRD 98 (2018) 096017

The partial TFF_{LCSR}

$$Q^2 F^{\gamma\pi}_{\text{LCSR;0}}(Q^2) = \boxed{\text{standard Born term +twist-4,6}} + \ldots$$

$$N_{\mathrm{T}} \Bigg\{ \boxed{ \int_0^{\bar{x}_0} \psi_0(x) \frac{dx}{\bar{x}} + \frac{Q^2}{m_\rho^2} \int_{\bar{x}_0}^1 \exp\left(\frac{m_\rho^2}{M^2} - \frac{Q^2}{M^2} \frac{\bar{x}}{x} \right) \psi_0(x) \frac{dx}{x} + \text{twist-4,6} } +$$

$$\left(\frac{\mathbb{A}_1(0, s_0; x)}{x} \right) \underset{x}{\otimes} \mathcal{T}^{(1)}(x,y) \underset{y}{\otimes} \psi_0(y) +$$

$$\frac{Q^2}{m_\rho^2} \int_{\bar{x}_0}^1 \exp\left(\frac{m_\rho^2}{M^2} - \frac{Q^2}{M^2} \frac{\bar{x}}{x} \right) \frac{dx}{x} \boldsymbol{\Delta}_1(0, \bar{x}) \mathcal{T}^{(1)}(\bar{x}, y) \otimes \psi_0(y) + O(\mathbb{A}_2) \Bigg\},$$

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$$\left(\frac{\mathbb{A}_1(0, s_0; x)}{x} \right) \underset{x}{\otimes} \mathcal{T}^{(1)}(x,y) \underset{y}{\otimes} \psi_0(y) +$$

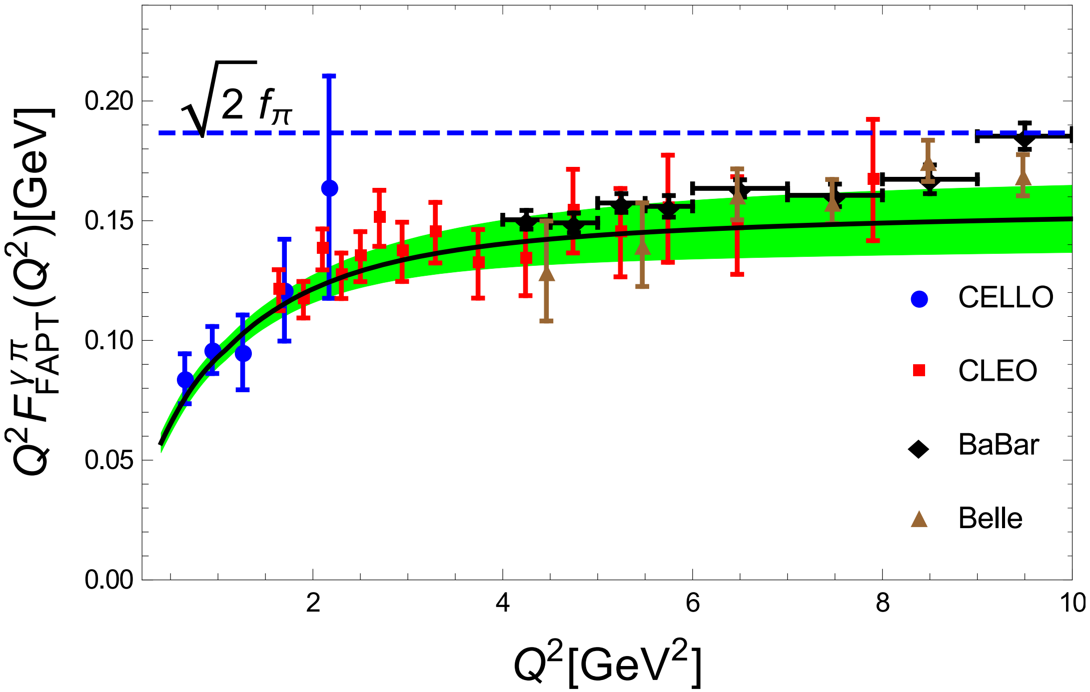
$$\left. \frac{Q^2}{m_\rho^2} \int_{\bar{x}_0}^1 \exp\left(\frac{m_\rho^2}{M^2} - \frac{Q^2}{M^2} \frac{\bar{x}}{x} \right) \frac{dx}{x} \boldsymbol{\Delta}_1(0, \bar{x}) \mathcal{T}^{(1)}(\bar{x}, y) \otimes \psi_0(y) + O(\mathbb{A}_2) \right\},$$

