Transcendental structure of multiloop beta-functions and anomalous dimensions

Konstantin Chetyrkin

Karlsruhe Institite of Tecnology & Hamburg University

in collaboration with Pavel Baikov
Institute of Nuclear Physics of Moscow University

based on P. Baikov, K.Ch.: arXiv:1804.10088, arXiv:1808.00237 and arXiv:1908.03012

QFTHEP–2019, Sochi, 28.09.2019

Our input data: numerous analytical results for 5-loop correlators and RG-functions

obtained during 18 years (2000 — 2017) by Kalsruhe-Moscow group

composed of

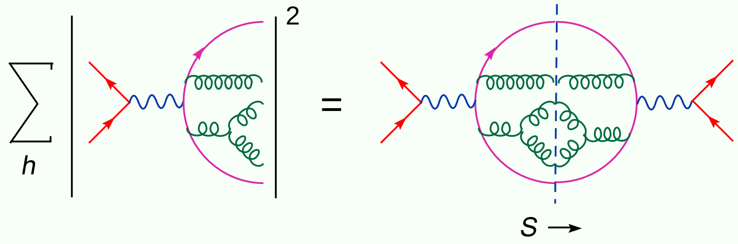
Pavel Baikov, Johann Kühn (KIT) and K.Ch.

Important: essentially all our main results were confirmed by independent calculations during 2017-2018

Massless Propagators (aka *p-integrals*) & Physics

- **2-points correlators at large energies (massless + $\mathcal{O}(m_q^{2n}/Q^{2n})$ corrections)**
  related via the optical theorem to

  $$R(s) = \frac{\sigma_{tot}(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



  semi-leptonic τ-decays

  $\Gamma(Z \to$ **hadrons**$)$,
  $\Gamma(H \to$ **hadrons**$)$, . . .

- **coefficient functions in OPE (DIS, SVZ sum rules,. . .)**

- **beta-functions and anomalous dimensions**

- **massless gluon, quark, etc. QCD propagators (useful for lattice)**

OUR TOOLS

- **CAS “FORM”, /Vermaseren /(1991—. . .)/**

- R^***-operation, V. Smirnov, K.Ch., /(1984—. . .)/, expresses (L+1)-loop RG functions via L-loop p-integrals**

- **very useful and important: NEW representation of Feynman Integrals which led to a new method of reduction $\to 1/D$ expansion /Baikov, (2000—. . .)/**

Interesting fact:

By now the representation is universally named as “the Baikov’s one” in literature

For experts: no IBP reduction was employed

New Representation of FI's /due to Baikov/:

Feynman parameters:

$$\frac{1}{m^2 - p^2} \approx \int d\ \alpha\ \mathrm{e}^{i\alpha(m^2-p^2)}$$

New parameters:

$$\frac{1}{m^2 - p^2} \approx \int \frac{d\ x}{x}\ \delta(x - (m^2 - p^2))$$

Now for a given topology one can make loop integrations once and forever with the result:

$$F(\underline{n}) \quad \sim \quad \int \dots \int \frac{\mathbf{d}x_1 \dots \mathbf{d}x_N}{x_1^{n_1} \dots x_N^{n_N}} \left[P(\underline{x})\right]^{(D-h-1)/2} \ ,$$

where $P(\underline{x})$ is a polynomial on $x_1, \dots, x_N$ (and masses and external momenta)

New representation obviously meets the same set IBP'id as the original integral but it has much

Our results were published, particularly, in 9 Physical Review Letters:

f a Chetyrkin and a baikov and j phys.rev.lett. | Brief format | Search | Easy Search | Advanced Search

find j "Phys.Rev.Lett.,105*" :: more

Sort by: earliest date | asc. | - or rank by -

Display results: 25 results | single list

HEP — **9** records found — Search took 0.18 seconds.

1. **The Cross section of e+ e- annihilation into hadrons of order alpha(s)**4 n(f)**2 in perturbative QCD**
P.A. Baikov (Moscow State U.), K.G. Chetyrkin (Freiburg U.), Johann H. Kuhn (Karlsruhe U., TTP). Aug 2001. 4 pp.
Published in **Phys.Rev.Lett. 88 (2002) 012001**
FREIBURG-THEP-01-13, TTP-01-19
DOI: 10.1103/PhysRevLett.88.012001
e-Print: **hep-ph/0108197** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service
Detailed record - Cited by 58 records 50+

2. **Strange quark mass from tau lepton decays with O(alpha(s)**3) accuracy**
P.A. Baikov (Moscow State U.), K.G. Chetyrkin, Johann H. Kuhn (Karlsruhe U.). Dec 2004. 5 pp.
Published in **Phys.Rev.Lett. 95 (2005) 012003**
SFB-CPP-04-72, TTP-04-28
DOI: 10.1103/PhysRevLett.95.012003
e-Print: **hep-ph/0412350** | **PDF**
References | BibTeX | LaTeX(US) | LaTeX(EU) | Harvmac | EndNote
ADS Abstract Service
Detailed record - Cited by 91 records 50+

3. **Scalar correlator at O(alpha(s)**4), Higgs decay into b-quarks and bounds on the light quark masses**
P.A. Baikov (Moscow, INR), K.G. Chetyrkin, Johann H. Kuhn (Karlsruhe U., TTP). Nov 2005. 5 pp.
Published in **Phys.Rev.Lett. 96 (2006) 012003**
SFB-CPP-05-33, TTP05-11
DOI: 10.1103/PhysRevLett.96.012003

