Specifics of renormalization for the quantum Yang – Mills theory in the four dimensional space – time

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Yang–Mills theory

The basic concepts of this work

Assume, G is a compact group of charges, $\mathfrak{G}$ is the Lie algebra.
Assume t^a are the generators of the lie algebra.
$\mathrm{tr}[\cdot,\cdot]$ is the Killing form.

$$A(x) = A^a_\mu(x) t^a dx^\mu$$

$$F = dA + A \wedge A$$

The classical action of the theory of Yang — Mills

$$S = \frac{1}{4g^2} \int \mathrm{tr}\, F \wedge F^*,$$

where $g = \frac{\sqrt{\alpha}}{2}$, a α is the coupling constant.

Effective action

The differential operators defining the quadratic form

$$M_1 = \nabla_\sigma^2 \delta_{\mu\nu} + 2[F_{\mu\nu}, \cdot\,]\,, \qquad M_0 = \nabla_\mu^2.$$

The action unfolds in a series of views

$$W(B,\alpha) = \frac{1}{\alpha} W_{-1}(B) + \sum_{k=0}^{\infty} \alpha^k W_k(B),$$

where

$$W_{-1}(B) = \int \mathrm{tr} F_{\mu\nu}^2 d^4x \qquad W_0(B) = -\tfrac{1}{2} \ln\det M_1 + \ln\det M_0$$

and W_n , $n = 1, 2...$ are defined as the contribution of strongly connected vacuum diagrams with $n+1$ loops, constructed via Green functions.

The first correction

The functional $W_{1loop}(B)$ can be defined via the proper time method of Fock giving formula

$$W_{1loop}(B) = \int\limits_0^\infty \frac{ds}{s}\left(T_0(B) + T_1(B,s)\right),$$

where

$$T_0(B) = \tfrac{1}{2}\beta_1 W_{-1} \qquad \lim_{s\to 0} T_1(s) = 0$$

Then

$$W_{1loop}^{reg}(B) = \int\limits_0^{1/\Lambda^2} \frac{ds}{s} T_1(B,s) + \int\limits_{1/\Lambda^2}^{\infty} \frac{ds}{s}(T_0(B) + T_1(B,s))$$

So the only divergence is proportional to the classical action and can be compensated by the renormalization of the coupling constant α.

$$W^{reg}_{1loop}(B,\Lambda) = W_{0,0}(B,\mu) + \beta_1 \log\left(\frac{\Lambda}{\mu}\right) W_{-1}$$

Input dependence of the coupling constant α on Λ as

$$\frac{1}{\alpha(\Lambda)} = -\beta_1 \log\left(\frac{\Lambda}{m}\right)$$

and define the renormalized running coupling constant as

$$\frac{1}{\alpha(\mu)} = -\beta_1 \log\left(\frac{\mu}{m}\right)$$

so that

$$W^{reg}_{1loop}(B) = \frac{1}{\alpha(\mu)} W_{-1} + W_{00}$$

This function $\alpha(\cdot)$ satisfies the first approximation to Gell-Mann Low equation

$$x\frac{d}{dx}\alpha(x) = \beta_1 \alpha^2(x)$$

Action before regularization

$$W(\alpha) = \frac{1}{\alpha} W_{-1} + \sum_{k=0}^{\infty} \alpha^k W_k$$

Regularization and renormalization

$$W(\alpha) \quad \xrightarrow{reg} \quad W_{reg}(\alpha, \Lambda, \mu) \quad \xrightarrow{ren} \quad W_{reg}(\alpha(\Lambda), \Lambda, \mu) = W_{ren}(\alpha_r(\mu), \mu)$$

The structure of the action after regularization

$$W_0 = W_{0,0} + L W_{0,1}$$
$$W_i = \sum_{k=0}^{i} L^k W_{ik} \quad i \geqslant 1$$

where $L = \log \frac{\Lambda}{\mu}$.

Problem

$$W_{reg}(\alpha(\Lambda), \Lambda, \mu) = \frac{1}{\alpha(\Lambda)} W_{-1} + L W_{0,1} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} W_{n,k} \alpha^n(\Lambda) L^k,$$

$$\Lambda \frac{d\alpha(\Lambda)}{d\Lambda} = \beta(\alpha(\Lambda)),$$

$$\frac{dW_{reg}(\alpha(\Lambda), \Lambda, \mu)}{d\Lambda} = 0,$$

where

$$\beta(x) = \sum_{i=0}^{\infty} \beta_{i+1} x^{i+2}$$

We want to find the recurrent relations on the coefficients and find the function $\alpha(\Lambda)$.

