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V.A. Fock's discovery of "hidden" O(4) symmetry of the H-atom and dynamical group theory

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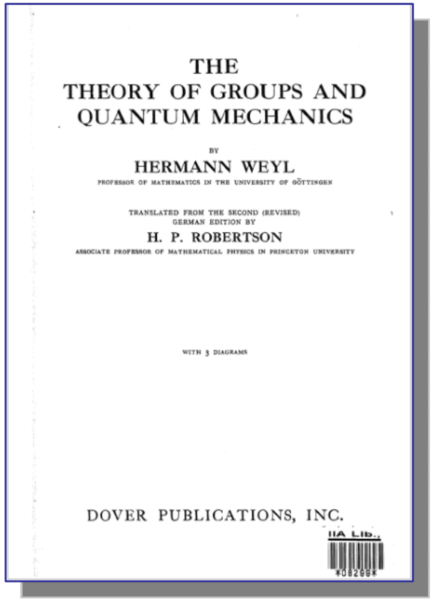
Contents

Introduction

"In the thirties, under the demoralizing influence of quantum – theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets." **R. Iost**


THE
THEORY OF GROUPS AND
QUANTUM MECHANICS

BY
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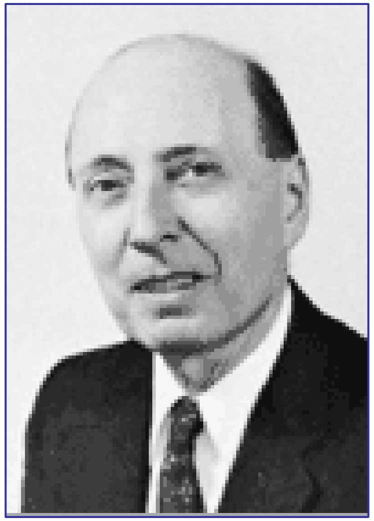

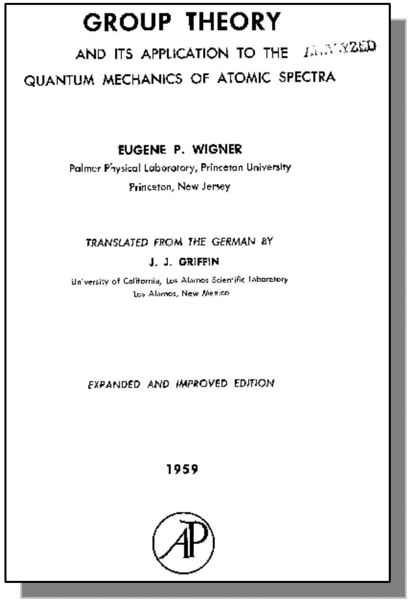
GROUP THEORY
AND ITS APPLICATION TO THE
QUANTUM MECHANICS OF ATOMIC SPECTRA

EUGENE P. WIGNER
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TRANSLATED FROM THE GERMAN BY
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EXPANDED AND IMPROVED EDITION

1959

Lie groups and Lie algebras have been successfully applied in quantum mechanics since its inception.

V.A. Fock and V. Bargmann approaches to the description of the H- atom symmetry

$$\hat{H}\Psi(\vec{r}) = \mathcal{E}\Psi(\vec{r});$$

$$\hat{H} = \frac{1}{2\mu}\hat{\vec{p}}^2 - \frac{Ze_0^2}{r}, \quad \mathcal{E}_n = -\frac{\mathcal{E}_0}{2n^2}, n = 1, 2, \ldots \quad \mathcal{E}_0 = \frac{Z^2 e_0^4 \mu}{\hbar^2} = 27 \cdot Z^2 \text{эв.}$$

$$R \in SO(3), \ T^l(R), \ \dim T^l = 2l+1;$$

$$\sum_{l=0}^{n-1}(2l+1) = n^2. \ \text{"Accidental" degeneracy of H-atom levels}$$

1935, V.A. Fock. Analytical approach. SO(4)

$$\left(\frac{p^2}{2\mu} - \mathcal{E}\right)\Psi(\vec{p}) = \frac{Ze_0^2}{2\pi^2\hbar}\int \frac{\Psi(\vec{p}^{\,\prime})\,d\vec{p}^{\,\prime}}{\left|\vec{p} - \vec{p}^{\,\prime}\right|^2}$$

ИЗВЕСТИЯ АКАДЕМИИ НАУК СССР. 1935

BULLETIN DE L'ACADÉMIE DES SCIENCES DE L'URSS

Classe des sciences mathématiques et naturelles

Отделение математических и естественных наук

АТОМ ВОДОРОДА И НЕ-ЕВКЛИДОВА ГЕОМЕТРИЯ*

*(Предварительное сообщение**)*

В. А. ФОКА

(Представлено академиком **С. И. Вавиловым**)

В работе показано, что уравнение Шредингера для атома водорода в пространстве импульсов приводится к интегральному уравнению для шаровых функций четырехмерного шара. Таким образом, допускаемая уравнением Шредингера группа преобразований оказывается тождественной с четырехмерной группой вращений; наличие этой группы объясняет так называемое вырождение уровней водорода по отношению к азимутальному квантовому числу. Следствия, вытекающие из сопоставления уравнения Шредингера с четырехмерной теорией потенциала (теорема сложения и т. д.), могут иметь разнообразные физические приложения. Так например, бесконечные суммы, встречающиеся в теории явления Комптона от связанных электронов и в других аналогичных задачах, получаются из нашей теории почти без всяких вычислений. Далее, теория позволяет построить упрощенную модель сложных атомов, на основании которой можно вывести явные выражения для смешанной плотности в пространстве импульсов, для атомных факторов, для экранирующего потенциала и т. д.