QCD β-function in FIVE loops: result

$$\mu^2 \frac{\partial}{\partial \mu^2} a_s = \beta(a_s)\, a_s, \quad a_s \equiv \frac{\alpha_s}{\pi}, \quad \beta(a_s) = \sum_{i \geq 1} \beta_i\, a_s^i$$

$$\begin{aligned}
4^5\, \beta_5 = & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad n_f & \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5 \right] \\
+ \quad n_f^2 & \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5 \right] \\
+ \quad n_f^3 & \boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5 \right]} + n_f^4 \boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3 \right]}
\end{aligned}$$

n_f^4 term is in **FULL AGREEMENT** with the 20 years old result by John Gracey (in the framework of the conformal bootstrap method of A. Vasiliev, Yu. Pis'mak and J. Honkonen (1981))

n_f^3 term is in **FULL AGREEMENT** with a result by Th. Luthe, A. Maier, P. Marquard and

QCD β-function in FIVE loops: Zeta's

In general any 5-loop beta in any theory will have the following "transcendental structure" (an obvious outcome of our knowledge of the corresponding masters)

1 and 2 loops: rational

3 loops: rationals + ζ_3

4 loops: rationals + $\zeta_3 + \zeta_4 + \zeta_5$

5 loops: rationals + $\zeta_3 + \zeta_4 + \zeta_5 + \zeta_6 + \zeta_3^2 + \zeta_7$

$$\beta_1 = \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \quad \beta_2 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \quad \beta_3 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\},$$

$$\beta_4 = \frac{1}{4^4}\left\{\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right\}$$

QCD β-function in FIVE loops: Zeta's

In reality, the QCD β-function displays a delayed appearance of zeta's (well-known at 3 and 4 loops) which happens also in 5 loops.

1 and 2 loops: rational

3 loops: rationals + $\cancel{\zeta_3}$

4 loops: rationals + ζ_3 + $\cancel{\zeta_4 + \zeta_5}$

5 loops: rationals + ζ_3 + ζ_4 + ζ_5 $\cancel{+ \zeta_6 + \zeta_3^2 + \zeta_7}$

$$
\begin{aligned}
\beta_1 &= \frac{1}{4}\left\{11 - \frac{2}{3}n_f,\right\}, \quad \beta_2 = \frac{1}{4^2}\left\{102 - \frac{38}{3}n_f\right\}, \quad \beta_3 = \frac{1}{4^3}\left\{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2\right\}, \\
\beta_4 &= \frac{1}{4^4}\left\{\left(\frac{149753}{6} + 3564\zeta_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)n_f\right. \\
&\qquad \left. + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)n_f^2 + \frac{1093}{729}n_f^3\right\}
\end{aligned}
$$

$$
\begin{aligned}
4^5\,\beta_5 = {} & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\zeta_5 \\
+ \quad & n_f \left[-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5 \right] \\
+ \quad & n_f^2 \left[\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5 \right] \\
+ \quad & n_f^3 \boxed{\left[-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5 \right]} + n_f^4 \boxed{\left[\frac{1205}{2916} - \frac{152}{81}\zeta_3 \right]}
\end{aligned}
$$

all possible irrationalities do appear in separate diagrams contributing to the β at 5 loops (about two and half million ($2.5 \cdot 10^6$))

A related puzzle

The seminal calculation /Gorishnii, Kataev, Larin/ of the $\mathcal{O}(\alpha_s^3)$ Adler function demonstrated for the first time a mysterious complete cancellation of **all** contributions proportional to ζ_4 (abounding in separate diagrams) while odd zetas ζ_3 and ζ_5 survive! The result is π-**free** ($\zeta_4 = \frac{\pi^4}{90}$ and $\zeta_6 = \frac{\pi^6}{945}$)

$$\mathbf{d}_2 = -\frac{3}{32}C_F^2 + C_F T_f \left[\zeta_3 - \frac{11}{8}\right] + C_F C_A \left[\frac{123}{32} - \frac{11\zeta_3}{4}\right],$$

$$d_3 = -\frac{69}{128}C_F^3 + C_F^2 T_f \left[-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right] + C_F T_f^2 \left[\frac{151}{54} - \frac{19}{9}\zeta_3\right] + C_F^2 C_A \left[-\frac{127}{64} - \frac{143}{16}\zeta_3 +$$

$$+C_F T_f C_A \left[-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_F C_A^2 \left[\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right],$$

the authors wrote: **"We would like to stress the cancellations of ζ_4 in the final results for $R(s)$. It is very interesting to find the origin of the cancellation of ζ_4 in the physical quantity."**

The situation got even more interesting about 20 years later: the $\mathcal{O}({\alpha_s}^4)$ contributions to the Adler function and to the coefficient function (CF) of

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$	$\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\zeta_3 + \frac{125}{6}\zeta_5 + 3\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\zeta_3 - \frac{170}{9}\zeta_5 - \frac{1}{2}\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\zeta_3 - \frac{5}{12}\zeta_5 + \frac{1}{6}\zeta_3^2 EQN$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\zeta_3 - \frac{125}{4}\zeta_5 + \frac{105}{4}\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\zeta_3 + \frac{145}{24}\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\zeta_3 - \frac{5155}{48}\zeta_5 - \frac{33}{4}\zeta_3^2 - \frac{105}{8}\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\zeta_3 - \frac{95}{144}\zeta_5 - \frac{35}{4}\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\zeta_3 + \frac{18805}{288}\zeta_5 - \frac{11}{2}\zeta_3^2 + \frac{35}{16}\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\zeta_3 + \frac{1855}{288}\zeta_5 - \frac{11}{12}\zeta_3^2 + \frac{35}{16}$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\zeta_3 + \frac{2255}{32}\zeta_5 - \frac{1155}{16}\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\zeta_3 - \frac{1045}{48}\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\zeta_3 + \frac{6505}{48}\zeta_5 + \frac{1155}{32}\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\zeta_3 + \frac{55}{9}\zeta_5 + \frac{385}{16}\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\zeta_3 - \frac{77995}{1152}\zeta_5 + \frac{605}{32}\zeta_3^2 - \frac{385}{64}\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\zeta_3 - \frac{12545}{1152}\zeta_5 + \frac{121}{96}\zeta_3^2 -$

