The XXII International Workshop High Energy Physics and Quantum Field Theory, Samara, June 24-July 1 2015

Theory of the Lamb shift in muonic helium ions

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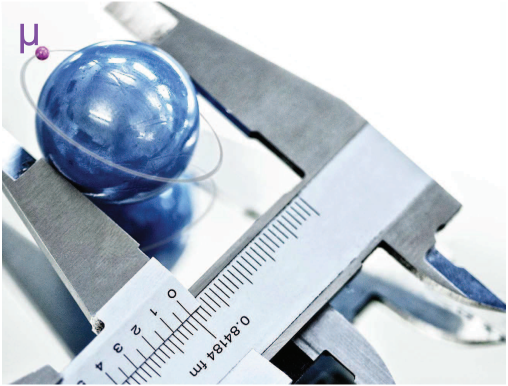
26 June 2015

Outline

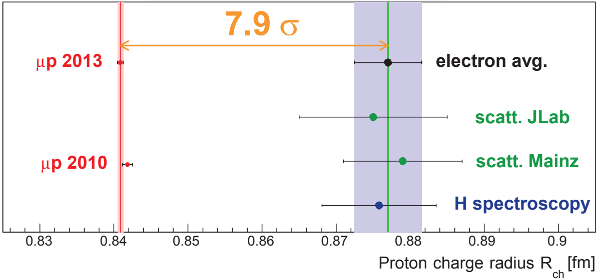
CREMA(Charge Radius Experiment with Muonic Atoms) collaboration 2010-2015

Task: to measure fine and hyperfine structure in light muonic atoms (muonic hydrogen, muonic deuterium, ions of muonic helium...); to determine charge radii of the proton, deuteron, helion, alpha-particle with the accuracy 0.0005 fm.



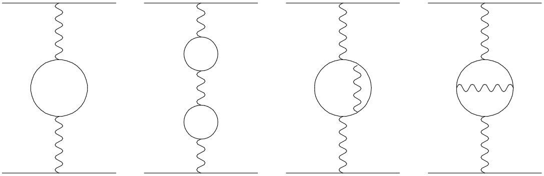
The proton radius puzzle

The proton rms charge radius measured with
electrons: 0.8770 ± 0.0045 fm
muons: 0.8409 ± 0.0004 fm



If the proton radius puzzle is caused by muon-electron universality breakdown $(\mu He)^+$ spectroscopy can reveal it! The transitions in $(\mu^4He_2)^+$ and $(\mu^3He_2)^+$ are planned to measure with $\lambda \in [800, 1000]$ nm

One- and two-loop VP corrections in 1γ interaction



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$$H_B = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r} - \frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1^2} + \frac{\delta_l}{m_2^2}\right)\delta(\mathbf{r}) -$$

$$-\frac{Z\alpha}{2m_1 m_2 r}\left(\mathbf{p}^2 + \frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2}\right) + \frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2} + \frac{1}{2m_1 m_2}\right)(\mathbf{L}\boldsymbol{\sigma}_1).$$

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-\frac{Wr}{2}}\left(1 - \frac{Wr}{2}\right), \psi_{2lm}(r) = \frac{W^{3/2}}{2\sqrt{6}} e^{-\frac{Wr}{2}} W r Y_{lm}(\theta,\phi).$$

$$W = \mu Z\alpha.$$

One-loop VP correction to the Lamb shift in 1γ interaction

$$V^C_{VP}(r) = \frac{\alpha}{3\pi}\int_1^\infty d\xi\rho(\xi)\left(-\frac{Z\alpha}{r}e^{-2m_e\xi r}\right),\ \rho(\xi) = \frac{\sqrt{\xi^2-1}(2\xi^2+1)}{\xi^4},$$

$$\Delta E_{VP}(2S) = -\frac{\mu(Z\alpha)^2\alpha}{6\pi}\int_1^\infty \rho(\xi)d\xi\int_0^\infty xdx\left(1-\frac{x}{2}\right)^2 e^{-x\left(1+\frac{2m_e\xi}{W}\right)} =$$

$$\frac{1}{12\left(1-k_1^2\right)^{5/2}}\left[\sqrt{1-k_1^2}\left(-168k_1^6+272k_1^4-49k_1^2+6\pi\left(k_1^2-1\right)^2\left(14k_1^2+3\right)k_1-28\right)+\right.$$

$$\left.+3\left(56k_1^8-128k_1^6+75k_1^4+10k_1^2-4\right)\ln\left(\frac{1-\sqrt{1-k_1^2}}{k_1}\right)\right] = \begin{cases}-2041.9990\ meV\\ -2077.2217\ meV\end{cases},$$

$$\Delta E_{VP}(2P) = -\frac{\mu(Z\alpha)^2\alpha}{72\pi}\int_1^\infty \rho(\xi)d\xi\int_0^\infty x^3dxe^{-x\left(1+\frac{2m_e\xi}{W}\right)} =$$

$$= \frac{1}{\left(1-k_1^2\right)^{5/2}}\left[\sqrt{1-k_1^2}\left(-120k_1^6+184k_1^4-23k_1^2+6\pi\left(k_1^2-1\right)^2\left(10k_1^2+3\right)k_1-32\right)\right.$$

$$\left.+3\left(40k_1^8-88k_1^6+45k_1^4+10k_1^2-4\right)\ln\left(\frac{1-\sqrt{1-k_1^2}}{k_1}\right)\right] = \begin{cases}-400.1128\ meV\\ -411.4486\ meV\end{cases},$$

$$\Delta E_{VP}(2P-2S) = \begin{cases}1641.8862\ meV\\ 1665.7730\ meV\end{cases}.$$

