On the background field method and quantum equation of motion

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Basic concepts of ϕ^3 theory

Lagrange function: $$\mathscr{L}[\phi](x) = \frac{1}{2}\partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x) - \frac{g}{6}\phi^3(x)$$

ϕ is a scalar field,

m is a mass, g is a coupling constant, $x \in \mathbb{R}^n$

Action: $$S[\phi] = \int_{\mathbb{R}^n} d^n x \, \mathscr{L}[\phi](x)$$

A.J. Macfarlane, G. Woo (1974)
J.C. Collins, DAMPT preprint 73/38
J.A. Gracey (2015)

Effective action: $$e^{-W} = \int_H \mathscr{D}\phi \, e^{-S[\phi]}$$

H is a functional set

Background field method

L.F. Abbott (1982)
I.Y. Aref'eva, A.A. Slavnov, L.D Faddeev (1974)
L.D. Faddeev, A.A. Slavnov (1980)

Background field: $B(x)$

Shift: $\phi \longrightarrow \phi + B$ $\qquad H \longrightarrow H_0$

ϕ decreases at infinity → **Crucial property:**

$$S[\phi + B] = S[B] + (M, \phi) + \frac{1}{2}(N\phi, \phi) - \frac{g}{6}\int_{\mathbb{R}^n} d^n x \phi^3(x)$$

$$N(x) = -\partial_\mu \partial^\mu - m^2 - gB(x)$$

$$M(x) = -\partial_\mu \partial^\mu B(x) - m^2 B(x) - \frac{g}{2}B^2(x)$$

$$e^{-W} = \int_H \mathscr{D}\phi e^{-S[\phi]}$$

→

$$e^{-W[B]} = e^{-S[B]} \int_{H_0} \mathscr{D}\phi e^{-(M,\phi) - \frac{1}{2}(N\phi,\phi) + \frac{g}{6}\int_{\mathbb{R}}^{n} d^n x \phi^3(x)}$$

Background field method

L.D. Faddeev (2009)
arXiv:0911.1013

One more shift: $\phi \longrightarrow \phi + G_n \eta$

G_n is integration operator with kernel $G_n(x, y)$

Green function: $N(x) G_n(x, y) = \delta(x, y)$

η is a smooth function

$$\int_{H_0} \mathscr{D}\phi \, e^{-(M,\phi)-\frac{1}{2}(N\phi,\phi)+\frac{g}{6}\int_{\mathbb{R}^n} d^n x\, \phi^3(x)} = \det(N)^{-1/2} e^{-(M,\frac{\delta}{\delta\eta})+\frac{g}{6}\int_{\mathbb{R}^n} d^n x \left(\frac{\delta}{\delta\eta(x)}\right)^3} e^{\frac{1}{2}(G_n\eta,\eta)}\Bigg|_{\eta=0}$$

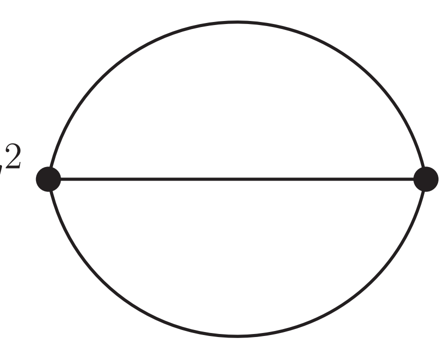
$$W[B] = S[B] + \tfrac{1}{2}\ln\det N - \tfrac{1}{12} g^2$$

$$+ \, O(g^4).$$

Diagram technique

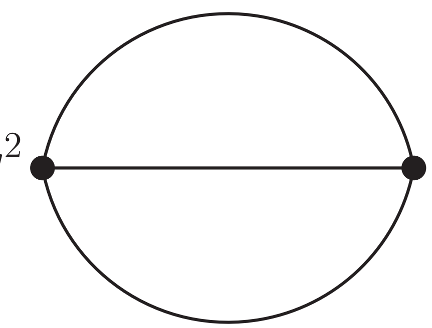
Rules:

$G_n(x, y) \leftrightarrow$ line

$integration \leftrightarrow$ dot

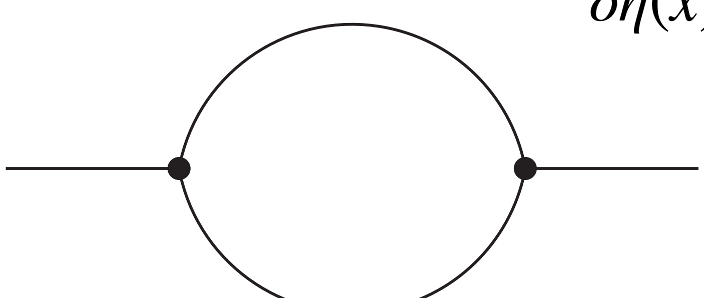
- $\rho(g, B)$ equals to one-particle irreducible (1PI) part of $e^{\frac{g}{6}\int_{\mathbb{R}^n} d^n x \left(\frac{\delta}{\delta\eta(x)}\right)^3} e^{\frac{1}{2}(G_n\eta,\eta)}\Big|_{\eta=0}$