Transcedentals: odd zetas: $\zeta_3, \zeta_5, \zeta_7$ BUT NOT even one ζ_4 or ζ_6 (both appear eventually in *every* separate input diagram /from about 20 thousand!/)

Very recently there has happened a breakthrough[*] in our understanding of the transcendental structure of all RG-functions, including β_{QCD}, as well as the Adler function and similar objects like C_{Bjp}. As a result, we do understand now and can even predict the exact form of π-dependent terms in RG-functions *in terms of π-independent ones*:

4-loops:

$$\boldsymbol{\beta_4^{\zeta_4} = \beta_1\, \beta_3^{\zeta_3}} \quad (= \mathbf{0} \ \text{ for QCD as } \boldsymbol{\beta_3^{\zeta_3} \equiv 0})$$

5-loops (the relation below is, in fact, sitting /in a disguised form!/ in an important paper

Jamin and Miravitllas, *Absence of even-integer ζ-function values in Euclidean physical quantities in QCD*, 1711.00787

which has triggered our work on the π-dependence of RG-functions)

$$\boldsymbol{\beta_5^{\zeta_4} = \frac{9}{8}\beta_1\, \beta_4^{\zeta_3}}$$

where ($\boldsymbol{F^{\zeta_i} = \lim_{\zeta_i \to 0} \frac{\partial}{\partial \zeta_i} F}$):

The factorization in the second formula is not trivial at all:

$$\beta_1 \qquad\qquad\qquad\qquad (\partial/\partial\zeta_3)\beta_4$$

$$\frac{\partial}{\partial\zeta_4}\beta_5 \;=\; \frac{9}{8}\Big(\frac{2}{3}n_f - 11\Big)\Big(-\frac{6472}{81}n_f^2 + \frac{6508}{27}n_f - 3564\Big)$$

while for a case of a generic gauge group it takes the form:

$$\beta_1 \qquad\qquad\qquad\qquad (\partial/\partial\zeta_3)\beta_4$$

$$\begin{aligned}\frac{\partial}{\partial\zeta_4}\beta_5 \;=\; & \frac{9}{8}\Big(\frac{4}{3}n_f T_F - \frac{11}{3}C_A\Big) \star \Big(\frac{44}{9}C_A^4 - \frac{136}{3}C_A^3 n_f T_F \\ & + \frac{656}{9}C_A^2 C_F n_f T_F - \frac{224}{9}C_A^2 n_f^2 T_F^2 - \frac{352}{9}C_A C_F^2 n_f T_F \\ & - \frac{448}{9}C_A C_F n_f^2 T_F^2 + \frac{704}{9}C_F^2 n_f^2 T_F^2 - \frac{704}{3}\frac{d_A^{abcd}d_A^{abcd}}{N_A} \\ & + \frac{1664}{3}\frac{d_F^{abcd}d_A^{abcd}}{N_A}n_f - \frac{512}{3}\frac{d_F^{abcd}d_F^{abcd}}{N_A}n_f^2\Big)\end{aligned}$$

A word about notations and conventions (goodbye β_0 and γ_0)

we use

1. $$\gamma(a) = \sum_{i\geq 1} \gamma_i \, a^i, \quad a = \frac{\alpha_s}{4\pi}$$

2. $$\beta(a) = \sum_{i\geq 1} \beta_i \, a^i$$

3. Landau gauge for QCD (for simplicity, could be relaxed)

4. G-scheme instead of $\overline{\text{MS}}$ one: all ADs and betas are *not* different from their $\overline{\text{MS}}$ versions but the simplest 1-loop p-integral is tuned to be maximally simple:

$$\frac{1}{i(2\pi)^D} \int \frac{d^D l}{(-l^2)(-(q-l)^2)} = \frac{1}{(4\pi)^2 \, (-q^2)^\epsilon} \frac{1}{\epsilon}$$

for finite renormalized quantities: $\left(\ln \frac{\mu^2}{Q^2}\right)_G \to \left(\ln \frac{\mu^2}{Q^2}\right)_{\overline{\text{MS}}} + 2$

π-structure of p-integrals

We will call a (bare) L-loop p-integral $F(Q^2, \epsilon)$ π-*safe* if the π-dependence of its pole in ϵ *and* constant part can be completely absorbed into the properly defined "hatted" odd zetas.

The first observation of a non-trivial class of π-safe p-integrals — all 3-loop ones — was made in /Broadhurst (1999)/ An extension of the observation on the class of all 4-loop p-integrals was performed in /Baikov, K.Ch. (2010)/ Here it was shown that, given an arbitrary 4-loop p-integral, its pole in ϵ *and* constant part depend on even zetas *only* via the following combinations:

$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6, \;\; \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2}\zeta_6 \quad \text{and} \quad \hat{\zeta}_7 := \zeta_7.$$

Exact meaning: for any 4-loop p-integral F_4:

$$F_4(\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7) = F_4(\hat{\zeta}_3, 0, \hat{\zeta}_5, 0, \hat{\zeta}_7) + \mathcal{O}(\epsilon) \qquad \star$$

A generalization of the $\star$ for L=5 has been recently constructed in /Georgoudis, Goncalves, Panzer, Pereira, [1802.00803]/ (and confirmed independently by us)

Remainder on connection between L-loop p-integrals and (L+1) loop Z-factors (a Minimal scheme is assumed!)