Two-loop VP correction to the Lamb shift in 1γ interaction

$$\frac{1}{k^2} \rightarrow \frac{\alpha}{3\pi} \int_1^\infty \rho(\xi) d\xi \frac{1}{k^2 + 4m_e^2\xi^2}.$$

$$V^C_{VP-VP}(r) = \frac{\alpha^2}{9\pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \left(-\frac{Z\alpha}{r}\right) \frac{1}{(\xi^2 - \eta^2)} \left(\xi^2 e^{-2m_e\xi r} - \eta^2 e^{-2m_e\eta r}\right).$$

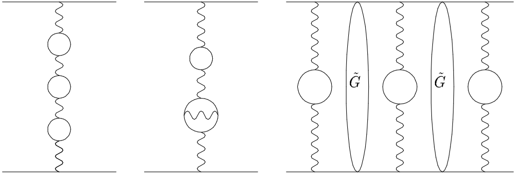
$$\Delta E_{VP-VP}(2P-2S) = \begin{cases} 3.7207\ meV \\ 3.7997\ meV \end{cases}.$$

$$\Delta V_{VP-MVP}(r) = -\frac{4(Z\alpha)\alpha^2}{45\pi^2 m_1^2} \int_1^\infty \rho(\xi) d\xi \left[\pi\delta(\mathbf{r}) - \frac{m_e^2\xi^2}{r} e^{-2m_e\xi r}\right].$$

$$\Delta E_{VP-MVP}(2P-2S) = \begin{cases} 0.0022\ meV \\ 0.0023\ meV \end{cases}.$$

$$\Delta V^C_{2-loop\ VP} = -\frac{2}{3}\frac{Z\alpha}{r}\left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{f(v)dv}{(1-v^2)} e^{-\frac{2m_e r}{\sqrt{1-v^2}}}.$$

$$\Delta E_{2-loop\ VP}(2P-2S) = \begin{cases} 7.6863\ meV \\ 7.7696\ meV \end{cases}.$$



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$$V^C_{VP-VP-VP}(r) = -\frac{Z\alpha}{r}\frac{\alpha^3}{(3\pi)^3}\int_1^\infty \rho(\xi)d\xi \int_1^\infty \rho(\eta d\eta \int_1^\infty \rho(\zeta)d\zeta \times$$

$$\times\left[e^{-2m_e\zeta r}\frac{\zeta^4}{(\xi^2-\zeta^2)(\eta^2-\zeta^2)} + e^{-2m_e\xi r}\frac{\xi^4}{(\zeta^2-\xi^2)(\eta^2-\xi^2)} + e^{-2m_e\eta r}\frac{\eta^4}{(\xi^2-\eta^2)(\zeta^2-\eta^2)}\right],$$

$$V^C_{VP-2-loop\ VP} = -\frac{4\mu\alpha^3(Z\alpha)}{9\pi^3}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \frac{f(\eta)d\eta}{\eta}\frac{1}{r(\eta^2-\xi^2)}\left(\eta^2 e^{-2m_e\eta r} - \xi^2 e^{-2m_e\xi r}\right),$$

$$\Delta E_{VP-VP-VP}(2P-2S) = \begin{cases} 0.0085\ meV \\ 0.0088\ meV \end{cases},$$

$$\Delta E_{VP-2-loop\ VP}(2P-2S) = \begin{cases} 0.0359\ meV \\ 0.0366\ meV \end{cases}.$$

There exists also a contribution with three-loop vacuum polarization operator. It was calculated in

T. Kinoshita and M. Nio, Phys. Rev. Lett. **62**, 3240 (1999); Phys. Rev. **D60**, 053008 (1999).

S.G. Karshenboim, V.G. Ivanov, E.Yu. Korzinin, and V.A. Shelyuto, Pisma v ZheTF **92**, 9 (2010); PRA **81**, 060501 (2010).

One-loop vacuum polarization corrections to the Breit Hamiltonian

$$\Delta V^B_{VP}(r) = \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi \sum_{i=1}^{4}\Delta V^B_{i,VP}(r),$$

$$\Delta V^B_{1,VP} = \frac{Z\alpha}{8}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\left[4\pi\delta(\mathbf{r}) - \frac{4m_e^2\xi^2}{r}e^{-2m_e\xi r}\right],$$

$$\Delta V^B_{2,VP} = -\frac{Z\alpha m_e^2\xi^2}{m_1 m_2 r}e^{-2m_e\xi r}(1-m_e\xi r),$$

$$\Delta V^B_{3,VP} = -\frac{Z\alpha}{2m_1 m_2}p_i\frac{e^{-2m_e\xi r}}{r}\left[\delta_{ij}+\frac{r_i r_j}{r^2}(1+2m_e\xi r)\right]p_j,$$

$$\Delta V^B_{4,VP} = \frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1 m_2}\right)e^{-2m_e\xi r}(1+2m_e\xi r)(\mathbf{L}\boldsymbol{\sigma}_1).$$