Как известно, уровни энергии атома водорода зависят только от главного квантового числа n и не зависят от азимутального квантового числа l. Если пользоваться общеупотребительным, но не совсем удачным термином, то можно сказать, что имеется вырождение (т. е. кратность уровней) относительно азимутального квантового числа. С другой стороны, можно установить общее правило, согласно которому кратность собственных значений уравнения Шредингера связана с инвариантностью его по отношению к определенной группе преобразований. Так например, инвариантность по отношению к обыкновенной группе вращений (сферическая симметрия) влечет за собой независимость уровней от

* Доложено 8 февраля 1935 г. в теоретическом семинарии Физического института ЛГУ и 23 марта 1935 г. на сессии Академии Наук СССР в Москве.

** Подробное изложение теории будет напечатано в Физическом журнале Советского Союза.

Zur Theorie des Wasserstoffatoms[1]).

Von **V. Fock** in Leningrad.

(Eingegangen am 5. August 1935.)

Die Schrödinger-Gleichung für das Wasserstoffatom im Impulsraum erweist sich als identisch mit der Integralgleichung für die Kugelfunktionen der vierdimensionalen Potentialtheorie. Die Transformationsgruppe der Wasserstoffgleichung ist also die *vierdimensionale* Drehgruppe; dadurch wird die Entartung der Wasserstoffniveaus in bezug auf die Azimutalquantenzahl l erklärt. Die aus der potentialtheoretischen Deutung der Schrödinger-Gleichung folgenden Beziehungen (Additionstheorem usw.) erlauben mannigfache physikalische Anwendungen. Die Methode ermöglicht, die unendlichen Summen, die in der Theorie des Compton-Effektes an gebundenen Elektronen und in verwandten Problemen auftreten, fast ohne Rechnung auszuwerten. Unter Zugrundelegung eines vereinfachten Atommodells lassen sich ferner explizite Ausdrücke für die Dichtematrix im Impulsraum, für Atomformfaktoren, für das Abschirmungspotential usw. aufstellen.

Es ist längst bekannt, daß die Energieniveaus des Wasserstoffatoms in bezug auf die Azimutalquantenzahl l entartet sind; man spricht gelegentlich von einer „zufälligen" Entartung. Nun ist aber jede Entartung der Eigenwerte mit der Transformationsgruppe der betreffenden Gleichung verbunden: so z. B. die Entartung in bezug auf die magnetische Quantenzahl m mit der gewöhnlichen Drehgruppe. Die Gruppe aber, welche der „zufälligen" Entartung der Wasserstoffniveaus entspricht, war bis jetzt unbekannt.

In dieser Arbeit wollen wir zeigen, daß diese Gruppe mit der vierdimensionalen Drehgruppe äquivalent ist.

1. Die Schrödinger-Gleichung eines wasserstoffähnlichen Atoms hat bekanntlich im Impulsraum die Form einer Integralgleichung

$$\frac{1}{2m} p^2 \psi(\mathfrak{p}) - \frac{Z e^2}{2 \pi^2 h} \int \frac{\psi(\mathfrak{p}')\,(d\mathfrak{p}')}{|\mathfrak{p} - \mathfrak{p}'|^2} = E \psi(\mathfrak{p}), \tag{1}$$

wo mit $(d\mathfrak{p}') = dp'_x\,dp'_y\,dp'_z$ das Volumelement im Impulsraum bezeichnet ist. Wir betrachten zunächst das Punktspektrum und bezeichnen mit p_0 den mittleren quadratischen Impuls

$$p_0 = \sqrt{-2mE}. \tag{2}$$

Wir wollen nun die durch p_0 dividierten Komponenten des Impulsvektors $\mathfrak{p}$ als Koordinaten in einer Hyperebene deuten, welche die stereo-

[1]) Vorgetragen am 8. Februar 1935 im theoretischen Seminar an der Universität Leningrad. Vgl. V. Fock, Bull. de l'ac. des sciences de l'URSS. 1935, Nr. 2, 169.