The connection is given by the following (35 years old!) Theorem (V. Smirnov, K. Ch, /1983/)

Theorem *Any (L+1)-loop UV counterterm for any Feynman integral may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

Corollary *Any (L+1)-loop anomalous dimension or a beta-function in any theory may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

$\hat{G}$-scheme

Let us define the $\widehat{\mathrm{G}}$-scheme by pretending that hatted zetas do not depend on ϵ. This means that all p-integrals are assumed to be expressed in term of the hatted zetas and that the extraction of the pole part of a p-integral is defined as:

$$\hat{K}\Big(\mathcal{P}(\epsilon)\prod_j \hat{\zeta}_j\Big) := \left(\sum_{i<0} \mathcal{P}_i\,\epsilon^j\right)\prod_j \hat{\zeta}_j,$$

with $\mathcal{P}(\epsilon) = \sum_i \epsilon^i \mathcal{P}_i$ being a polynomial in ϵ with rational coefficients. The corresponding coupling constant will be denoted as $\hat{a}$.

The $\hat{G}$-scheme has some remarkable features. Indeed, one can see just from its definition that the corresponding "hatted" Green function, ADs and Z-factors can be obtained from the normal (that is computed with the G-scheme) by very simple rules.

- As a first step we make a formal replacement of the coupling constant a by $\hat{a}$ in every G-renormalized Green function, AD and Z-factor we want to transform to the $\hat{G}$-scheme.
- Renormalized Green function $\hat{F}(\hat{a})$ is obtained from $F(\hat{a})$ by setting to zero *all* even zetas in the latter (both are assumed as taken at $\epsilon = 0$).
- The same rule works for ADs and β-functions.
- If Z is a (G-scheme) renormalization constant then one should not only nullify all even zetas in $Z(\hat{a})$ but also replace every odd zeta term in it with its "hatted" counterpart.

$\hat{G}$-scheme: useful properties and benefits

1. All 2-point (masless, <u>but not necessarily SI</u>) correlators (at least to 5 loops), β-functions and ADs (at least to 6 loops) are π-free in $\hat{G}$-scheme

2. As $\hat{G}$-scheme is related in a unique way to the normal G-scheme we arrive to conclusion that π-dependent terms in G-(and $\overline{\text{MS}}$ too!) (renormalized) correlators, and RG-functions should be restorable from the π-free contributions *and* the structures of the hatted representations of π-free generators

$\hat{G}$-scheme: constraints on even zetas

Suppose we know a result for an AD $\hat{\gamma} := (\gamma)_{\hat{G}-\text{scheme}}$ as well as the precise way how hatted zetas are related to the normal ones. The infromation should be enough to construct the result in normal, say, $\overline{\text{MS}}$-scheme Thus, all terms proportional to even zetas in γ should be possible to recover. To do this let us consider the relation between $\hat{a}$ and a:

$$\hat{a} = a \left(1 + \sum_{1 \le i \le L} c_i \, a^i \right),$$

As the bare charge must not depend on the choice of the renormalization scheme the coefficients c_i are fixed by requiring that

$$Z_a \, a = \hat{Z}_a(\hat{a})\hat{a}$$

For simplicity we start from the case of 4 loops. On general grounds we can write

$$\beta = \beta_1 a + \beta_2 a^2 + (r_3 + \beta_3^{\zeta_3} \, \zeta_3) \, a^3 + (r_4 + \beta_4^{\zeta_3} \, \zeta_3 + \beta_4^{\zeta_4} \, \zeta_4 + \beta_4^{\zeta_5} \, \zeta_5) \, a^4$$

The corresponding RCs Z_a and $\hat{Z}_a$ read:

$$\begin{aligned} Z_a &= 1 + \frac{a\beta_1}{\epsilon} + a^2\,(\frac{1}{2\epsilon}\,\beta_2 + \frac{1}{\epsilon^2}\,\beta_1^2) + a^3\,(\frac{1}{3\epsilon}\,(r_3 + \beta_3^{\zeta_3}\,\zeta_3) + \frac{7}{6\epsilon^2}\,\beta_1\,\beta_2 + \frac{1}{\epsilon^3}\,\beta_1^3) \\ &\quad + a^4\,(\frac{1}{4\epsilon}\,(r_4 + \beta_4^{\zeta_3}\,\zeta_3 + \beta_4^{\zeta_4}\,\zeta_4 + \beta_4^{\zeta_5}\,\zeta_5) + \frac{1}{\epsilon^2}\,(\frac{5}{6}\,\beta_1\,r_3 + \frac{5}{6}\,\beta_1\,\beta_3^{\zeta_3}\,\zeta_3 + \frac{3}{8}\,\beta_2^2) \\ &\quad + \frac{23}{12\epsilon^3}\,\beta_1^2\,\beta_2 + \frac{1}{\epsilon^4}\,\beta_1^4) \end{aligned} \tag{1}$$

and

$$\begin{aligned} \hat{Z}_a &= 1 + \frac{\hat{a}}{\epsilon}\,\beta_1 + \hat{a}^2\,(\frac{1}{2\epsilon}\,\beta_2 + \frac{1}{\epsilon^2}\,\beta_1^2) + \hat{a}^3\,(\frac{1}{3\epsilon}\,(r_3 + \beta_3^{\zeta_3}\,\hat{\zeta}_3) + \frac{7}{6\epsilon^2}\,\beta_1\,\beta_2 + \frac{1}{\epsilon^3}\,\beta_1^3) \\ &\quad + \hat{a}^4\,(\frac{1}{4\epsilon}\,(r_4 + \beta_4^{\zeta_3}\,\hat{\zeta}_3 + \beta_4^{\zeta_5}\,\hat{\zeta}_5) + \frac{1}{\epsilon^2}\,(\frac{5}{6}\,\beta_1\,r_3 + \frac{5}{6}\,\beta_1\,\beta_3^{\zeta_3}\,\hat{\zeta}_3 + \frac{3}{8}\,\beta_2^2) \\ &\quad + \frac{23}{12\epsilon^3}\,\beta_1^2\,\beta_2 + \frac{1}{\epsilon^4}\,\beta_1^4). \end{aligned} \tag{2}$$