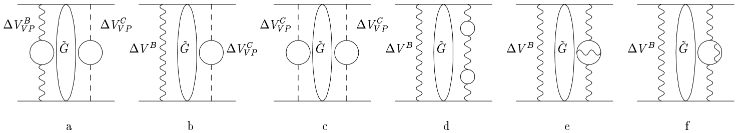
$$\Delta E^B_{1,VP}(2P-2S) = \begin{cases} -0.8670\ meV \\ -0.8931\ meV \end{cases},$$

$$\Delta E^B_{2,VP}(2P-2S) = \begin{cases} 0.0150\ meV \\ 0.0116\ meV \end{cases},$$

$$\Delta E^B_{3,VP}(2P-2S) = \begin{cases} 0.0281\ meV \\ 0.0219\ meV \end{cases},$$

$$\Delta E^B_{4,VP}(2P-2S) = \begin{cases} -0.0860\ meV \\ -0.0876\ meV \end{cases}.$$

Relativistic and VP corrections in second order perturbation theory



$$\Delta E^{VP}_{SOPT} =<\psi|\Delta V^C_{VP}\tilde G\Delta V^C_{VP}|\psi> +2<\psi|\Delta V^B\tilde G\Delta V^C_{VP}|\psi> .$$

$$\tilde G(2S) = -\frac{Z\alpha\mu^2}{4x_1x_2}e^{-\frac{x_1+x_2}{2}}\frac{1}{4\pi}g_{2S}(x_1,x_2),$$

$$g_{2S}(x_1,x_2) = 8x_< - 4x_<^2 + 8x_> + 12x_<x_> - 26x_<^2x_> + 2x_<^3x_> - 4x_>^2 - 26x_<x_>^2 + 23x_<^2x_>^2 -$$

$$-x_<^3x_>^2 + 2x_<x_>^3 - x_<^2x_>^3 + 4e^x(1-x_<)(x_>-2)x_> + 4(x_<-2)x_<(x_>-2)x_>\times$$

$$\times[-2C + Ei(x_<) - \ln(x_<) - \ln(x_>)],$$

$$\tilde G(2P) = -\frac{Z\alpha\mu^2}{36x_1^2x_2^2}e^{-\frac{x_1+x_2}{2}}\frac{3}{4\pi}\frac{(\mathbf{x}_1\mathbf{x}_2)}{x_1x_2}g_{2P}(x_1,x_2),$$

$$g_{2P}(x_1,x_2) = 24x_<^3 + 36x_<^3x_> + 36x_<^3x_>^2 + 24x_>^3 + 36x_<x_>^3 + 36x_<^2x_>^3 + 49x_<^3x_>^3 - 3x_<^4x_>^3 -$$

$$-12e^x_<(2 + x_< + x_<^2)x_>^3 - 3x_<^3x_>^4 + 12x_<^3x_>^3[-2C + Ei(x_<) - \ln(x_<) - \ln(x_>)],$$

First term $<\psi|\Delta V^C_{VP}\tilde{G}\Delta V^C_{VP}|\psi>$

$$\Delta E^{VP,VP}_{SOPT}(2S)=-\frac{\mu\alpha^2(Z\alpha)^2}{72\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\times$$

$$\times\int_0^\infty\left(1-\frac{x}{2}\right)e^{-x\left(1-\frac{2m_e\xi}{W}\right)}dx\int_0^\infty\left(1-\frac{x'}{2}\right)e^{-x'\left(1-\frac{2m_e\eta}{W}\right)}dx'g_{2S}(x,x')=\begin{cases}-1.8640\ meV\\-1.9017\ meV\end{cases},$$

$$\Delta E^{VP,VP}_{SOPT}(2P)=-\frac{\mu\alpha^2(Z\alpha)^2}{7776\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\times$$

$$\times\int_0^\infty e^{-x\left(1+\frac{2m_e\xi}{W}\right)}dx\int_0^\infty e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}dx'g_{2P}(x,x')=\begin{cases}-0.1867\ meV\\-0.1942\ meV\end{cases},$$

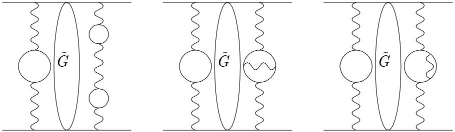
Second term $<\psi|\Delta V^B\tilde{G}\Delta V^C_{VP}|\psi>$

$$<\psi|\frac{\mathbf{p}^4}{(2\mu)^2}{\sum_m}'\frac{|\psi_m><\psi_m|}{E_2-E_m}\Delta V^C_{VP}|\psi>=<\psi|(E_2+\frac{Z\alpha}{r})(\hat{H}_0+\frac{Z\alpha}{r}){\sum_m}'\frac{|\psi_m><\psi_m|}{E_2-E_m}\Delta V^C_{VP}|\psi>=$$

$$=<\psi|\left(E_2+\frac{Z\alpha}{r}\right)^2\tilde{G}\Delta V^C_{VP}|\psi>-<\psi|\frac{Z\alpha}{r}\Delta V^C_{VP}|\psi>+<\psi|\frac{Z\alpha}{r}|\psi><\psi|\Delta V^C_{VP}|\psi>.$$

$$\Delta E^{B,VP}_{SOPT}(2P-2S)=\begin{cases}1.4192\ meV\\1.4682\ meV\end{cases}.$$