Specific couplings for the case of LCSR, $x_0 = s_0/(s_0 + Q^2)$**,**

$$\begin{aligned} \mathbb{A}_\nu(0, s_0; x) &= \theta(x \geqslant x_0) \left[\mathcal{A}_\nu(Q(x)) - \mathcal{A}_\nu(0) \right] + \\ &\quad \theta(x < x_0) \left[\mathcal{I}_\nu(s_0(x), Q(x)) - \mathfrak{A}_\nu(s_0(x)) \right], \\ \boldsymbol{\Delta}_\nu(0, x) &= \mathbb{A}_\nu(0; x) - \mathbb{A}_\nu(0, s_0; x), \\ s_0(x) &= s_0 \bar{x} - Q^2 x;\; s_0(x_0) = 0\,. \end{aligned}$$

TFF_{LCSR} in FAPT vs the experimental data

$$F^{\gamma\pi}_{\text{LCSR}}\left(Q^2\right)=F^{\gamma\pi}_{\text{LCSR};0}\left(Q^2\right)+\sum_{n=2,4} b_n(\mu^2)\; F^{\gamma\pi}_{\text{LCSR};n}\left(Q^2\right)+\text{Tw-4,6}$$

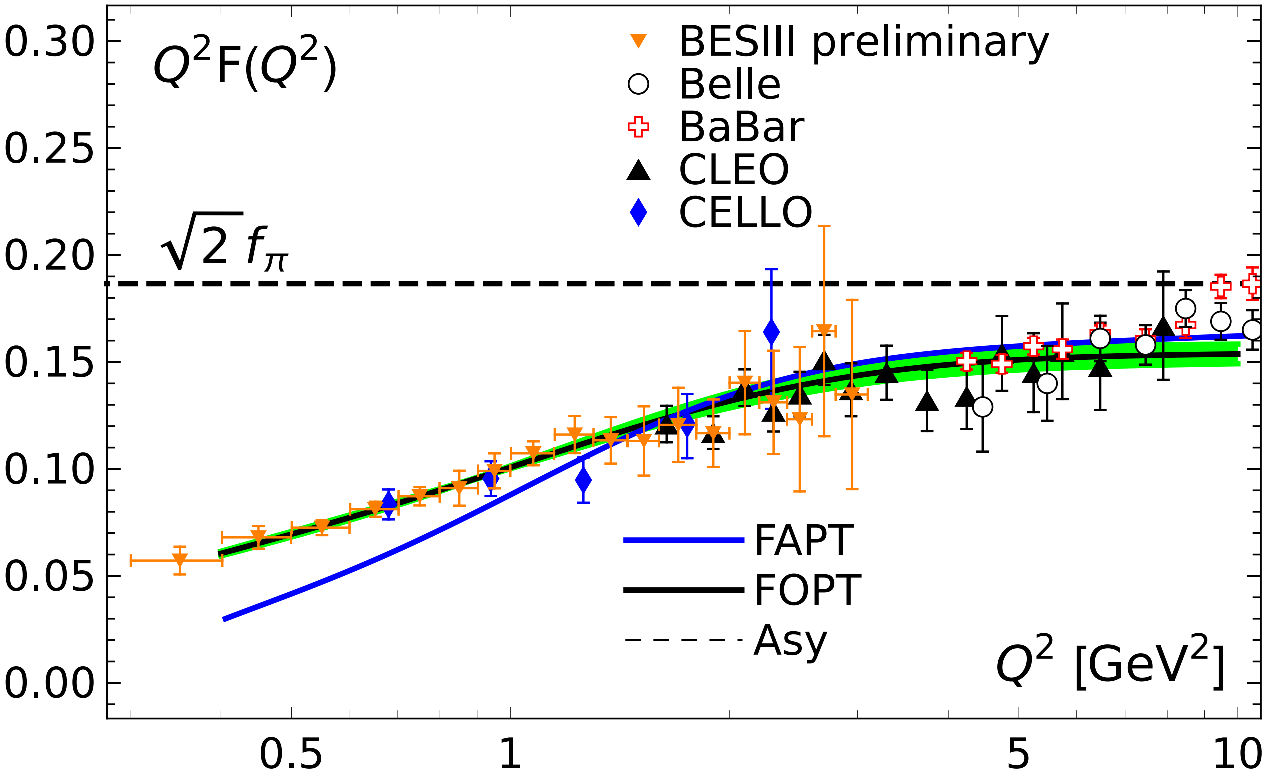


Green strip shows the theoretical uncertainties of $Q^2 F^{\gamma\pi}_{\text{LCSR}}(Q^2)$ at the BMS DA $\{1, b_2, b_4\}$

Ayala &M.S. &Stefanis PRD 98,096017 (2018)

TFF_{LCSR} in FAPT vs the experimental data

$$F^{\gamma\pi}_{\text{LCSR}}(Q^2) = F^{\gamma\pi}_{\text{LCSR};0}(Q^2) + \sum_{n=2,4} b_n(\mu^2)\, F^{\gamma\pi}_{\text{LCSR};n}(Q^2) + \textbf{Tw-4,6}$$



Black line&green strip around - FAPT predictions to $Q^2F^{\gamma\pi}_{\text{LCSR}}$

Blue line - FOPT prediction at N^{2}LO to $Q^2F^{\gamma\pi}_{\text{LCSR}}$

The single fitted parameter is the scale of Tw-6 $\langle\bar{q}q\rangle^2$, taken at its high admissible bound $(0.25)^6$ GeV^2.

