Description of gravity in the model with independent nonsymmetric connection

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Introduction

General Relativity - current description of gravity in modern physics

Consider modified gravity:

- presence of torsion: $S^{\xi}_{\mu\nu} = \Gamma^{\xi}_{\mu\nu} - \Gamma^{\xi}_{\nu\mu} \neq 0$;
- metric and connection are independent variables (Palatini Formalism): $\Gamma^{\alpha}_{\rho\nu} \neq \frac{1}{2} g^{\alpha\sigma}(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho})$.

the geometry of the space-time $\longrightarrow$ electromagnetic potential

Gravity field equations without matter

The Einstein-Hilbert action:

$$S_1 = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}R, \qquad (1)$$

where κ - Einstein's constant, R - the scalar curvature, $g \equiv \det g_{\mu\nu}$.
The minimal action principle gives:

$$\begin{cases} R^{\nu\mu} + R^{\mu\nu} - Rg^{\mu\nu} = 0; \\ D_\rho g^{\sigma\nu} = \frac{1}{3}g^{\sigma\nu}S^{\alpha}_{\rho\alpha} + \frac{1}{3}g^{\sigma\xi}S^{\alpha}_{\xi\alpha}\delta^{\nu}_{\rho} + g^{\xi\sigma}S^{\nu}_{\rho\xi}. \end{cases} \qquad (2)$$

Gravity field equations without matter

Consider equations

$$D_\rho g^{\sigma\nu} = \frac{1}{3} g^{\sigma\nu} S^{\alpha}_{\rho\alpha} + \frac{1}{3} g^{\sigma\xi} S^{\alpha}_{\xi\alpha} \delta^{\nu}_{\rho} + g^{\xi\sigma} S^{\nu}_{\rho\xi}. \tag{3}$$

It is solved by

$$\Gamma^{\rho}_{\mu\nu} = \widehat{\Gamma}^{\rho}_{\mu\nu} + f_\mu \delta^{\rho}_{\nu} \quad \longrightarrow \quad \Gamma^{\rho}_{\mu\nu} - \frac{1}{4} \Gamma^{\alpha}_{\mu\alpha} \delta^{\rho}_{\nu} = \widehat{\Gamma}^{\rho}_{\mu\nu} - \frac{1}{4} \widehat{\Gamma}^{\alpha}_{\mu\alpha} \delta^{\rho}_{\nu}, \tag{4}$$

where $\widehat{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} (\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho})$ - Christoffel symbols,
f_μ - arbitrary vector.
The invariance of the action under transformation:

$$\Gamma^{\rho}_{\mu\nu} \longrightarrow \Gamma^{\rho}_{\mu\nu} + \theta_\mu \delta^{\rho}_{\nu}. \tag{5}$$

Addition of matter

Point-like particle

$$S_2 = -m \int d\tau \sqrt{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}}. \tag{6}$$

Interaction with the field of connection

$$S_3 = -q \int d\tau \dot{x}^\mu \Gamma^\nu_{\mu\nu} \tag{7}$$

Remark. Now the action $S_1 + S_2 + S_3$ is invariant under a narrow group of transformations:

$$\Gamma^\rho_{\mu\nu} \longrightarrow \Gamma^\rho_{\mu\nu} + \delta^\rho_\nu \partial_\mu \theta. \tag{8}$$

Addition of kinetic term

Problem: incompatible equation of motion

$$\dot{x} = 0.$$

Consequently, we need to add the kinetic term:

$$S_4 = -\frac{1}{16\pi} \int d^4x \sqrt{-g} \Omega_{\alpha\beta} \Omega_{\delta\gamma} g^{\alpha\delta} g^{\beta\gamma}, \tag{9}$$

where $\Omega_{\lambda\rho} \equiv R^{\mu}_{\mu\lambda\rho} = (\partial_\lambda \Gamma^{\xi}_{\rho\xi} - \partial_\rho \Gamma^{\xi}_{\lambda\xi})$.
The total action

$$S = S_1 + S_2 + S_3 + S_4 = -\frac{1}{2\kappa} \int d^4x \sqrt{-g} R - m \int d\tau \sqrt{\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}} - \\ - q \int d\tau \dot{x}^\mu \Gamma^{\nu}_{\mu\nu} - \frac{1}{16\pi} \int d^4x \sqrt{-g} \Omega_{\alpha\beta} \Omega_{\delta\gamma} g^{\alpha\delta} g^{\beta\gamma} \tag{10}$$