Group SO(4) as symmetry group of H – atom:

$$\psi(\xi)=\left(\vec{p}^{\,2}+p_0^2\right)\Psi(\vec{p}),$$

$$\xi=(\xi_0,\vec{\xi})=\left(\cos\chi,\sin\chi\sin\theta\cos\phi,\sin\chi\sin\theta\sin\phi,\sin\chi\cos\theta\right),\ \xi_0^2+\vec{\xi}^{\,2}=1;$$

$$\xi_0=\frac{p_0^2-\vec{p}^{\,2}}{p_0^2+\vec{p}^{\,2}},\vec{\xi}=\frac{2p_0\vec{p}}{p_0^2+\vec{p}^{\,2}},p_0=\sqrt{2\mu|\mathcal{E}|},\ \mathcal{E}=-\frac{p_0^2}{2\mu}.$$

$$\psi(\xi)=\frac{\eta}{2\pi^2}\int\frac{\psi(\xi')}{\left(\xi-\xi'\right)^2}\frac{\mathrm{d}\xi'}{\xi'}\Rightarrow\psi(\chi,\theta,\phi)=\frac{\eta}{2\pi^2}\int\frac{\psi(\chi',\theta',\phi')\mathrm{d}\Omega'}{4\sin^2\left(\omega/2\right)};$$

$$\eta=\frac{\alpha\cdot\mu}{p_0},\alpha=\frac{Ze_0^2}{\hbar};\ \mathrm{d}\Omega=\sin^2\chi\mathrm{d}\chi\sin\theta\mathrm{d}\theta\mathrm{d}\phi,$$

$$\cos\omega=\cos\chi\cos\chi'+\sin\chi\sin\chi'\cos\gamma,$$

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\left(\phi-\phi'\right).$$

$$\psi_{nlm}(\chi,\theta,\phi)=\Pi_l(n,\chi)\cdot\mathrm{Y}_{lm}(\theta,\phi),\quad\frac{2}{\pi}\int_0^\pi\Pi_1^2(n,\chi)\sin^2\chi\mathrm{d}\chi=1.$$

$$\Psi_{nlm}\left(\vec{p}\right)=\mathrm{N}_{nlm}\cdot\left(\vec{p}^{\,2}+p_0^2\right)^{-2}\cdot\mathrm{Y}_{nlm}\left(\xi\right)$$

Zur Theorie des Wasserstoffatoms.

Bemerkungen zur gleichnamigen Arbeit von V. Fock.

Von **V. Bargmann** in Zürich.

(Eingegangen am 1. Februar 1936.)

Es wird gezeigt, daß die Matrixgleichungen, die Pauli seiner Behandlung des Wasserstoffatoms zugrunde gelegt hat, bei gruppentheoretischer Deutung auf die kürzlich von Fock entwickelte Methode führen. — Die Separation der Schrödinger-Gleichung in parabolischen Koordinaten wird in diesen Zusammenhang eingeordnet.

In einer sehr interessanten Arbeit[1]) führt Fock die Schrödinger-Gleichung des Wasserstoffatoms im Impulsraum zurück auf die Integralgleichung der Kugelfunktionen in vier Variablen, indem er den Impulsraum auf die vierdimensionale Einheitskugel stereographisch projiziert. Diese Überlegungen stehen nun, wie wir zeigen wollen, in engem Zusammenhang mit der Behandlung des Wasserstoffatoms nach der Matrizenrechnung durch Pauli[2]); die gruppentheoretische Deutung der von Pauli abgeleiteten Beziehungen führt zwangsläufig zu der Fockschen Methode.

1. Wir beginnen mit einigen Vorbemerkungen über infinitesimale Transformationen. Gegeben seien n Variable y_ϱ. Unterwirft man sie einer infinitesimalen Transformation

$$\delta y_\varrho = -\varepsilon \cdot \alpha_\varrho(y), \qquad (\varrho = 1, \ldots, n),$$

so erleidet eine Funktion $f(y)$ die Transformation

$$\delta f = f(y - \delta y) - f(y) = \varepsilon\, T f$$

mit

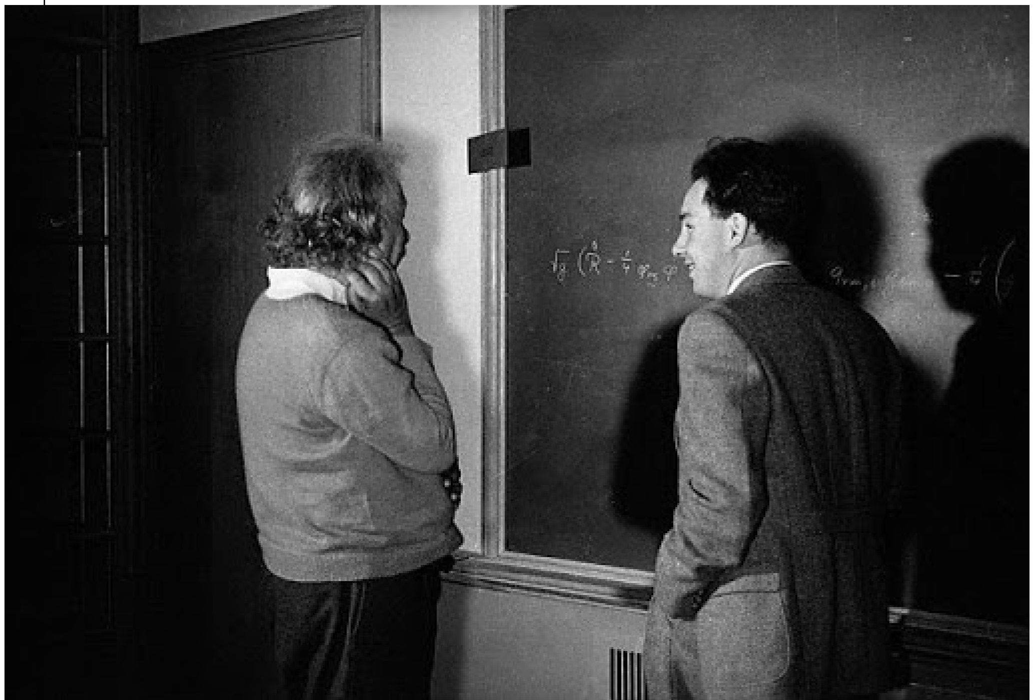
$$T f = \sum_\varrho \alpha_\varrho(y) \frac{\partial f}{\partial y_\varrho}. \tag{1}$$