CONCLUSIONS

- Fractional APT provides a natural tool to apply APT approach for renormalization group improved perturbative amplitudes.
- The applicability of the FAPT to exclusive processes demands new boundary conditions for the FAPT couplings, $\mathcal{A}_\nu(0) = \mathfrak{A}_\nu(0) = 0, \forall \nu$ as a "feedback"
- LCSRs augmented with RG summation of radiative corrections yield (with endpoint-suppressed BMS DA) transition FF with improved Q^2 behavior and extends the domain of QCD applicability below 1 GeV^2
- This approach of LCSR with FAPT is best-suited for announced BESIII data with high precision and good describes them.

Generelized FAPT:

STORE

Analytic Perturbation Theory in QCD,

Inclusive processes.

“Take care of Principles
and the Principles
will take care of you”

D. Shirkov & I. Solovtsov, PRL79 (1997) 1209;
Theor. Math. Phys. 150 (2007) 132

Basics of APT– formal conditions

- **Different effective couplings in Euclidean, $\mathcal{A}_n$, and Minkowskian, $\mathfrak{A}_n$, regions**

$$\overline{\alpha}_s^n \to \{\mathcal{A}_n, \mathfrak{A}_n\}$$

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- **Based on RG $\oplus$ Causality**

$\Downarrow$ **UV asymptotics** (RG)

$\Downarrow$ **Dispersive represent.** (Causality)

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  $\Downarrow$ **UV asymptotics** $\Downarrow$ **Dispersive represent.**

- Euclid.: $-q^2 = Q^2,\ L = \ln(Q^2/\Lambda^2),\ a_s^n[L] \to \{\mathcal{A}_n[L]\}_{n\in\mathbb{N}}$
- Minkowsk.: $q^2 = s,\ L_s = \ln(s/\Lambda^2),\ a_s^n[L] \to \{\mathfrak{A}_n[L_s]\}_{n\in\mathbb{N}}$

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- Minkowsk.: $q^2 = s,\ L_s = \ln\left(s/\Lambda^2\right),\ a_s^n[L] \to \{\mathfrak{A}_n[L_s]\}_{n\in\mathbb{N}}$
- $$\mathcal{A}_n^{(l)} = \hat{D}\left[\mathfrak{A}_n^{(l)}\right] \equiv Q^2 \int_0^\infty \frac{\mathfrak{A}_n^{(l)}(\sigma)}{\left(\sigma+Q^2\right)^2}\, d\sigma$$

$$\mathfrak{A}_n^{(l)} = \hat{R}\left[\mathcal{A}_n^{(l)}\right] \equiv \int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{\mathcal{A}_n^{(l)}(\sigma)}{\sigma}\, d\sigma\,,$$

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  $\Downarrow$ UV asymptotics $\Downarrow$ Dispersive represent.

- Euclid.: $-q^2 = Q^2,\ L = \ln(Q^2/\Lambda^2),\ a_s^n[L] \to \{\mathcal{A}_n[L]\}_{n\in\mathbb{N}}$
- Minkowsk.: $q^2 = s,\ L_s = \ln(s/\Lambda^2),\ a_s^n[L] \to \{\mathfrak{A}_n[L_s]\}_{n\in\mathbb{N}}$
- $$\mathcal{A}_n^{(l)} = \hat{D}\left[\mathfrak{A}_n^{(l)}\right] \equiv Q^2 \int_0^\infty \frac{\mathfrak{A}_n^{(l)}(\sigma)}{\left(\sigma+Q^2\right)^2}\, d\sigma$$

$$\mathfrak{A}_n^{(l)} = \hat{R}\left[\mathcal{A}_n^{(l)}\right] \equiv \int_{-s-i\varepsilon}^{-s+i\varepsilon} \frac{\mathcal{A}_n^{(l)}(\sigma)}{\sigma}\, d\sigma\,,$$

- On the set of the pars $\{\mathcal{A}_n, \mathfrak{A}_n\}:\ \hat{D}\hat{R} = \hat{R}\hat{D} = 1$

Basics of APT– formal conditions

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- **PT** $\sum_m d_m a_s^m(Q^2) \Rightarrow \sum_m d_m \mathcal{A}_m(Q^2)$ **APT**