Results

The new coupling constant has the form

$$\alpha_r(\mu) = e^{-L\beta(\alpha)\partial_\alpha}\alpha\Big|_{\alpha=\alpha(\Lambda)},$$

The renormalized action has the form

$$W_{ren}(\alpha_r(\mu),\mu) = e^{-L\beta(\alpha)\partial_\alpha}\left(\sum_{n=-1}^{\infty} W_{n,0}\alpha^n\right)\Bigg|_{\alpha=\alpha(\Lambda)}$$

The solution of recurrent relations has the form

$$W_{0,1} = W_{-1}\beta_1 \,,$$

$$W_{m,n} = W_{-1}\Xi_{n,-1,m-n+1} + \sum_{i=0}^{m-n} W_{i,0}\Xi_{n,i,m-n-i} \;,\; m \geqslant 1\,,\; n = 1\ldots m\,,$$

where

$$\Xi_{n,k,i} = \frac{k(-1)^n}{n!} \sum_{j_{n-1}=0}^{i} \ldots \sum_{j_1=0}^{j_2} \beta_{j_1+1}\beta_{j_2-j_1+1}\ldots\beta_{i-j_{n-1}+1} \cdot$$
$$\cdot (j_1 + 2 + (k-1))\ldots(j_{n-1} + 2 + (n-2) + (k-1))$$

Green function

The fundamental solution for the operator $A_0 = -\partial_\mu \partial_\mu$ satisfies the equation

$$A_0 G(x, y) = \delta(x - y).$$

BUT according to the method of the heat kernel we can find a function $K_0(x, y; \tau)$ that satisfies the problem

$$\begin{cases} \left(\frac{\partial}{\partial \tau} + A_0\right) K_0(x, y; \tau) = 0 \\ K_0(x, y; 0) = \delta(x - y), \end{cases}$$

Then we have

$$G_0(x, y) = \int_0^\infty d\tau K_0(x, y; \tau).$$

Determinant of operator

Suppose that we have two "good"operators: A_0 and A_1. Using these operators, we can construct heat kernels

$$\begin{array}{ccc} A_0 & \longrightarrow & K_0(x,y;\tau) \\ A_1 & \longrightarrow & K_1(x,y;\tau). \end{array}$$

In this case we can write formula

$$\ln\det A_1/A_0 = -\int\limits_0^\infty \frac{d\tau}{\tau} \mathrm{tr}\left(e^{-A_1\tau} - e^{-A_0\tau}\right)$$

GOOD!

But how to build a heat kernal???

System of differential equations

Suppose that $B_\mu(x) \in \mathfrak{G}$ for $\forall x$. Then we define the operator $A = D_\mu D_\mu$, where $D_\mu = \partial_\mu + B_\mu$. In this case we can find our solution in the form

$$K(x,y;\tau) = K_0(x,y;\tau) \sum_{n=0}^{\infty} \tau^n a_n(x,y),$$

where $K_0(x,y;\tau)$ is heat kernel for Laplace operator A_0.
The relations have the form:

$$(x-y)^\mu D_\mu a_0(x,y) = 0$$
$$((n+1) + (x-y)^\mu D_\mu)\, a_{n+1}(x,y) = A a_n(x,y)$$

But how to calculate an arbitrary order???

The first equation

The path-ordered exponential

$$\Phi(x,y) := 1 + \sum_{n=1}^{\infty}(-1)^n \int_0^1 \int_0^1 \ldots \int_0^1 ds_1 \ldots ds_n \frac{dz_1^{\nu_1}}{ds_1} \ldots \frac{dz_n^{\nu_n}}{ds_n} B_{\nu_1}(z_1) \ldots B_{\nu_n}(z_n)$$

satisfies the integral equation

$$\Phi(x,y) = 1 + \int_0^1 ds \frac{dz^{\nu}(s)}{ds} B_{\nu}(z(s))\, \Phi(z(s),y)$$

and the differential equation with the boundary condition

$$(x-y)^{\mu} D_{\mu}\Phi(x,y) = 0\ , \ \ \Phi(x,x) = 1$$

The classical solution for the problem

$$\begin{cases} \big((n+1)+(x-y)^{\mu}\nabla_{\mu}\big)\, a_{n+1}(x,y) = Aa_n(x,y) \\ W_{n+1}(x,y) \in L_{\infty}(\mathbb{R}^d). \end{cases}$$

has the form

$$a_{n+1}(x,y) = \int\limits_0^1 dss^n \Phi(x,z(s))Aa_n(z(s),y)$$

BUT how to calculate?

CAN we write an arbitrary order?