Equation for c_i can be now easily solved with the result

$$\begin{aligned}
c_1 &= c_2 = 0, \\
c_3 &= -\frac{1}{2}\,\beta_3^{\zeta_3}\,\zeta_4 + \frac{5\epsilon^2}{6}\,\beta_3^{\zeta_3}\,\zeta_6 - \frac{7\epsilon^4}{2}\,\beta_3^{\zeta_3}\,\zeta_8, \\
c_4 &= \frac{1}{4\epsilon}\,(\beta_4^{\zeta_4} - \beta_1\,\beta_3^{\zeta_3})\,\zeta_4 - \frac{3}{8}\,\beta_4^{\zeta_3}\,\zeta_4 - \frac{5}{8}\,\beta_4^{\zeta_5}\,\zeta_6 \\
&\quad + \frac{5\epsilon}{12}\,\beta_1\,\beta_3^{\zeta_3}\,\zeta_6 + \epsilon^2\,(\frac{5}{8}\,\beta_4^{\zeta_3}\,\zeta_6 + \frac{35}{16}\,\beta_4^{\zeta_5}\,\zeta_8) - \frac{7\epsilon^3}{4}\,\beta_1\,\beta_3^{\zeta_3}\,\zeta_8 - \frac{21\epsilon^4}{8}\,\beta_4^{\zeta_3}\,\zeta_8
\end{aligned}$$

As the coefficients c_i have to be finite at $\epsilon \to 0$ we arrive at the exact connection

$$\boldsymbol{\beta_4^{\zeta_4} = \beta_1\,\beta_3^{\zeta_3}}$$

Repeating the same reasoning for L=5 and 6 (and similar one for the case of an AD) we arrive at a host of new exact identities for even zetas terms

Model independent predictions for β and γ for any 1-charge theory

$$\beta_4^{\zeta_4} = \beta_1 \beta_3^{\zeta_3}$$

$$\beta_5^{\zeta_4} = \frac{1}{2}\beta_3^{\zeta_3}\beta_2 + \frac{9}{8}\beta_1\beta_4^{\zeta_3}$$

$$\beta_5^{\zeta_6} = \frac{15}{8}\beta_1\beta_4^{\zeta_5}$$

$$\beta_5^{\zeta_3\zeta_4} = 0$$

$$\beta_6^{\zeta_4} = \frac{3}{4}\beta_2\beta_4^{\zeta_3} + \frac{6}{5}\beta_1\beta_5^{\zeta_3}$$

$$\beta_6^{\zeta_6} = \frac{5}{4}\beta_2\beta_4^{\zeta_5} + 2\beta_1\beta_5^{\zeta_5} - \beta_1^3\beta_3^{\zeta_3}$$

$$\beta_6^{\zeta_3\zeta_4} = \frac{12}{5}\beta_1\beta_5^{\zeta_3^2}$$

$$\gamma_4^{\zeta_4} = -\frac{1}{2}\beta_3^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_1\gamma_3^{\zeta_3}$$

$$\gamma_5^{\zeta_4} = -\frac{3}{8}\beta_4^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_2\gamma_3^{\zeta_3} - \beta_3^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_1\gamma_4^{\zeta_3}$$

$$\gamma_5^{\zeta_6} = -\frac{5}{8}\beta_4^{\zeta_5}\gamma_1 + \frac{5}{2}\beta_1\gamma_4^{\zeta_5}$$

$$\gamma_5^{\zeta_3\zeta_4} = 0$$

$$\gamma_6^{\zeta_4} = \frac{3}{2}\beta_3^{(1)}\gamma_3^{\zeta_3} - \frac{3}{10}\beta_5^{\zeta_3}\gamma_1 - \frac{3}{4}\beta_4^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_2\gamma_4^{\zeta_3} - \frac{3}{2}\beta_3^{\zeta_3}\gamma_3^{(1)} + \frac{3}{2}\beta_1\gamma_5^{\zeta_3}$$

$$\gamma_6^{\zeta_6} = -\frac{1}{2}\beta_5^{\zeta_5}\gamma_1 - \frac{5}{4}\beta_4^{\zeta_5}\gamma_2 + \frac{5}{2}\beta_2\gamma_4^{\zeta_5} + \frac{5}{2}\beta_1\gamma_5^{\zeta_5} + \frac{3}{2}\beta_1^2\beta_3^{\zeta_3}\gamma_1 - \frac{5}{2}\beta_1^3\gamma_3^{\zeta_3}$$

$$\gamma_6^{\zeta_3\zeta_4} = -\frac{3}{5}\beta_5^{\zeta_3^2}\gamma_1 + 3\beta_1\gamma_5^{\zeta_3^2}$$

$$\beta_6^{\zeta_8} = \frac{14}{5}\beta_1\beta_5^{\zeta_7}$$

$$\beta_6^{\zeta_3\zeta_6} = 0$$

$$\beta_6^{\zeta_4\zeta_5} = 0$$

$$\gamma_6^{\zeta_8} = -\frac{7}{10}\beta_5^{\zeta_7}\gamma_1 + \frac{7}{2}\beta_1\gamma_5^{\zeta_7}$$

$$\gamma_6^{\zeta_3\zeta_6} = 0$$

$$\gamma_6^{\zeta_4\zeta_5} = 0$$

The above constraints have been sucessfully checked on the following examples:

L=4 and 5: numerous QCD RG functions (including gauge-dependent ones taken in the Landau gauge) recently computed in

/K.Ch, Falcioni, Herzog and J Vermaseren [1709.08541] .