Three-loop vacuum polarization correction in second order perturbation theory



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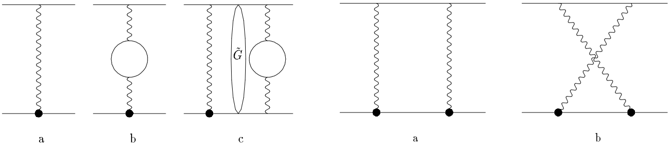
$$\Delta E^{VP-VP,VP}_{SOPT}(2S)=-\frac{\mu\alpha^3(Z\alpha)^2}{108\pi^3}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\int_1^\infty\rho(\zeta)d\zeta\int_0^\infty dx(1-\frac{x}{2})\times$$

$$\int_0^\infty dx'(1-\frac{x'}{2})e^{-x'(1+\frac{2m_e\zeta}{W})}\frac{1}{\xi^2-\eta^2}\left[\xi^2e^{-x(1+\frac{2m_e\xi}{W})}-\eta^2e^{-x(1+\frac{2m_e\eta}{W})}\right]g_{2S}(x,x')=\begin{cases}-0.0104\ meV\\-0.0107\ meV\end{cases},$$

$$\Delta E^{2-loop\ VP,VP}_{SOPT}(2S)=-\frac{\mu\alpha^3(Z\alpha)^2}{18\pi^3}\int_0^1\frac{f(v)dv}{1-v^2}\int_1^\infty\rho(\xi)d\xi\times$$

$$\times\int_0^\infty dx\left(1-\frac{x}{2}\right)e^{-x(1+\frac{2m_e}{\sqrt{1-v^2}W})}\int_0^\infty dx'\left(1-\frac{x'}{2}\right)e^{-x'(1+\frac{2m_e\xi}{W})}g_{2S}(x,x')=\begin{cases}-0.0168\ meV\\-0.0171\ meV\end{cases},$$

Nuclear structure correction in 1γ and 2γ interaction



$$\Delta E_{str}(2P-2S) = -\frac{\mu^3 (Z\alpha)^4}{12} < r_N^2 >= \begin{cases} -396.2669\ meV \\ -295.8478\ meV \end{cases}.$$

$$\Delta E_{str}^{2\gamma}(nS) = -\frac{\mu^3 (Z\alpha)^5}{\pi n^3}\delta_{l0}\int_0^\infty \frac{dk}{k}V(k),\ \Delta E_{G,str}^{2\gamma}(2P-2S) = \begin{cases} 10.28 \pm 0.10\ meV \\ 6.61 \pm 0.07\ meV \end{cases}.$$

$$V(k) = \frac{2(F^2-1)}{m_1 m_2} + \frac{8m_1[-F(0)+4m_2^2 F'(0)]}{m_2(m_1+m_2)k} + \frac{k^2}{2m_1^3 m_2^3}\times$$

$$\times\left[2(F^2-1)(m_1^2+m_2^2) - F^2 m_1^2\right] + \frac{\sqrt{k^2+4m_1^2}}{2m_1^3 m_2 (m_1^2-m_2^2)k}\times$$

$$\times\left\{k^2\left[2(F^2-1)m_2^2 - F^2 m_1^2\right] + 8m_1^4 F^2 + \frac{16 m_1^4 m_2^2 (F^2-1)}{k^2}\right\} -$$

$$-\frac{\sqrt{k^2+4m_2^2}\, m_1}{2m_2^3(m_1^2-m_2^2)k}\left\{k^2\left[2(F^2-1) - F^2\right] + 8m_2^2 F^2 + \frac{16 m_2^4 (F^2-1)}{k^2}\right\}.$$

Nuclear structure and one-loop VP correction in second order PT

$$\Delta V^{VP}_{str}(r) = \frac{2}{3}\pi Z\alpha < r_N^2 > \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi\left[\delta(\mathbf{r}) - \frac{m_e^2\xi^2}{\pi r}e^{-2m_e\xi r}\right].$$

$$\Delta E^{VP}_{str}(2S) = \frac{\alpha(Z\alpha)^4 < r_N^2 > \mu^3}{36\pi}\int_1^\infty \rho(\xi)d\xi\left[1 - \frac{4m_e^2\xi^2}{W^2}\int_0^\infty xdx(1-\frac{x}{2})^2 e^{-x(1+\frac{2m_e\xi}{W})}\right] =$$

$$= \begin{cases} 1.2493\ meV \\ 0.9365\ meV \end{cases},$$

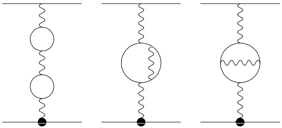
$$\Delta E^{VP}_{str}(2P) = -\frac{\alpha(Z\alpha)^4\mu^3 < r_N^2 >}{108\pi}\frac{m_e^2}{W^2}\int_1^\infty \xi^2\rho(\xi)d\xi\int_0^\infty x^3 e^{-x(1+\frac{2m_e\xi}{W})}dx = \begin{cases} -0.0300\ meV \\ -0.0225\ meV \end{cases},$$

$$\Delta E^{VP}_{str}(2P-2S) = \begin{cases} -1.2793 \pm 0.0130\ meV \\ -0.9590 \pm 0.0092\ meV \end{cases}.$$

$$\Delta E^{VP}_{str,SOPT}(2P-2S) = -\frac{\alpha(Z\alpha)^4\mu^3 < r_N^2 >}{36\pi}\int_1^\infty \rho(\xi)d\xi\times$$

$$\times\int_0^\infty dxe^{-x(1+\frac{2m_e\xi}{W})}(1-\frac{x}{2})\left[4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4\right] = \begin{cases} -2.0083\ meV \\ -1.5063\ meV \end{cases}.$$

Nuclear structure and two-loop VP correction



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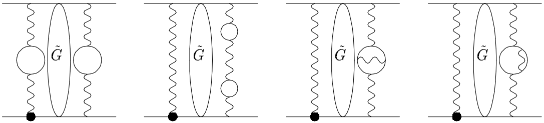