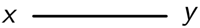
The basic properties of path-ordered exponential:

- $\Phi(x,x)=1$
- $\Phi^{-1}(x,y)=\Phi(y,x)$
- $\Phi(x,z)\Phi(z,y)=\Phi(x,y)$ if $x\in\{(1-s)y+sx, s\in\mathbb{R}\}$
- $D_\mu\Phi(x,y)=\int\limits_0^1 dss\frac{dz_\nu}{ds}\Phi(x,z)F_{\nu,\mu}(z)\Phi(z,y)$
- $\partial_{y_\mu}\Phi(x,y)=\Phi(x,y)B_\mu(y)+\int\limits_0^1 ds(1-s)\frac{dz_\nu}{ds}\Phi(x,z)F_{\nu,\mu}(z)\Phi(z,y)$
- $\int\limits_0^1 dss^p\Phi(x,z(s))f(z(s))\xrightarrow{y\longrightarrow x} f(x)\int\limits_0^1 dss^p=f(x)\frac{1}{p+1}$

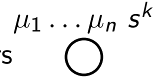
Let's use these properties to build a diagram technique!!!

Definitions

Definition 1

The function $\Phi(x,y)$ corresponds to 

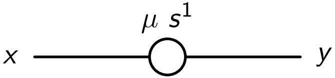
Definition 2

Circle contains two parameters 

:

- The set $\mu_1 \dots \mu_n$ corresponds to $\nabla_1 \dots \nabla_{n-1}(d(z-y)^{\rho} F_{\rho,\mu_n}(z))$, where the variable z is differentiable.
- The factor s^k corresponds to a parametrization parameter in power "k".

Example

$$\int\limits_y^x dz_\nu s \Phi(x,z) F_{\nu,\mu}(z) \Phi(z,y) =$$ 

Definitions

Definition 3

The construction

$$\int_0^1 ds s^n \, x \;\text{———}\; z(s)\cdot z(s) \;\text{———}\; (\forall\text{diagram})\,,$$

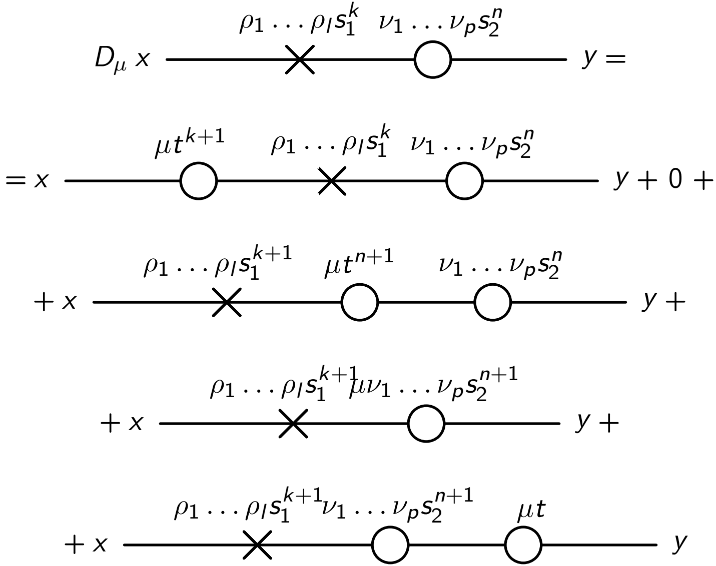
where $z_\nu = (1-s)y_\nu + sx_\nu$ corresponds to

$$x \;\text{——}\overset{s^{n+1}}{\times}\text{——}\; (\forall\text{diagram})$$

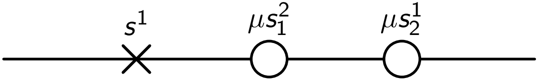
Example

$$\int_0^1 ds \Phi(x, z(s))\Phi(z(s), y) \; = x \;\text{——}\overset{s^{1}}{\times}\text{——}\; y$$

Example


$D_\mu\, x$
$\rho_1 \dots \rho_l s_1^k$
$\nu_1 \dots \nu_p s_2^n$
$y =$
$= x$
μt^{k+1}
$\rho_1 \dots \rho_l s_1^k$
$\nu_1 \dots \nu_p s_2^n$
$y + 0 +$
$+ x$
$\rho_1 \dots \rho_l s_1^{k+1}$
μt^{n+1}
$\nu_1 \dots \nu_p s_2^n$
$y +$
$+ x$
$\rho_1 \dots \rho_l s_1^{k+1} \mu \nu_1 \dots \nu_p s_2^{n+1}$
$y +$
$+ x$
$\rho_1 \dots \rho_l s_1^{k+1} \nu_1 \dots \nu_p s_2^{n+1}$
μt
y

Standart examples

$a_1(x,y) = 2\ x$  $y\ +$

$+\ x$  y

Hence we have $a_1(x,x) = 0$

$a_2(x,y) = 4\ x$  y

$+\ldots$

Hence we have $a_2(x,x) = \frac{1}{12} F_{\rho,\mu}(x) F_{\rho,\mu}(x) + (trace = 0)$