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FAPT:

Karanikas A.& Stefanis N., PLB504 (2001), 225; 636 (2006) 330;
Bakulev A. & M.S. & Stefanis N.,PRD72 (2005) 074014; PRD75 (2007) 056005; JHEP06 (2010) 085
G. Cvetic & A. Kotikov, J.Phys.G39 (2012) 065005

Spectral representation

By analytization we mean “Källen–Lehmann” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

the main hero is the spectral density $\rho_n(\sigma) = \frac{\textbf{Im}}{\pi}\left[a_s^n(-\sigma)\right]\beta_0^n$ **:**

$$\mathcal{A}_n[L] = \int_0^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma \stackrel{1l}{=} \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} \mathcal{A}_1[L]$$

$$\mathfrak{A}_n[L_s] = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma}\, d\sigma \stackrel{1l}{=} \frac{1}{(n-1)!}\left(-\frac{d}{dL_s}\right)^{n-1} \mathfrak{A}_1[L_s]$$

$$a_s^n[L] \stackrel{1l}{=} \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1} a_s[L]$$

Inequality: $a_s^n[L] \;\geqslant\; \left(\mathcal{A}_n[L],\, \mathfrak{A}_n[L]\right) \stackrel{L\to\infty}{\longrightarrow} a_s^n[L]$

FAPT(M): Properties of $\mathfrak{A}_\nu[L]$

Now, Minkowskian coupling ($L = L(s)$) is elementary function:

$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

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Properties:
The charge $\mathfrak{A}_\nu$ is bounded for $\nu \geqslant 1$,

- $\mathfrak{A}_0[L] = 1$;
- $\mathfrak{A}_{-1}[L] = L; \mathfrak{A}_{-2}[L] = L^2 - \frac{\pi^2}{3}, \ \ldots \ ;$
- $\mathfrak{A}_m[L] = (-1)^m \mathfrak{A}_m[-L]$ for $m \geq 2\,,\ m \in \mathbb{N}$;
- $\mathfrak{A}_\nu[\pm\infty] = 0$ for $\nu > 1$

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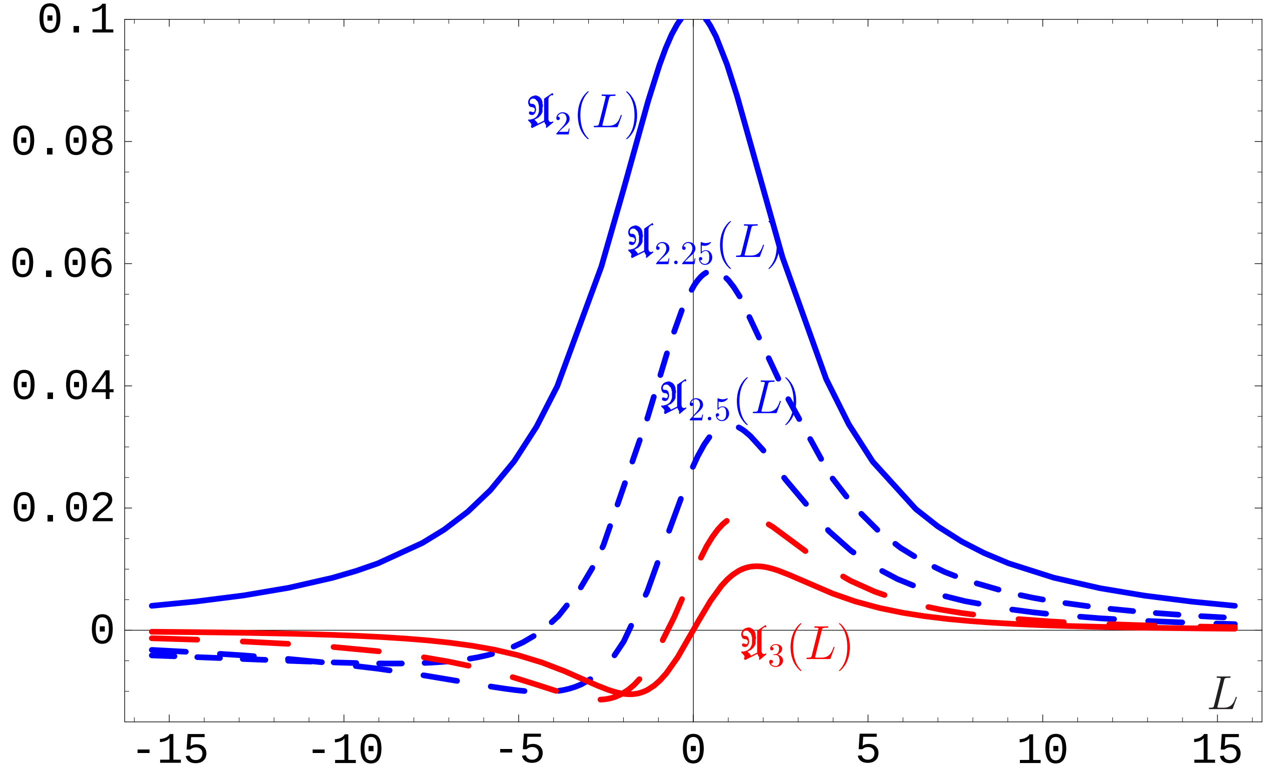
$\mathfrak{A}_\nu[-\infty] = (\infty^2 + \pi^2)^{(1-\nu)/2}$ **for** $\nu < 1$