Eine *infinitesimale Drehung*, d. h. eine lineare Transformation, die $R^2 = \sum_\varrho y_\varrho^2$ ungeändert läßt, kann stets linear aus den folgenden zusammengesetzt werden:

$$D_{\varrho\sigma} f = y_\varrho \frac{\partial f}{\partial y_\sigma} - y_\sigma \frac{\partial f}{\partial y_\varrho}. \tag{2}$$

[1]) V. Fock, ZS. f. Phys. **98**, 145, 1935. — [2]) W. Pauli, ebenda **36**, 336, 1926.

V. Bargmann, 1936



V. Bargmann, 1936. Algebraic approach

$$\hat{H},\ \hat{\vec{L}} = \vec{r}\times\hat{\vec{p}},\ \ \hat{\vec{p}} = -i\hbar\nabla,\ \ \hat{\vec{A}} = \sqrt{\mu}\left(\frac{1}{2\mu}\left(\hat{\vec{L}}\times\hat{\vec{p}} - \hat{\vec{p}}\times\hat{\vec{L}}\right) + \alpha\frac{\vec{r}}{r}\right)$$

$$\left[\hat{H},\hat{\vec{L}}\right] = \left[\hat{H},\hat{\vec{A}}\right] = 0.$$

$$\left[\hat{L}_i,\hat{L}_j\right] = i\hbar\varepsilon_{ijk}\hat{L}_k,\left[\hat{L}_i,\hat{A}_j\right] = i\hbar\varepsilon_{ijk}\hat{A}_k,\left[\hat{A}_i,\hat{A}_j\right] = -2\hat{H}\cdot i\hbar\varepsilon_{ijk}\hat{L}_k.$$

$$\mathcal{E} < 0,\ \hbar = 1.$$

$$\hat{N}_i = \frac{1}{\sqrt{-2\hat{H}}}\hat{A}_i,$$

$$\left[\hat{L}_i,\hat{L}_j\right] = i\varepsilon_{ijk}\hat{L}_k,\left[\hat{L}_i,\hat{N}_j\right] = i\varepsilon_{ijk}\hat{N}_k,\left[\hat{N}_i,\hat{N}_j\right] = i\varepsilon_{ijk}\hat{L}_k.$$

$$\hat{\vec{L}}\cdot\hat{\vec{N}}=\hat{\vec{N}}\cdot\hat{\vec{L}}=0,\ \hat{\vec{L}}^2+\hat{\vec{N}}^2=-\frac{\mathcal{E}_0}{2\hat{H}}.$$

$$\hat{\vec{J}}^{(1)}=\frac{1}{2}\left(\hat{\vec{L}}+\hat{\vec{N}}\right),\ \hat{\vec{J}}^{(2)}=\frac{1}{2}\left(\hat{\vec{L}}-\hat{\vec{N}}\right);\ \left(\hat{\vec{J}}^{(1)}\right)^2=\left(\hat{\vec{J}}^{(2)}\right)^2$$

$$\left[\hat{J}_i^{(a)},\hat{J}_j^{(b)}\right]=i\delta_{ab}\cdot\varepsilon_{ijk}\hat{J}_k^{(b)},\ (a,b=1,2.)$$

$$SO(4)=SO(3)\times SO(3)\rightarrow T^{(j_1,j_2)}=T^{j_1}\otimes T^{j_2};\ j_1=j_2=\frac{n-1}{2}.$$

$$\dim T^{(j_1,j_2)}=(2j_1+1)\cdot(2j_2+1)\equiv n^2.$$

$$\mathcal{E}>0,\ SO(4)\rightarrow SO(3,1);$$

$$\mathcal{E}=0,\ SO(4)\rightarrow \mathcal{G}_0,$$

SOVIET PHYSICS JETP VOLUME 23, NUMBER 1 JULY, 1966

THE LORENTZ GROUP AS A DYNAMIC SYMMETRY GROUP OF THE HYDROGEN ATOM

A. M. PERELOMOV and V. S. POPOV

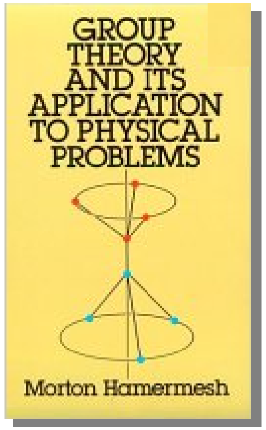
Institute of Theoretical and Experimental Physics, State Atomic Energy Commission