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$$\Delta V_{str}^{VP-VP}(r)=\frac{2}{3}Z\alpha<r_N^2>\left(\frac{\alpha}{3\pi}\right)^2\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\times$$

$$\times\left[\pi\delta(\mathbf{r})-\frac{m_e^2}{r(\xi^2-\eta^2)}\left(\xi^4e^{-2m_e\xi r}-\eta^4e^{-2m_e\eta r}\right)\right],$$

$$\Delta V_{str}^{2-loop\ VP}(r)=\frac{4}{9}Z\alpha<r_N^2>\left(\frac{\alpha}{\pi}\right)^2\int_0^1\frac{f(v)dv}{1-v^2}\left[\pi\delta(\mathbf{r})-\frac{m_e^2}{r(1-v^2)}e^{-\frac{2m_er}{\sqrt{1-v^2}}}\right].$$

$$\Delta E_{str}^{VP,VP}(2P-2S)=\begin{cases}-0.0102\ meV\\-0.0076\ meV\end{cases}.$$

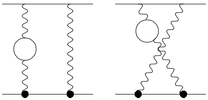
$$\Delta E^{VP,VP\,(1)}_{str,SOPT}(2S) = \frac{\alpha^2(Z\alpha)^4\mu^3 r_N^2}{108\pi^2}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\times$$

$$\int_0^\infty \left(1-\frac{x}{2}\right)dxe^{-x\left(1+\frac{2m_e\eta}{W}\right)}\left[4x(x-2)(\ln x + C) + x^3 - 13x^2 + 6x + 4\right],$$

$$\Delta E^{VP,VP\,(2)}_{str,SOPT}(2S) = -\frac{\alpha^2(Z\alpha)^4\mu^3 r_N^2 m_e^2}{54\pi^2 W^2}\int_1^\infty \xi^2\rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\times$$

$$\int_0^\infty \left(1-\frac{x}{2}\right)dxe^{-x\left(1+\frac{2m_e\xi}{W}\right)}\int_0^\infty\left(1-\frac{x'}{2}\right)dx'e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}g_{2S}(x,x').$$

$$\Delta E^{VP,VP}_{str,SOPT}(2P-2S) = \begin{cases} -0.0086\ meV \\ -0.0065\ meV \end{cases}.$$



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$$\Delta E^{2\gamma}_{str,VP}(nS) = -\frac{2\mu^3\alpha(Z\alpha)^5}{\pi^2 n^3}\int_0^\infty kV(k)dk\int_0^1\frac{v^2(1-\frac{v^2}{3})dv}{k^2(1-v^2)+4m_e^2},$$

$$\Delta E^{2\gamma}_{str,VP}(2P-2S) = \begin{cases} 0.2214 \pm 0.0022\ meV \\ 0.1270 \pm 0.0013\ meV \end{cases}.$$

Numerical results, comparison with other calculations

- E. Borie, Ann. Phys. (NY) **72**, 052511 (2012).
- E.Yu. Korzinin, V.G. Ivanov and S.G. Karshenboim, PRD **88**, 125019 (2013); S.G. Karshenboim, V.G. Ivanov, E.Yu. Korzinin, and V.A. Shelyuto, PRA **81**, 060501 (2010).
- U.D. Jentschura, Ann.Phys. **326**, 500 (2011); U.D. Jentschura, PRA **84**, 012505 (2011); U.D. Jentschura, EPJD **61**, 7 (2011).

Our one-loop VP result coincides with the calculation KKIS.

KKIS, meV

- First order VP: 1665.7729

Our result, meV

- VP contribution of order $\alpha(Z\alpha)^2$ in 1γ interaction: 1665.7730

Total two-loop contribution from KKIS is equal to

- 13.2769 meV, $(\mu_2^4 He)^+$

This agrees with our results

- 13.2789 meV, $(\mu_2^4 He)^+$

with the accuracy 0.002 meV (a number of two-loop corrections to the Breit Hamiltonian were estimated approximately).

Our three-loop VP result is also in agreement with the calculation KKIS.

KKIS, meV

- 0.074 $(\mu_2^4 He)^+$

Our result, meV

- 0.0703 $(\mu_2^4 He)^+$

Relativistic corrections with vacuum polarization effects (FOPT, SOPT) in our work coincide with the results of Jentschura.

Jentschura, meV

- $\delta E_{vp} = 0.521$

Our results, meV

- Relativistic-VP correction of order $\alpha(Z\alpha)^4$ in FOPT: -0.9472
- Relativistic-VP correction of order $\alpha(Z\alpha)^4$ in SOPT: 1.4682

Total: 0.521

There exists the only calculation of E. Borie where total results for the Lamb shift in muonic helium ions were obtained. In the case of $(\mu_2^4He)^+$:

Borie, meV

- Uehling: 1666.305

Our results, meV

- VP contribution of order $\alpha(Z\alpha)^2$ in 1γ interaction: 1665.7730
- Relativistic-VP contribution of order $\alpha(Z\alpha)^4$ in FOPT: -0.9472
- Relativistic-VP contribution of order $\alpha(Z\alpha)^4$ in SOPT: 1.4682

Total: 1666.2940

Borie, meV

- Kallen-Sabry: 11.573

Our results, meV

- 2-loop VP contribution of order $\alpha^2(Z\alpha)^2$ in 1γ interaction: 11.5693
- Relativistic-2loop VP contribution of order $\alpha^2(Z\alpha)^4$ in FOPT: -0.0037
- Relativistic-2loop VP contribution of order $\alpha^2(Z\alpha)^4$ in SOPT: 0.0058

Total: 11.5714

The small difference may be related with recoil terms accounted in our calculation.

We can easily compare our results for nuclear structure corrections with Borie's results.