***i.e.*, $\mathfrak{A}_\nu(Q^2 \to 0)$ becomes Unbounded**

FAPT(M): Graphics of $\mathfrak{A}_\nu[L]$ vs. L

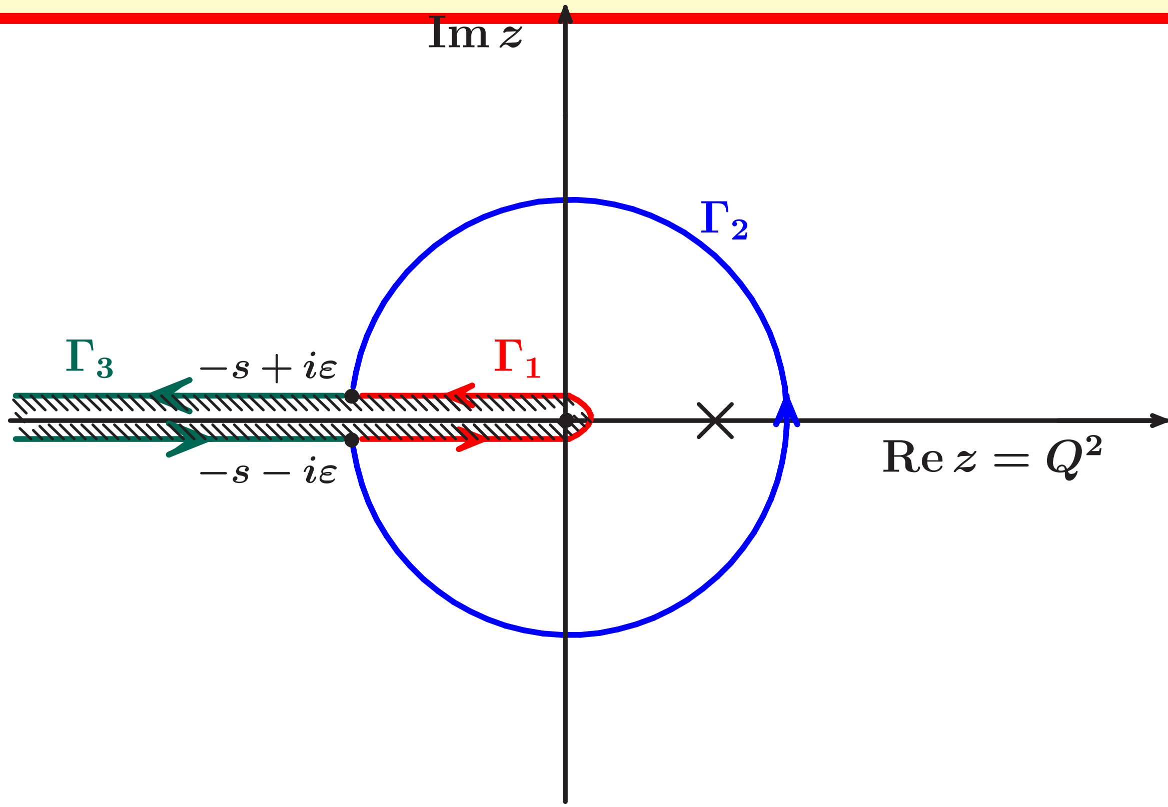
$$\mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(L/\sqrt{\pi^2+L^2}\right)\right]}{\pi(\nu-1)\left(\pi^2+L^2\right)^{(\nu-1)/2}}$$

Compare with graphics in Minkowskian region :



Equivalence CIPT and APT for $R(s)$

$$\mathcal{R} = \text{CIPT}\left\{\oint_{\Gamma_2}\frac{D(z)dz}{z}\right\} = \text{APT}\left\{\oint_{\Gamma_3}\frac{D(z)dz}{z}\right\} = d_0 + \sum_{n=1} d_n \mathfrak{A}_n[L_s]$$



$$\mathcal{R}[L] = d_0 + d_1\int_0^\infty \mathfrak{A}_1[L-t]P(t)dt$$ – **Resummed, finite.**