  sum of 1PI vacuum diagrams

$\rho(g, B) = 1 + \frac{1}{12} g^2$  $+ O(g^4).$

- Extended Green function $\mathscr{G}_n(x, y)$: sum of all 1PI contributions

$$\frac{\delta}{\delta\eta(x)} \frac{\delta}{\delta\eta(y)} e^{\frac{g}{6}\int_{\mathbb{R}^n} d^n x \left(\frac{\delta}{\delta\eta(x)}\right)^3} e^{\frac{1}{2}(G_n\eta,\eta)}\Big|_{\eta=0}$$

$\mathcal{G}(x, y) = x$ $y + \frac{1}{4} g^2 x$  $y + O(g^4).$

Quantum equation of motion

Lemma 1

The 1PI coefficient for $(G_n M, M)$ in the $W[B]$ equals to $\frac{1}{2}\rho(g, B)$.

Lemma 2

The contribution to the $W[B]$, which is proportional to M and represented as product of 1PI diagrams without consideration of M-vertex, equals to $-\frac{g}{2}\int_{\mathbb{R}^n} d^n x\, G_n M(x) \mathcal{G}_n(x, x)$.

$$2\,\rho\,(g, B)\, M(x) = g\, \mathcal{G}_n(x, x)$$

Quantum equation of motion

$$\rho(g,B)\,\frac{\delta}{\delta B(x)}W[B] = -\,\rho(g,B)\,M(x) + \frac{g}{2}\,\mathcal{G}_n(x,x)$$

Regularization

Rules:

The factor r^{-k} with $k \in \mathbb{N}$ $\longrightarrow$ $\chi_{r\Lambda>1} r^{-k}$;

The factor $\ln r$ $\longrightarrow$ $\chi_{r\Lambda>1} \ln r - \chi_{r\Lambda\leqslant 1} \ln \Lambda$,

where $\chi_{(a,b)}$ is a characteristic function of (a,b);

and Λ is a parameter of regularization.

$$\Lambda \longrightarrow +\infty \, , G_n^{\Lambda} \longrightarrow G_n$$

Renormalization

n=6

Renormalizable case ⟶ infinite number of divergencies ⟶ find renormalization constants

$$\phi \to \sqrt{Z}\phi, \quad g \to Z_0 Z^{-\frac{3}{2}} g, \quad m^2 \to Z_m Z^{-1} m^2$$

$$Z_0(g) = 1 + a_{12} g^2 \ln L + a_{14} g^4 \ln L + a_{24} g^4 \ln^2 L + o(g^4)$$

$$Z_m(g) = 1 + b_{12} g^2 \ln L + b_{14} g^4 \ln L + b_{24} g^4 \ln^2 L + o(g^4)$$

$$Z(g) = 1 + c_{12} g^2 \ln L + c_{14} g^4 \ln L + c_{24} g^4 \ln^2 L + o(g^4)$$

⟶ find the coefficients

where $L = \dfrac{\Lambda}{\mu}$

$$c_{12} = \frac{1}{6(4\pi)^3}, \quad b_{12} = \frac{1}{(4\pi)^3}, \quad a_{12} = \frac{1}{(4\pi)^3}$$

$$c_{14} = \frac{11}{36(4\pi)^6}, \quad b_{14} = \frac{3}{2(4\pi)^6}, \quad a_{14} = \frac{3}{2(4\pi)^6}$$

$$c_{24} = \frac{5}{36(4\pi)^6}, \quad b_{24} = \frac{5}{4(4\pi)^6}, \quad a_{24} = \frac{5}{4(4\pi)^6}$$

Renormalization

n=6

$$\frac{g\Lambda^2}{2(4\pi)^3}\int_{\mathbb{R}^6} d^6x \, \frac{\delta}{\delta B(x)} \ln\det(N) + \frac{g^2\Lambda^2}{2(4\pi)^6}\int_{\mathbb{R}^6} d^6x \, a_2(x,x)$$

↓

quantum equation of motion $\longleftrightarrow \frac{\delta}{\delta B(x)} \ln\det N \sim -gB^2(x)$

$$-\frac{g^2\Lambda^2}{(4\pi)^3}\left(1-\frac{g^2}{(4\pi)^3}\right)\frac{(B,B)}{2}$$

- has a different nature
- can be eliminated by renormalization of the mass parameter

Renormalization

n=5, n=4

Superrenormalizable case $\longrightarrow$ finite number of divergencies

shift of the mass parameter: $$m^2 \longrightarrow m^2 - \frac{g^4}{12(4\pi)^4} \ln \Lambda \qquad \text{n=5}$$

$$m^2 \longrightarrow m^2 - \frac{g^2}{(4\pi)^2} \ln \Lambda \qquad \text{n=4}$$

Many thanks!