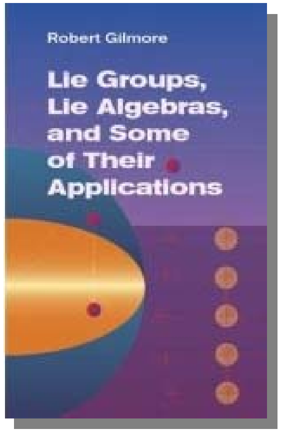
Submitted to JETP editor July 19, 1965

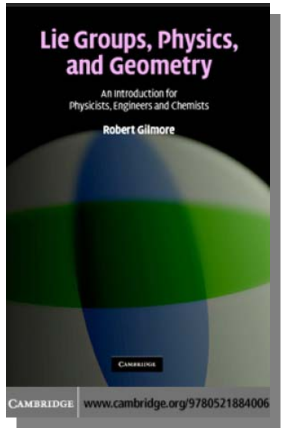
J. Exptl. Theoret. Phys. (U.S.S.R.) 50, 179-198 (January, 1966)

Symmetry group applications

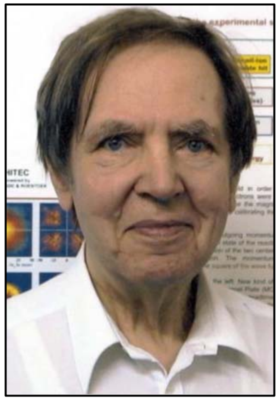
Group symmetry of N- dimensional oscillator – SU(N), Jauch, Hill, 1940. Classification of states, selection rules for atoms, molecules solids.
Lorentz and Poincaré groups unitary representation and classification of elementary particles.







J.L. Birman,
CUNY, USA

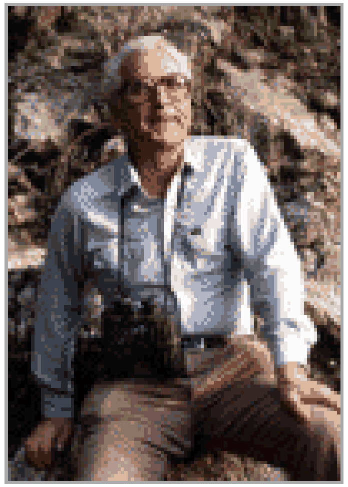


Yu.N. Demkov,
Leningrad University

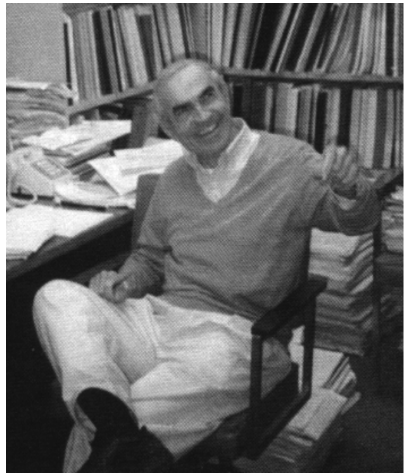
Ya.A. Smorodinsky, JIRN,
Dubna

Dynamical Symmetries

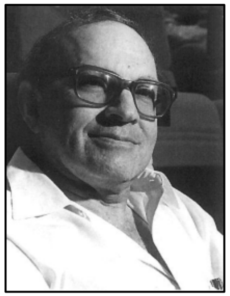
- Classification of hadrons: SU (3), SU (6) symmetry of the flavors, the quarks, the mass formulas
- Dynamical symmetries of quantum systems
- Spectrum generating algebra
- Coherent states