L=4,5 and 6: β-function and ADs of $O(n)$ ϕ^4 model recently computed in
Batkovich, K. Ch. and Kompaniets, [1601.01960] (γ_2 only)
Schnetz, [1606.08598] ($\beta, \gamma_2, \gamma_m$)
Kompaniets and Panzer, [1705.06483] ($\beta, \gamma_2, \gamma_m$)

Predictions for 6-loop QCD RG functions:

$$\beta_6 \stackrel{\pi}{=} \boxed{\frac{608}{405} n_f^5 \zeta_4} + n_f^4 (\frac{164792}{1215} \zeta_4 - \frac{1840}{27} \zeta_6) + n_f^3 (-\frac{4173428}{405} \zeta_4 + \frac{1800280}{243} \zeta_6)$$
$$+n_f^2 (\frac{68750632}{405} \zeta_4 - \frac{13834700}{81} \zeta_6) + n_f (-\frac{146487538}{135} \zeta_4 + \frac{40269130}{27} \zeta_6)$$
$$+99 (44213 \zeta_4 - 64020 \zeta_6)$$

$$\gamma_6^m \stackrel{\pi}{=} \boxed{\frac{320}{243} n_f^5 \zeta_4 + n_f^4 (-\frac{90368}{405} \zeta_4 + \frac{22400}{81} \zeta_6)}$$
$$+n_f^3 (-\frac{92800}{27} \zeta_3 \zeta_4 - \frac{2872156}{405} \zeta_4 + \frac{503360}{243} \zeta_6)$$
$$+n_f^2 (\frac{661760}{9} \zeta_3 \zeta_4 + \frac{155801234}{405} \zeta_4 - \frac{378577520}{729} \zeta_6 + \frac{12740000}{81} \zeta_8)$$
$$+n_f (-\frac{1413280}{3} \zeta_3 \zeta_4 - \frac{4187656168}{1215} \zeta_4 + \frac{5912758120}{729} \zeta_6 - \frac{96071360}{27} \zeta_8)$$
$$+3194400 \zeta_3 \zeta_4 + \frac{272688530}{81} \zeta_4 - \frac{6778602160}{243} \zeta_6 + 15889720 \zeta_8$$

boxed terms are in FULL AGREEMENT with the well-known results by /Gracey (1996)/ and /Ciuchini, Derkachov, Gracey and Manashov (1999-2000)/

all other terms are new

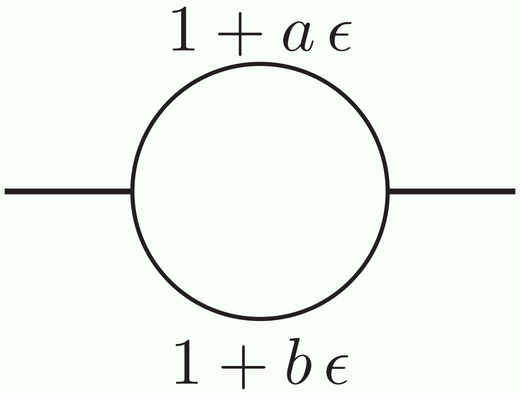
New developments: 6 (and 7 loops)

PROBLEM:

direct way: finding and evaluation of master p-integrals was fully implemented only for L=4, and (semi)-fully at L=5. The case with $L > 5$ is excuded for the moment due to their overwhelming complexity!

Hopeless? NO!

A lot of information can be get from 1LR diagrams like



$$\sim G(\alpha,\beta) = \frac{\Gamma(\alpha+\beta-2+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(2-\alpha-\epsilon)\,\Gamma(2-\beta-\epsilon)}{\Gamma(4-\alpha-\beta-2\epsilon)}$$

$$\epsilon\, G(1,1+\epsilon) = \frac{1}{2}+\frac{1}{2}\epsilon+\frac{3}{2}\epsilon^2+\left(\frac{9}{2}-3\zeta_3\right)\epsilon^3+\left(-3\zeta_3+\frac{27}{2}-\frac{\pi^4}{20}\right)\epsilon^4+\left(-9\zeta_3-21\zeta(5)+\frac{81}{2}-\frac{\pi^4}{20}\right)$$

$$+$$

$$\left(-27\zeta_3+9\zeta_3^2-21\zeta(5)+\frac{243}{2}+-\frac{3\pi^4}{20}-\frac{\pi^6}{21}\right)\epsilon^6$$

$$+$$

$$\left(-81\zeta_3+\frac{3\pi^4\zeta_3}{10}+9\zeta_3^2-63\zeta(5)-147\zeta(7)+\frac{729}{2}-\frac{9\pi^4}{20}-\frac{\pi^6}{21}\right)\epsilon^7+\mathcal{O}(\epsilon^8)$$

Summary of 1LR case:

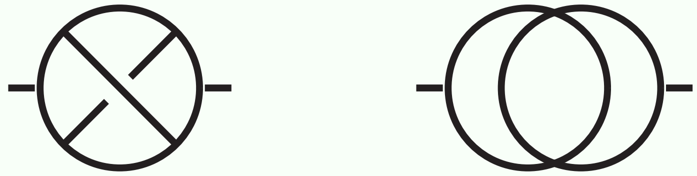
Hatted repesentations of *all* normal (that is SZV) odd zetas can be found from just expanding deeply in ϵ $G(1, 1+\epsilon)$ for arbitrary large number of loops (was proved in general by Kotikov and