Bakulev&M.S.&Stefanis JHEP06(2010)085

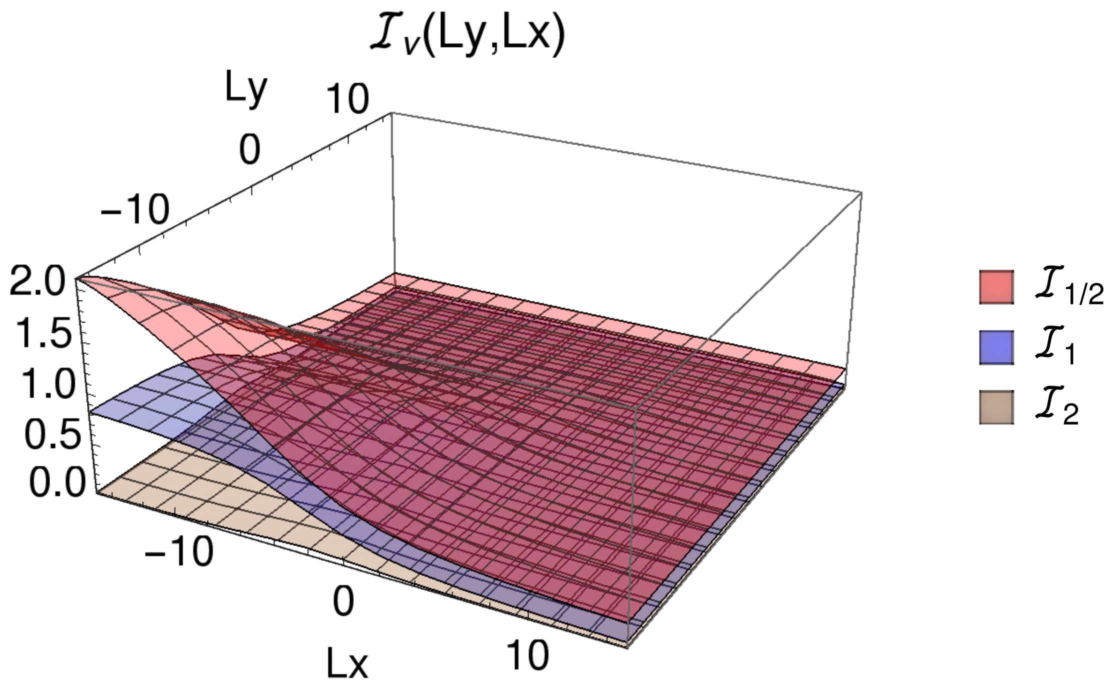
Comparison of PT, APT, and FAPT

Theory	PT	APT	FAPT
Set	$\left\{a^{\nu}\right\}_{\nu\in\mathbb{R}}$	$\left\{\mathcal{A}_m, \mathfrak{A}_m\right\}_{m\in\mathbb{N}}$	$\left\{\mathcal{A}_{\nu}, \mathfrak{A}_{\nu}\right\}_{\nu\in\mathbb{R}}$
Series	$\sum_m f_m\, a^{m+\nu}$	$\sum_m f_m\, \mathcal{A}_m$	$\sum_m f_m\, \mathcal{A}_{m+\nu}$
Inv. powers	$(a[L])^{-m}$	—	$\mathcal{A}_{-m}[L] = L^m$
Products	$a^{\mu}a^{\nu} = a^{\mu+\nu}$	—	—
Index deriv.	$a^{\nu}\ln^k a$	—	$\mathcal{D}^k\mathcal{A}_{\nu}$
Logarithms	$a^{\nu}L^k$	—	$\mathcal{A}_{\nu-k}$
Resummation	—	$\langle\langle\mathcal{A}_1[L-t]\rangle\rangle_{P(t)}$	$\langle\langle\mathcal{A}_{1+\nu}[L-t]\rangle\rangle_{P_{\nu}(t)}$