We used the same value for charge radius of α-particle

$$r_{He} = 1.676\ fm$$

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We also use the same Gaussian parametrization for the formfactors.

Our results, meV

- Nuclear structure of order $(Z\alpha)^4$: -295.848± 2.83
- Nuclear structure of order $(Z\alpha)^5$ in 2γ interaction: 6.605± 0.07
- Nuclear structure-VP of order $\alpha(Z\alpha)^4$ (FOPT): -0.960± 0.0092
- Nuclear structure and VP correction of order $\alpha(Z\alpha)^4$ (SOPT): -1.5063± 0.0092
- Nuclear structure-2-loop VP correction of order $\alpha^2(Z\alpha)^4$ in 1γ interaction: -0.0076
- Nuclear structure and 2-loop VP correction of order $\alpha^2(Z\alpha)^4$ (SOPT): -0.0182
- Nuclear structure-VP contribution in 2γ interaction: 0.1279± 0.0013

Total: -291.844, Borie, meV:-292.045

Comparison between total results for $(\mu_2^4He)^+$ of

Borie: $\Delta E = 1379.2479$ *meV*

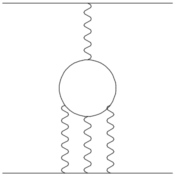
Our result: $\Delta E = 1379.1107$ *meV*

Discrepancy is equal 0.1 meV.

Thank you for your attention.

Appendix

The Wichmann-Kroll correction to the Lamb shift



$$\Delta V^{WK}(r) = \frac{\alpha(Z\alpha)^3}{\pi r}\int_0^\infty \frac{d\zeta}{\zeta^4} e^{-2m_e\zeta r}\left[-\frac{\pi^2}{12}\sqrt{\zeta^2-1}\,\theta(\zeta-1) + \int_0^\zeta dx\sqrt{\zeta^2-x^2} f^{WK}(x)\right].$$

$$\Delta E^{WK}(2P-2S) = \begin{cases} -0.0197\ meV \\ -0.0199\ meV \end{cases}.$$

Three-loop vacuum polarization correction in third order PT

$$\Delta E =< \psi_2|\Delta V^C \tilde{G} \Delta V^C \tilde{G} \Delta V^C|\psi_2 > - < \psi_2|\Delta V^C|\psi_2 >< \psi_2|\Delta V^C \tilde{G}\tilde{G} \Delta V^C|\psi_2 > .$$

$$\Delta E_{TOPT,1}(2S) = -\frac{\mu Z^2 \alpha^5}{864\pi^3} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_1^\infty \rho(\zeta) d\zeta \int_0^\infty \left(1 - \frac{x}{2}\right) e^{-x(1+2m_e\xi/W)} dx \times$$

$$\int_0^\infty \left(1 - \frac{x''}{2}\right) e^{-x''(1+2m_e\zeta/W)} dx'' \int_0^\infty \frac{dx'}{x'} e^{-x''(1+2m_e\zeta/W)} g(x,x')g(x',x'') = \begin{cases} -0.0044\ meV \\ -0.0045\ meV \end{cases},$$

$$\Delta E_{TOPT,2}(2S) = \frac{\alpha^2}{288\pi^2} \int_1^\infty \rho(\xi) d\xi \int_1^\infty \rho(\eta) d\eta \int_0^\infty \left(1 - \frac{x}{2}\right) e^{-x(1+2m_e\xi/W)} dx \times$$

$$\int_0^\infty \left(1 - \frac{x''}{2}\right) e^{-x''(1+2m_e\eta/W)} dx'' \int_0^\infty dx' g(x,x')g(x',x'') \begin{cases} 2041.9990\ meV \\ 2077.2217\ meV \end{cases} = \begin{cases} 0.0037\ meV \\ 0.0038\ meV \end{cases}.$$

Replacement $\Delta V^C_{VP} \to \Delta H_B$, $\Delta V^C_{VP} \to \Delta V^C_{VP,VP}$, $\Delta H_{B,1} = (\pi Z\alpha/2)(1/m_1^2 + \delta_I/m_2^2)\delta(\mathbf{r})$

$$\Delta E^{VP-VP,\Delta H_{B,1}}_{SOPT}(2S) = \frac{\mu^3(Z\alpha)^4\alpha^2}{144\pi^2}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\frac{1}{\xi^2-\eta^2}\times$$

$$\int_0^\infty\left(1-\frac{x}{2}\right)dx[4x(x-2)(\ln x+C)+x^3-13x^2+6x+4]\left[\xi^2 e^{-x(1+\frac{2m_e\xi}{W})}-\eta^2 e^{-x(1+\frac{2m_e\eta}{W})}\right]=$$

$$=\begin{cases}0.0050\ meV\\ 0.0051\ meV\end{cases},$$

$$\Delta E^{2-loop\ VP,\Delta H_{B,1}}_{SOPT}(2S) = \frac{\mu^3(Z\alpha)^4\alpha^2}{24\pi^2}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\int_0^1\frac{f(v)dv}{1-v^2}\times$$

$$\int_0^\infty(1-\frac{x}{2})dx[4x(x-2)(\ln x+C)+x^3-13x^2+6x+4]e^{-x(1+\frac{2m_e}{W\sqrt{1-v^2}}}=\begin{cases}0.0056\ meV\\ 0.0058\ meV\end{cases}.$$