M. Gell-Mann

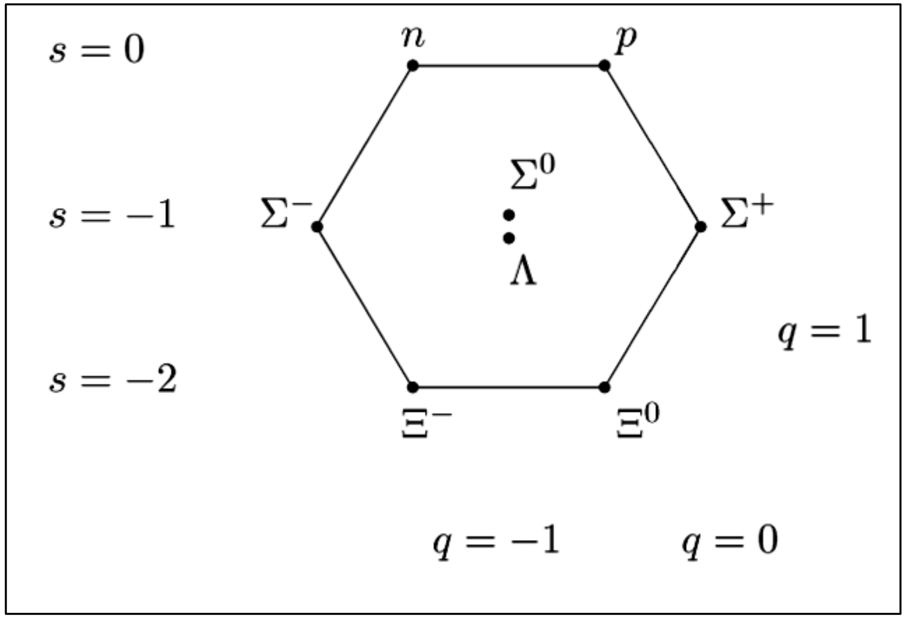


Asim Orhan Barut



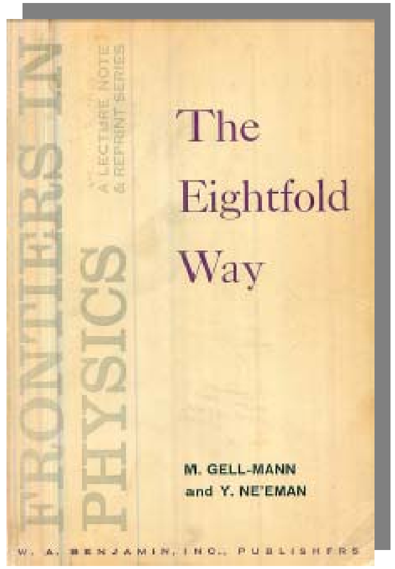
Y. Neeman

Eightfold way and dynamical (spectrum generating) groups



SU(3) ⊃ SU$_I$(2)

$8 = 1 \oplus 2 \oplus 2 \oplus 3$



<u>The baryon octet</u>

<u>Eightfold Way</u>, classification of hadrons into groups on the basis of their symmetrical properties, the number of members of each group being 1, 8 (most frequently), 10, or 27. The system was proposed in 1961 by M. Gell-Mann and Y. Ne˒eman. It is based on the mathematical symmetry group **SU(3)**; however, the name of the system was suggested by analogy with the Eightfold Path of Buddhism because of the centrality of the number eight..

A Simple Example - Harmonic Oscillator

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \hat{x}^2\right), \quad (\hbar = m = \omega = 1)$$

$$\hat{K}_0 = \frac{1}{4}\left(\hat{p}^2 + \hat{x}^2\right),\ \hat{K}_1 = \frac{1}{4}\left(\hat{p}^2 - \hat{x}^2\right), \hat{K}_2 = \frac{1}{4}\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right);$$

$$\hat{K}_\pm = \hat{K}_1 \pm i\,\hat{K}_2, \quad \left\{\hat{K}_+ = \frac{1}{2}\hat{a}^+\hat{a}^+, \hat{K}_- = \frac{1}{2}\hat{a}\hat{a}\right\},$$

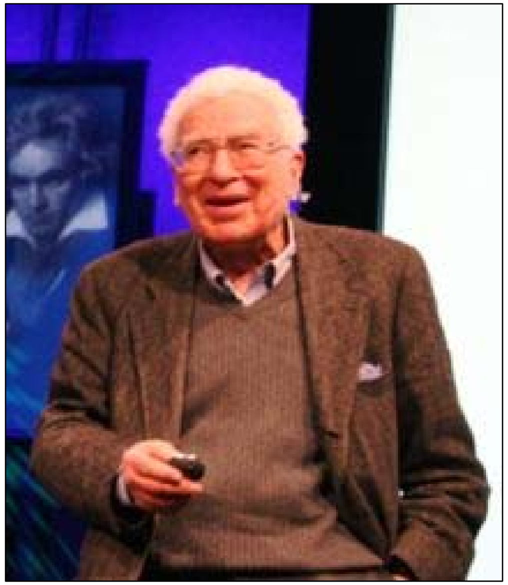
$$\left[\hat{K}_0, \hat{K}_\pm\right] = \pm\hat{K}_\pm, \left[\hat{K}_+, \hat{K}_-\right] = -2\hat{K}_0.$$

$$SU(1,1) = SL(2,R) = Sp(2,R) \approx SO(2,1).$$

$$\mathrm{T}_+^{\mathrm{k}},\ k = \frac{1}{4}, \frac{3}{4}.$$

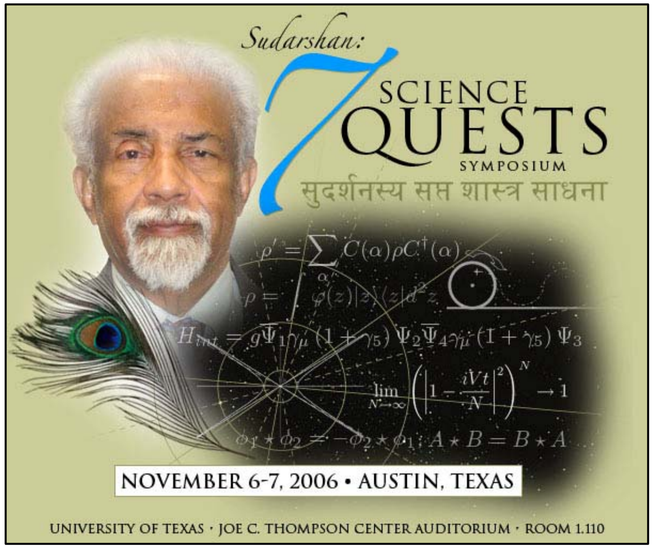
$$N - \mathrm{dim.oscillator:} \quad \mathrm{W_N} \wedge Sp(2N,R)$$