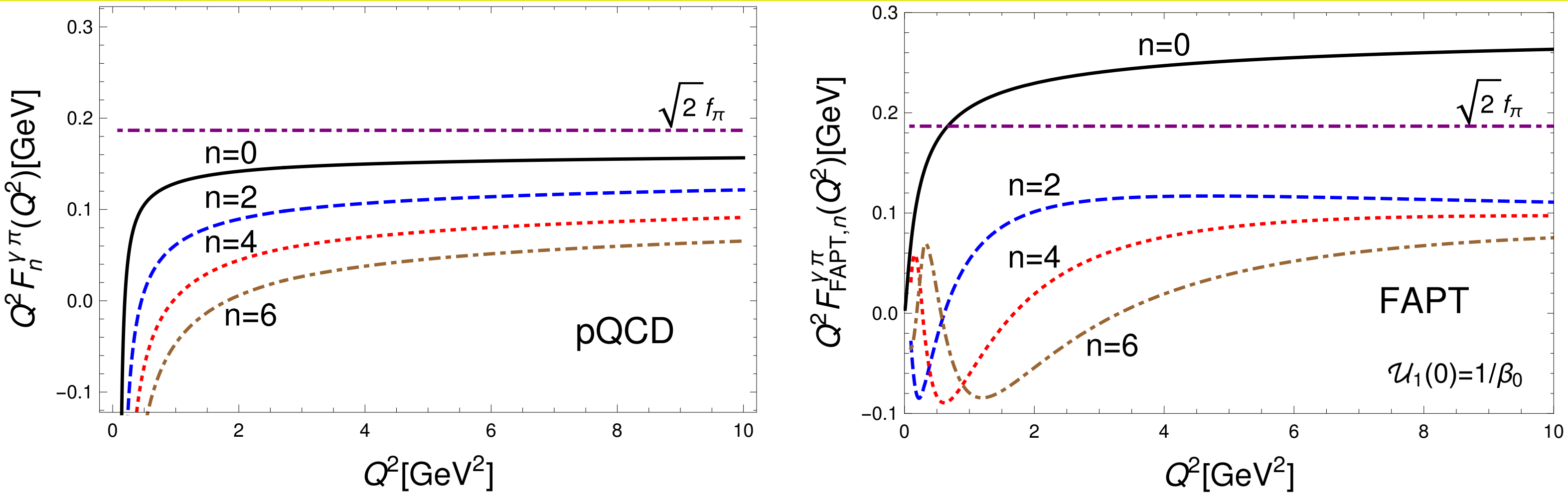
New Expansion charge $\mathcal{I}(y,x)$

$$\mathcal{I}_\nu(y,x) \stackrel{def}{=} \int_y^\infty \frac{d\sigma}{\sigma+x}\rho_\nu^{(l)}(\sigma)$$

$$\mathcal{A}_\nu(x) = \mathcal{I}_\nu(y \to 0, x),\ \mathfrak{A}_\nu(Y) = \mathcal{I}_\nu(y, x \to 0),\ \mathcal{A}_1(0) = \mathfrak{A}_1(0)$$