Thus, if we just assume that the $L = 4$ case is described completely by SVZ's (which is true!) then the corresponding hatted representation

$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6, \;\; \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2}\zeta_6 \quad \text{and} \quad \hat{\zeta}_7 := \zeta_7.$$

can be derived just from the properties of $G(1, 1 + a\,\epsilon)$!
But no MZV's ever appear! (because they are absent in the normal Γ-function expanded around integer values of its argument)

Next step: consider 3-loop case: then MZV's do show up in corresponding two 1LI masters:



but the resulting 2 eqs. for every π dependent term do fix the hatted form the only MZV, namely $\zeta(5,3)$ appearing at 5-loops (or, equivalently, φ) in full agreement to the result of /Georgoudis, Goncalves, Panzer, Pereira, [1802.00803]/ as it should be (as we take into account is a small, essentially trivial, subset of around 150 higly nontrivial 5-loop masters considered there).

Lesson: higher orders in ϵ of 3-loop masters do "know" everything about π-structure of 4- and 5-loop masters!

What about 4-loop case? Luckily, many orders in ϵ are known from
R. N. Lee, A. V. Smirnov and V. A. Smirnov, *Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve, Nucl. Phys.* B856 (2012) 95–110,

Hatted form for the 6-loop case /transcendental level ≤ 11/

$$\underbrace{\hat{\zeta}_3 := \boxed{\zeta_3} + \frac{3\epsilon}{2}\zeta_4}_{L=3} \quad \underbrace{-\frac{5\epsilon^3}{2}\zeta_6}_{\delta(L=4)} \quad \underbrace{+\frac{21\epsilon^5}{2}\zeta_8}_{\delta(L=5)} \quad \underbrace{-\frac{153\epsilon^7}{2}\zeta_{10}}_{\delta(L=6)}, \tag{3}$$

$$\underbrace{\hat{\zeta}_5 := \boxed{\zeta_5} + \frac{5\epsilon}{2}\zeta_6}_{(L=4)} \quad \underbrace{-\frac{35\epsilon^3}{4}\zeta_8}_{\delta(L=5)} \quad \underbrace{+63\epsilon^5\zeta_{10}}_{\delta(L=6)}, \tag{4}$$

$$\underbrace{\hat{\zeta}_7 := \boxed{\zeta_7}}_{L=4} \quad \underbrace{+\frac{7\epsilon}{2}\zeta_8}_{\delta(L=5)} \quad \underbrace{-21\epsilon^3\zeta_{10}}_{\delta(L=6)}, \tag{5}$$

$$\underbrace{\hat{\varphi} := \boxed{\varphi} - 3\epsilon\,\zeta_4\,\zeta_5 + \frac{5\epsilon}{2}\zeta_3\,\zeta_6}_{L=5} \quad \underbrace{-\frac{24\,\epsilon^2}{47}\zeta_{10} + \epsilon^3\left(-\frac{35}{4}\zeta_3\zeta_8 + 5\zeta_5\zeta_6\right)}_{\delta(L=6)}, \tag{6}$$

$$\underbrace{\hat{\zeta}_9 := \boxed{\zeta_9}}_{L=5} \quad \underbrace{+\frac{9}{2}\epsilon\,\zeta_{10}}_{\delta(L=6)}, \tag{7}$$

$$\underbrace{\hat{\zeta}_{7,3} := \boxed{\zeta_{7,3} - \frac{793}{94}\zeta_{10}} + 3\epsilon(-7\zeta_4\zeta_7 - 5\zeta_5\zeta_6)}_{L=6}, \tag{8}$$

$$\underbrace{\hat{\zeta}_{11} := \boxed{\zeta_{11}}}_{L=6}, \tag{9}$$

$$\underbrace{\hat{\zeta}_{5,3,3} := \boxed{\zeta_{5,3,3} + 45\zeta_2\zeta_9 + 3\zeta_4\zeta_7 - \frac{5}{2}\zeta_5\zeta_6}}_{L=6}. \tag{10}$$

The boxed terms are in agreement with the results of F. Brown, D. Broadhurst, D. Kreimer, E, Panzer, O. Schnetz ...

Now we can upgrade our formulas for π-dependent terms in AD's and β-functions at the next 7-loop level!

$$\beta_7^{\zeta_4} = \frac{3}{8}\,\beta_4^{\zeta_3}\beta_3^{(1)} + \frac{9}{10}\,\beta_2\beta_5^{\zeta_3} - \frac{1}{2}\,\beta_3^{\zeta_3}\beta_4^{(1)} + \frac{5}{4}\,\beta_1\beta_6^{\zeta_3},$$

$$\beta_7^{\zeta_6} = \frac{5}{8}\,\beta_4^{\zeta_5}\beta_3^{(1)} + \frac{3}{2}\,\beta_2\beta_5^{\zeta_5} + \frac{25}{12}\,\beta_1\beta_6^{\zeta_5} - 2\beta_1^2\beta_3^{\zeta_3}\beta_2 - \frac{5}{4}\,\beta_1^3\beta_4^{\zeta_3},$$

$$\beta_7^{\zeta_3\zeta_4} = \frac{9}{5}\,\beta_2\beta_5^{\zeta_3^2} - \frac{1}{8}\,\beta_3^{\zeta_3}\beta_4^{\zeta_3} + \frac{5}{2}\,\beta_1\beta_6^{\zeta_3^2},$$