Variation

Varying with respect to the metric δg:

$$\widehat{R}^{\mu\nu} - \frac{1}{2}\widehat{R}g^{\mu\nu} = \kappa T^{\mu\nu}, \tag{11}$$

where $\widehat{R}^{\mu\nu}$ and $\widehat{R}$ - Riemannian geometry objects,

$$T^{\mu\nu} = \rho_m u^\mu u^\nu - \frac{1}{4\pi}(\Omega^{\mu\alpha}\Omega^\nu_\alpha - \frac{1}{4}g^{\mu\nu}\Omega_{\alpha\beta}\Omega^{\alpha\beta}) \tag{12}$$

stress-energy tensor,
$u^\mu = \dot{x}^\mu \frac{1}{\sqrt{\dot{x}^\alpha \dot{x}^\beta g_{\alpha\beta}}}$ -four-velocity, $\rho_m = m\int ds\delta(x - x(s))\frac{1}{\sqrt{-g(x(s))}}$- mass density.

Variation

Minimizing geodesic $x^\mu(\tau)$

$$mu^\mu \widehat{D}_\mu u^\alpha = -q u_\xi \Omega^{\xi\alpha}. \tag{13}$$

Varying with respect to the connection $\Gamma^\mu_{\nu\rho}$

$$\begin{cases} \Gamma^\rho_{\mu\nu} - \frac{1}{4}\Gamma^\alpha_{\mu\alpha}\delta^\rho_\nu = \widehat{\Gamma}^\rho_{\mu\nu} - \frac{1}{4}\widehat{\Gamma}^\alpha_{\mu\alpha}\delta^\rho_\nu, \\ \widehat{D}_\mu \Omega^{\nu\mu} = 4\pi j^\mu, \end{cases} \tag{14}$$

where $j^\mu = q \int ds u^\mu \delta(x - x(s)) \frac{1}{\sqrt{-g(x(s))}}$.

Gravity field equations with matter

Finally we arrived at

$$\begin{cases} \widehat{R}^{\mu\nu} - \frac{1}{2}\widehat{R}g^{\mu\nu} = \kappa T^{\mu\nu}; \\ mu^{\mu}\widehat{D}_{\mu}u^{\alpha} = -qu_{\xi}\Omega^{\xi\alpha}; \\ \Gamma^{\rho}_{\mu\nu} - \frac{1}{4}\Gamma^{\alpha}_{\mu\alpha}\delta^{\rho}_{\nu} = \widehat{\Gamma}^{\rho}_{\mu\nu} - \frac{1}{4}\widehat{\Gamma}^{\alpha}_{\mu\alpha}\delta^{\rho}_{\nu}; \\ \widehat{D}_{\mu}\Omega^{\nu\mu} = 4\pi j^{\mu}, \end{cases} \tag{15}$$

where $\Omega_{\lambda\rho} \equiv R^{\mu}_{\mu\lambda\rho} = (\partial_{\lambda}\Gamma^{\xi}_{\rho\xi} - \partial_{\rho}\Gamma^{\xi}_{\lambda\xi})$.

If q - charge of the particle, and $\Gamma^{\xi}_{\rho\xi} \equiv A_{\rho}$ - the potential of the electromagnetic field, thus we constructed electrodynamics in a gravitational field.

Historical remark

- Hermann Weyl (1918): gauge invariance $g'_{\mu\nu}(x) = e^{2\lambda(x)} g_{\mu\nu}(x)$;
- Arthur Eddington (1921): the natural gauge of the world $R_{\alpha\beta} + R_{\beta\alpha} = \Lambda g_{\alpha\beta}$
- Schouten, Friedmann (1923-24): presence of torsion: $\Gamma^{\xi}_{\mu\nu} - \Gamma^{\xi}_{\nu\mu} \neq 0$.

Historical remark

Albert Einstein (1917 - 1925): $\Gamma^{\xi}_{\mu\nu} - \Gamma^{\xi}_{\nu\mu} \neq 0;$

$$g_{\mu\nu} - g_{\nu\mu} \neq 0;$$

$$\Gamma^{\alpha}_{\rho\nu} \neq \frac{1}{2} g^{\alpha\sigma}(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}).$$

Obuhov, Krechet (1978): $\Gamma^{\xi}_{\mu\nu} - \Gamma^{\xi}_{\nu\mu} \neq 0;$

$$\Gamma^{\alpha}_{\rho\nu} \neq \frac{1}{2} g^{\alpha\sigma}(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}).$$

Results

- In the model of gravity with independent nonsymmetric connection with presence of matter - point-like particle, we get a unified theory of gravity and electromagnetism.