Dynamical symmetries: condensed matter, quantum optics



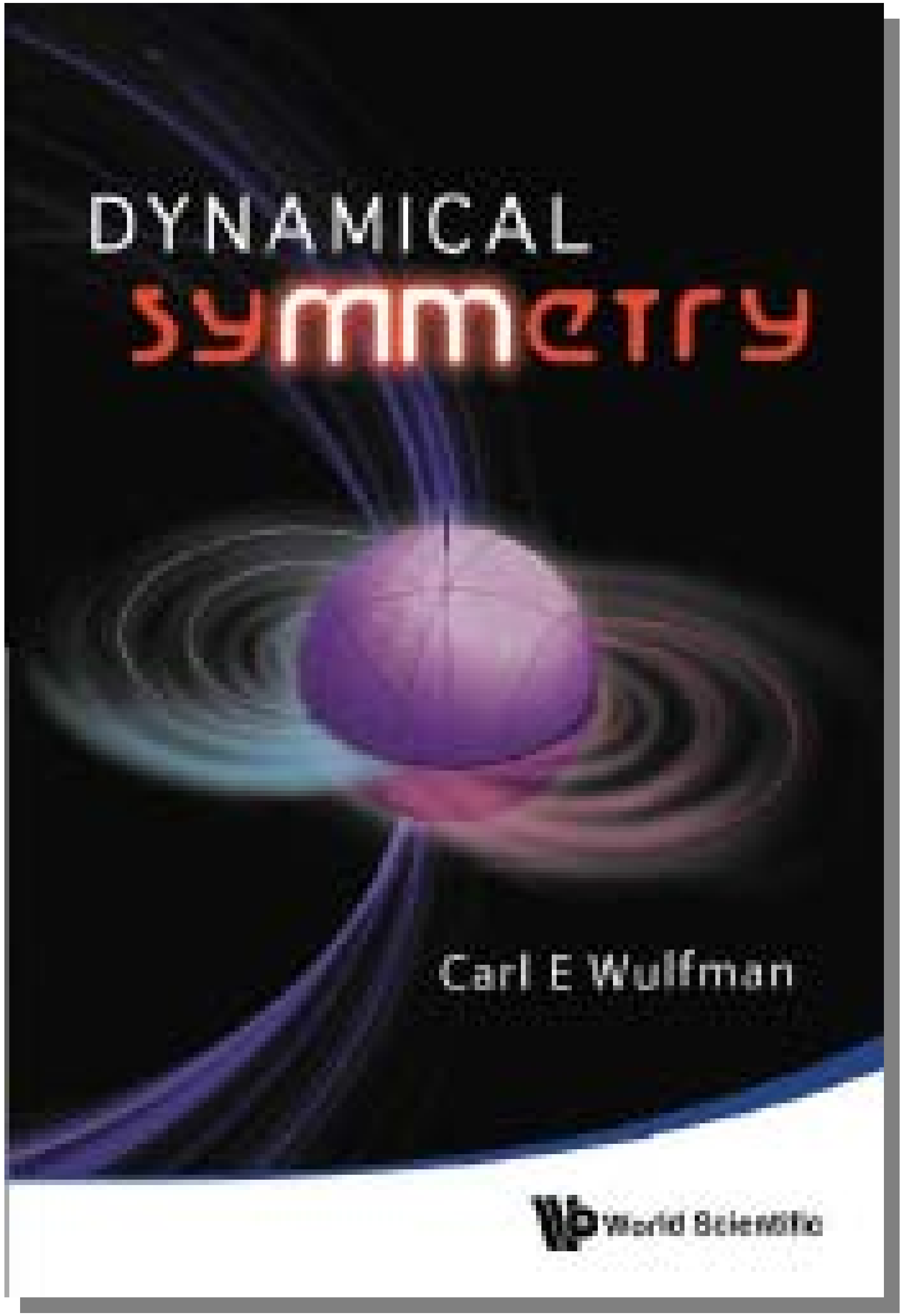
Murray Gell-Mann,

lecturing in 2007,



Ennackal Chandy George Sudarshan

SU(2), SU(3), SU(n), SO(4,2), SU(m,n), Sp(2N,R), W(N)^Sp(2N,R),…


DYNAMICAL
symmetry
Carl E Wulfman
World Scientific

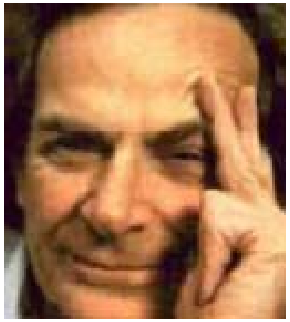
Dynamical symmetry

$$\hat{H} = \sum_{s_1,\dots,s_r} \omega_{s_1\dots s_r} \hat{A}_1^{s_1} \cdots \hat{A}_r^{s_r}$$

$$[\hat{A}_k, \hat{A}_l] = i\, C_{kl}^{m}\, \hat{A}_m$$

$$\hat{H} = \sum \omega_k \hat{A}_k$$

R. Feynman's method of operator exponent disentanglement



$$\hat{U}(t, t_0) = \hat{T}(g(t, t_0)) = \exp\left[-i \sum_k \lambda_k(t, t_0) \hat{A}_k\right]$$

In a linear case it is possible to find exact solution:

- Energy levels and corresponding wave functions (time independent Hamiltonian);
- Transition probabilities (time dependent Hamiltonian);
- Quasi energy and quasi energy states (periodic Hamiltonian).