$$\beta_7^{\zeta_8} = \frac{21}{10}\,\beta_2\beta_5^{\zeta_7} + \frac{35}{12}\,\beta_1\beta_6^{\zeta_7} - \frac{7}{24}\,\beta_1(\beta_3^{\zeta_3})^2 + \frac{7}{4}\,\beta_1^2\beta_5^{\zeta_3^2} - \frac{35}{8}\,\beta_1^3\beta_4^{\zeta_5},$$

$$\beta_7^{\zeta_3\zeta_6} = \frac{5}{8}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\zeta_3\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\phi},$$

$$\beta_7^{\zeta_4\zeta_5} = -\frac{1}{2}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{5}{4}\beta_1\beta_6^{\zeta_3\zeta_5} - \frac{5}{2}\beta_1\beta_6^{\phi},$$

$$\beta_7^{\zeta_{10}} = \frac{15}{4}\beta_1\beta_6^{\zeta_9},$$

$$\beta_7^{\zeta_4\zeta_3^2} = \frac{15}{4}\beta_1\beta_6^{\zeta_3^3},$$

$$\beta_7^{\zeta_4\zeta_7} = \beta_7^{\zeta_5\zeta_6} = \beta_7^{\zeta_3\zeta_8} = 0.$$

Tests of pur predictions for AD's at L=7 loop: I

We have checked that the π-dependent contributions to the terms of order $n_f^6\alpha_s^7$ in the the QCD β-function as well as to the terms of order $n_f^6\alpha_s^7$ and of order $n_f^5\alpha_s^7$ contributing to the quark mass AD, all computed in

J. Gracey, *The QCD Beta function at $\mathcal{O}(1/N_f)$*, *Phys.Lett.* **B373** (1996) 178–184, [hep-ph/9602214].

M. Ciuchini, S. E. Derkachov, J. Gracey and A. Manashov, *Quark mass anomalous dimension at O(1/N(f)**2) in QCD*, *Phys.Lett.* **B458** (1999) 117–126, [hep-ph/9903410].

M. Ciuchini, S. E. Derkachov, J. Gracey and A. Manashov, *Computation of quark mass anomalous dimension at O(1 / N**2(f)) in quantum chromodynamics*, *Nucl.Phys.* **B579** (2000) 56–100, [hep-ph/9912221].

are in agreement with our predictions

Tests of pur predictions for AD's at L=7 loop, cont-ed

Significantly more complicated test is provided by the recent calculation of the full 7-loop RG functions in the φ^4-model

O. Schnetz, ***Numbers and Functions in Quantum Field Theory, Phys. Rev.*** **D97 (2018) 085018, [1606.08598]**

We have reproduced successfully all π-dependent constants appearing in the β-function and anomalous dimensions γ_m and γ_2 of the $O(n)$ φ^4 at 7 loops

Oliver Schnetz, PRD 97 (2018): 7! loop result for ϕ^4 RG functions:

$$\beta = \Big(\frac{195654269}{23040} + \frac{15676169}{720}\zeta(3) - \frac{316009}{3840}\pi^4 \frac{18326039}{480}\zeta(5) - \frac{129631}{5040}\pi^6$$
$$+ \frac{516957}{20}\zeta(3)^2 - \frac{4453}{60}\pi^4\zeta(3) + \frac{1536173}{20}\zeta(7) - \frac{20425591}{1260000}\pi^8$$
$$+ 116973\zeta(3)\zeta(5) + \frac{947214}{25}\zeta(5,3) - \frac{1010}{63}\pi^6\zeta(3) + \frac{613}{5}\pi^4\zeta(5) + 4176\zeta(3)^3$$
$$+ \frac{547118}{3}\zeta(9) - \frac{45106}{43659}\pi^{10} - 48\pi^4\zeta(3)^2 + \frac{84231}{2}\zeta(3)\zeta(7) - \frac{273030}{7}\zeta(5)^2$$
$$+ \frac{8460}{7}\zeta(7,3) - \frac{174}{25}\pi^8\zeta(3) + \frac{6227}{35}\pi^6\zeta(5) - \frac{56043}{25}\pi^4\zeta(7)$$
$$- 504387\pi^2\zeta(9) + 46845\zeta(3)^2\zeta(5) + 27216\zeta(3)\zeta(5,3) - \frac{336258}{5}\zeta(5,3,3)$$
$$+ \frac{52756839}{10}\zeta(11) + 24P_{7,11}\Big)\, g^8 + \ldots$$

All π-dependent terms follow from $\beta/.\pi \to 0$: first (partial) check of both the 7-loop β for the ϕ^4-model and on the hatted representation of $\mathcal{P}_6$. The same is true for γ_m, γ_2 and the 6-loop self-energy

Conclusions

- **all π-dependent terms in a generic (L+1)-loop $\overline{\text{MS}}-$ (or, equivalently, G-) anomalous dimension γ are completely fixed by π-independent contributions to γ (and corresponding β) with loop number L or less *provided* the (all) L-loop p-master integrals are π-safe**

- **The π-safeness holds for L=4 and L=5 and, probably, for L=6. It is known that for L=7 the property (partially) stops to be valid★ and, thus, our predictions should be modified (at astronomically large for QCD level of L=8 RG functions)**

- **All available results at 5 (QCD), and 6 and 7 loops (large n_f QCD and the ϕ^4-model) do meet all our constraints**

★ communicated to us by Oliver Schnetz

(the problem is an appearence of the ζ_{12} as indepenent term of some 7-loop finite p-integral, see works by (F.Brown, O.Schnetz, E.Panzer . . . on Feynman periods)