Replacement $\Delta V^C_{VP} \to \Delta H_B$, $\Delta V^C_{VP} \to \Delta V^C_{VP,VP}$, $\Delta H_{B,2} = -\mathbf{p}^4(1/m_1^3 + 1/m_2^3)$

$$\Delta E^{VP-VP,\Delta H_{B,2}}_{SOPT,1}(2S) = -\frac{\mu^4(Z\alpha)^4\alpha^2}{72\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\frac{1}{\xi^2-\eta^2}\times$$

$$\int_0^\infty\left(1-\frac{x}{2}\right)xdx\left(\frac{1}{x}-\frac{1}{8}\right)^2\int_0^\infty(1-\frac{x'}{2})dx'g(x,x')\left[\xi^2 e^{-x'(1+\frac{2m_e\xi}{W})}-\eta^2 e^{-x'(1+\frac{2m_e\eta}{W})}\right]=$$

$$=\begin{cases}-0.0029\ meV\\ -0.0031\ meV\end{cases},$$

$$\Delta E^{2-loop\ VP,\Delta H_{B,2}}_{SOPT,2}(2S) = -\frac{\mu^4(Z\alpha)^4\alpha^2}{12\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\int_1^\infty \rho(\xi)d\xi\int_0^1\frac{f(v)dv}{1-v^2}\times$$

$$\int_0^\infty(1-\frac{x}{2})xdx\left(\frac{1}{x}-\frac{1}{8}\right)^2\int_0^\infty(1-\frac{x'}{2})dx'g(x,x')e^{-x'(1+\frac{2m_e}{W\sqrt{1-v^2}})}=\begin{cases}-0.0045\ meV\\ -0.0047\ meV\end{cases},$$

$$\Delta E^{VP-VP,\Delta H_{B,2}}_{SOPT,3}(2S) = -\frac{\mu^4(Z\alpha)^4\alpha^2}{18\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\frac{1}{\xi^2-\eta^2}\times$$

$$\int_0^\infty\left(1-\frac{x}{2}\right)^2dx\left[\xi^2 e^{-x(1+\frac{2m_e\xi}{W})}-\eta^2 e^{-x(1+\frac{2m_e\eta}{W})}\right]=\begin{cases}-0.0072\ meV\\ -0.0075\ meV\end{cases},$$

$$\Delta E^{2-loop\ VP,\Delta H_{B,2}}_{SOPT,4}(2S) = -\frac{\mu^4(Z\alpha)^4\alpha^2}{3\pi^2}\left(\frac{1}{m_1^3}+\frac{1}{m_2^3}\right)\int_1^\infty\frac{f(v)dv}{1-v^2}\times$$

$$\int_0^\infty\left(1-\frac{x}{2}\right)^2dxe^{-x(1+\frac{2m_e}{W\sqrt{1-v^2}})}=\begin{cases}-0.0083\ meV\\ -0.0086\ meV\end{cases}.$$

$$\Delta E^{VP,VP;\Delta V^B}_{SOPT}(2P-2S)=\begin{cases}0.0120\ meV\\ 0.0127\ meV\end{cases},\ \Delta E^{VP,\Delta V^B_{VP}}_{SOPT}(2P-2S)=\begin{cases}-0.0066\ meV\\ -0.0069\ meV\end{cases}.$$

Recoil correction of order $(Z\alpha)^4$

$$\Delta E_{rec}(2P-2S)=\begin{cases}\frac{\mu^3(Z\alpha)^4}{48m_2^2}, & \delta_l=1\\ \frac{\mu^3(Z\alpha)^4}{12m_2^2}, & \delta_l=0\end{cases}=\begin{cases}0.1265\ meV\\ 0.2952\ meV\end{cases}.$$

Recoil correction of order $(Z\alpha)^5$

$$\Delta E_{rec}^{(Z\alpha)^5}=\frac{\mu^3(Z\alpha)^5}{m_1 m_2\pi n^3}\left[\frac{2}{3}\delta_{l0}\ln\frac{1}{Z\alpha}-\frac{8}{3}\ln k_0(n,l)-\frac{1}{9}\delta_{l0}-\frac{7}{3}a_n-\frac{2}{m_2^2-m_1^2}\delta_{l0}(m_2^2\ln\frac{m_1}{\mu}-m_1^2\ln\frac{m_2}{\mu})\right],$$

$$\ln k_0(2S)=2.811769893120563,\ \ln k_0(2P)=-0.030016708630213,$$

$$a_n=-2\left[\ln\frac{2}{n}+(1+\frac{1}{2}+\ldots+\frac{1}{n}+1-\frac{1}{2n}\right]\delta_{l0}+\frac{(1-\delta_{l0})}{l(l+1)(2l+1)}.$$

$$\Delta E_{rec}^{(Z\alpha)^5}(2P-2S)=\begin{cases}-0.5581\ meV\\ -0.4330\ meV\end{cases}.$$

Recoil correction of order $(Z\alpha)^6$

$$\Delta E_{rec}^{(Z\alpha)^6}(2P-2S)=\frac{(Z\alpha)^6 m_1^2}{8m_2}\left(\frac{23}{6}-4\ln 2\right)=\begin{cases}0.0051\ meV\\ 0.0038\ meV\end{cases}.$$