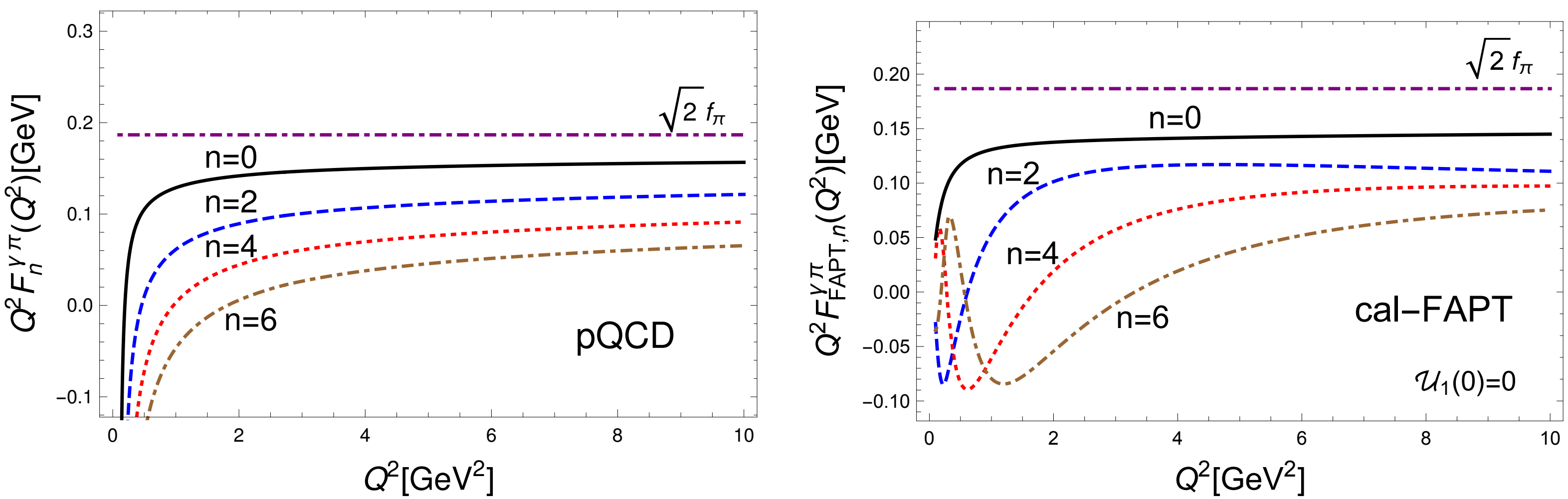


PT vs FAPT for partial TFF

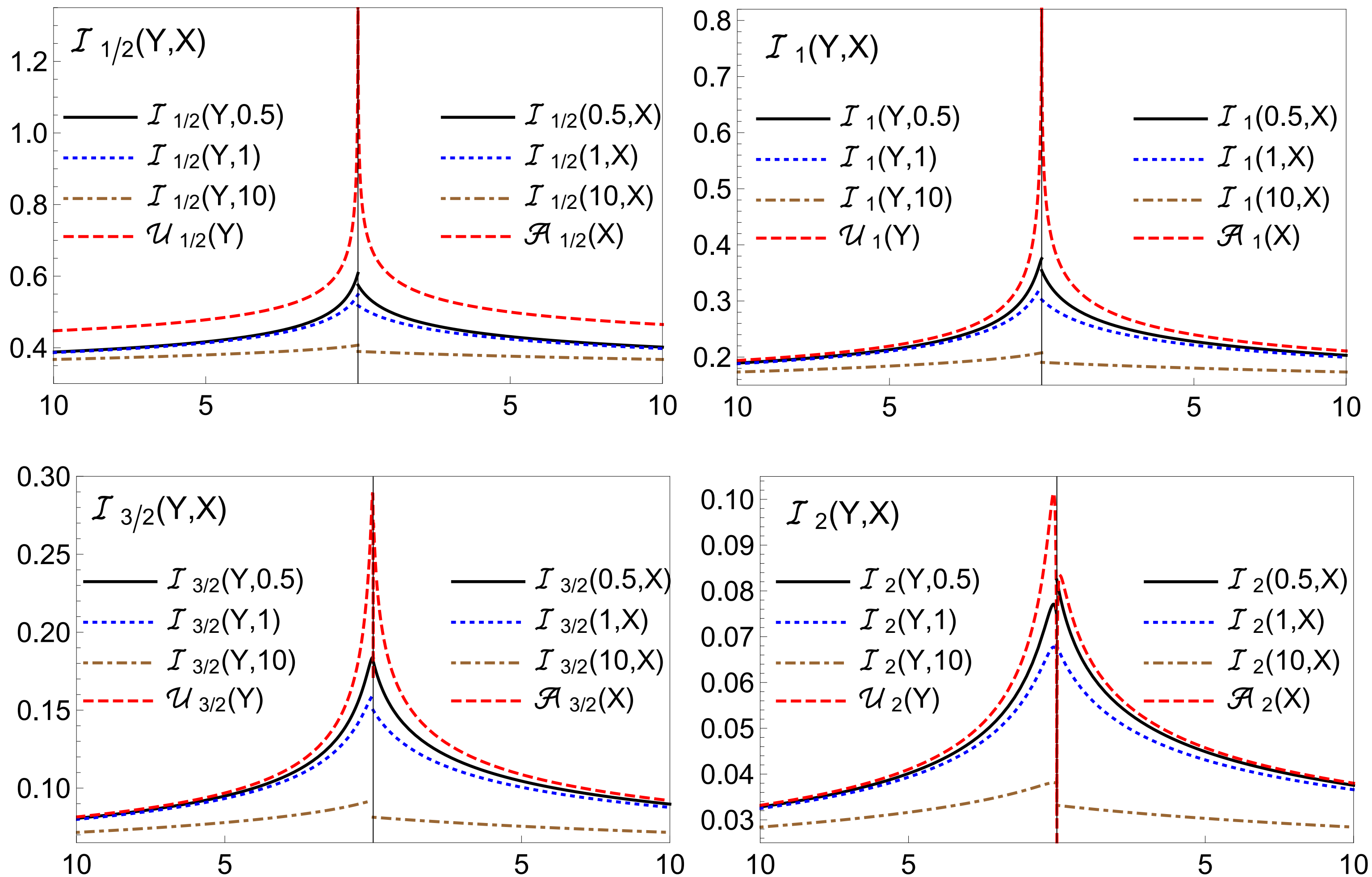


To hold the correspondence with PT asymptotics

$\mathcal{A}_\nu(0) = \mathfrak{A}_\nu(0) = 0$ **for** $0 < \nu \leqslant 1$ **- "calibrated FAPT"**



“Distorting mirror” symmetry



The 2D projections of the 3D plots of $\mathcal{I}_\nu$. The couplings $\mathcal{I}_\nu(y,\textbf{fixed}), \mathcal{I}_\nu(\textbf{fixed}, x)$ are taken for different values of the index $\nu = 1/2, 1, 3/2, 2$.