Muon vacuum polarization, muon self-energy correction

$$\Delta E_{MVP,MSE}(2S) = \frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left[\frac{4}{3}\ln\frac{m_1}{\mu(Z\alpha)^2} - \frac{4}{3}\ln k_0(2S) + \frac{38}{45} + \right.$$

$$\left. + \frac{\alpha}{\pi}\left(-\frac{9}{4}\zeta(3) + \frac{3}{2}\pi^2\ln 2 - \frac{10}{27}\pi^2 - \frac{2179}{648}\right) + 4\pi Z\alpha\left(\frac{427}{384} - \frac{\ln 2}{2}\right)\right] = \begin{cases} 10.6633\ meV \\ 10.9392\ meV \end{cases},$$

$$\Delta E_{MVP,MSE}(2P) = \frac{\alpha(Z\alpha)^4}{8\pi}\frac{\mu^3}{m_1^2}\left[-\frac{4}{3}\ln k_0(2P) - \frac{m_1}{6\mu} - \right.$$

$$\left. - \frac{\alpha}{3\pi}\frac{m_1}{\mu}\left(\frac{3}{4}\zeta(3) - \frac{\pi^2}{2}\ln 2 + \frac{\pi^2}{12} + \frac{197}{144}\right)\right] = \begin{cases} -0.1653\ meV \\ -0.1678\ meV \end{cases}.$$

Radiative-recoil corrections of orders $\alpha(Z\alpha)^5$, $(Z^2\alpha)(Z\alpha)^4$

$$\Delta E_{rad-rec}(2S) = -1.324\frac{\alpha(Z\alpha)^5 m_1^2}{8m_2} + \left(\frac{2\pi^2}{9} - \frac{70}{27}\right)\frac{\alpha(Z\alpha)^5 m_1^2}{8\pi^2 m_2} +$$

$$+ \left[\frac{1}{3}\ln\frac{\Lambda(Z\alpha)^{-2}}{\mu} + \frac{11}{72} - \frac{1}{24} - \frac{7\pi}{32}\frac{\Lambda^2}{4m_2^2} + \frac{2}{3}\left(\frac{\Lambda^2}{4m_2^2}\right)^2 - \frac{1}{3}\ln k_0(2S)\right]\frac{4(Z^2\alpha)(Z\alpha)^4\mu^3}{8\pi m_2^2},$$

$$\Delta E_{rad-rec}(2P) = -\frac{1}{3}\ln k_0(2P)\frac{4(Z^2\alpha(Z\alpha)^4\mu^3}{8\pi m_2^2}.$$

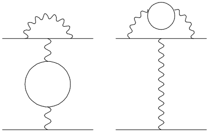
$$\Delta E_{rad-rec}(2P-2S) = \begin{cases} -0.0656\ meV \\ -0.0377\ meV \end{cases}.$$

Nuclear structure corrections of orders $(Z\alpha)^6$, $\alpha(Z\alpha)^5$

$$\Delta E^{(Z\alpha)^6}_{str}(2P-2S)=\frac{(Z\alpha)^6}{12}\mu^3\left\{r_N^2\left[\langle \ln \mu Z\alpha r\rangle + C-\frac{3}{2}\right]-\frac{1}{2}r_N^2+\frac{1}{3}\langle r^3\rangle\langle\frac{1}{r}\rangle-\right.$$

$$\left.-I_2^{rel}-I_3^{rel}-\mu^2F_{NR}+\frac{1}{40}\mu^2\langle r^4\rangle\right\}=\begin{cases}-0.005064\cdot r_h^2+0.11445=-0.3882\ meV\\ -0.00533\cdot r_\alpha^2+0.07846=-0.3063\ meV\end{cases},$$

$$\Delta E^{\alpha(Z\alpha)^5}_{str}(2P-2S)=\begin{cases}0.0940\ meV\\ 0.0702\ meV\end{cases}.$$



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$$\Delta E_{rad+VP}(nS)=\frac{\mu^3}{m_1^2}\frac{(Z\alpha)^4}{n^3}\left[4m_1^2F_1'(0)\delta_{l0}+F_2(0)\frac{C_{jl}}{2l+1}\right],\ C_{jl}=\delta_{l0}+(1-\delta_{l0})\frac{j(j+1)-l(l+1)-\frac{3}{4}}{l(l+1)}.$$

$$m_1^2F_1'(0)=\left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{9}\ln^2\frac{m_1}{m_e}-\frac{29}{108}\ln\frac{m_1}{m_e}+\frac{1}{9}\zeta(2)+\frac{395}{1296}\right],$$

$$F_2(0)=\left(\frac{\alpha}{\pi}\right)^2\left[\frac{1}{3}\ln\frac{m_1}{m_e}-\frac{25}{36}+\frac{\pi^2}{4}\frac{m_e}{m_1}-4\frac{m_e^2}{m_1^2}\ln\frac{m_1}{m_e}+3\frac{m_e^2}{m_1^2}\right].$$

$$\Delta E_{rad+VP}(2P-2S)=\begin{cases}-0.0299\ meV\\ -0.0307\ meV\end{cases}.$$

$$\Delta E^{VP}_{MSE} = \frac{\alpha}{3\pi m_1^2} \ln \frac{m_1}{\mu(Z\alpha)^2} \left[< \psi_n | \Delta \cdot \Delta V^C_{VP} | \psi_n > +2 < \psi_n | \Delta V^C_{VP} \tilde{G} \Delta \left(-\frac{Z\alpha}{r} \right) | \psi_n > \right].$$

$$\Delta E^{VP}_{MSE}(2P - 2S) = \begin{cases} -0.1008\ meV \\ -0.1074\ meV \end{cases}.$$

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HVP and nuclear polarizability contributions

$$\Delta E^{HVP} = \begin{cases} 0.2170\ meV \\ 0.2229\ meV \end{cases}.$$

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$$\Delta E^{NP} = \begin{cases} 4.9 \pm 1.0\ meV \\ 2.47 \pm 0.15\ meV \end{cases}.$$

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