Aharonov - Anandan geometric phase

Lie groups and an energy levels calculation

H-atom, SO(4), SO(4,2) — V.A. Fock, (1935) V. Bargmann, (1936); A.O. Barut, H. Kleinert, Yu.B. Rumer, A.I. Fet (1971)

SU(1,1), SU(N,1), Sp(2n,R), quantum optics, superfluidity

$$\hat{H} = f(\hat{A}_k),\ \ \hat{A}_k \in \mathrm{A},$$

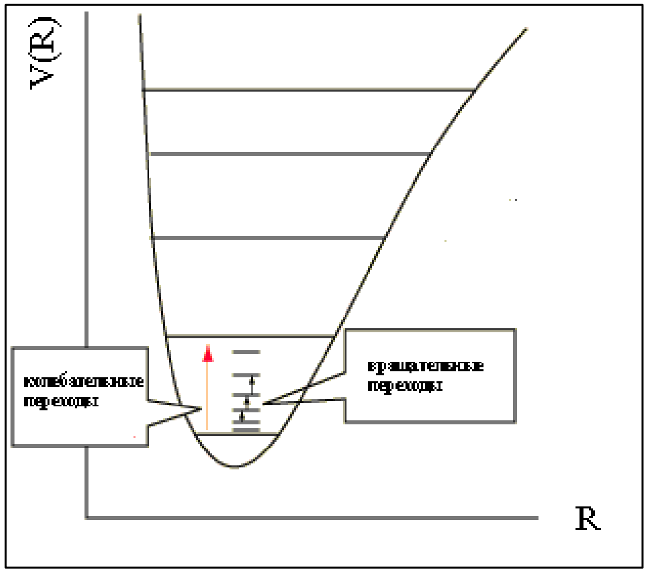
$$\mathrm{A} \supset \mathrm{A}' \supset \mathrm{A}'' \supset \ldots,$$

$$\hat{H} = \alpha f_1\left(C\left(\mathrm{A}\right)\right) + \alpha_1 f_2\left(C\left(\mathrm{A}'\right)\right) + \alpha_2 f_2\left(C\left(\mathrm{A}''\right)\right) + \ldots$$

Molecular rotational- vibrational levels

F. Iachello, R.D. Levine

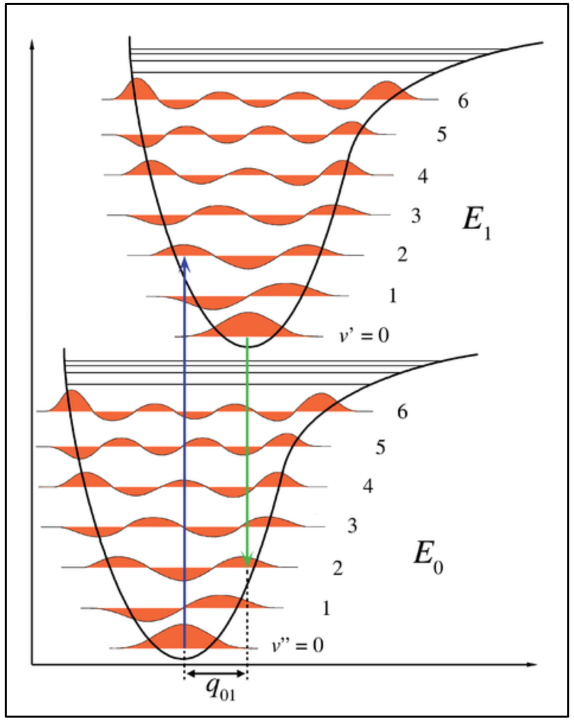
$$u(4) \supset so(4) \supset so(3) \supset so(2)$$



Molecular spectra & vibronic transitions

$W_n \wedge Sp(2N,\mathbb{R})$ -group and Franck – Condon overlap Integrals,

(harmonic approximation for vibrations)



$$\hat{H}' = \hat{V}\hat{H}\hat{V}^{+};$$

$$\vec{q}\,' = \hat{R}\vec{q} + \Delta\vec{q}, \quad \vec{q} = (q_1, q_2, \ldots, q_N);$$

$$\hat{R} = \left(\hat{L}'\right)^{-1}\hat{L}, \; \Delta\vec{q} = \left(\hat{L}'\right)^{-1}\Delta\vec{r}, \; \Delta\vec{r} = \vec{r}_0 - \vec{r}_0^{\,'}$$

$$\hat{\mathrm{V}} = \mathrm{T}(\vec{\delta})\,\hat{\mathrm{D}}(\vec{\rho})\,\hat{\mathrm{U}}(\hat{\mathrm{R}})$$

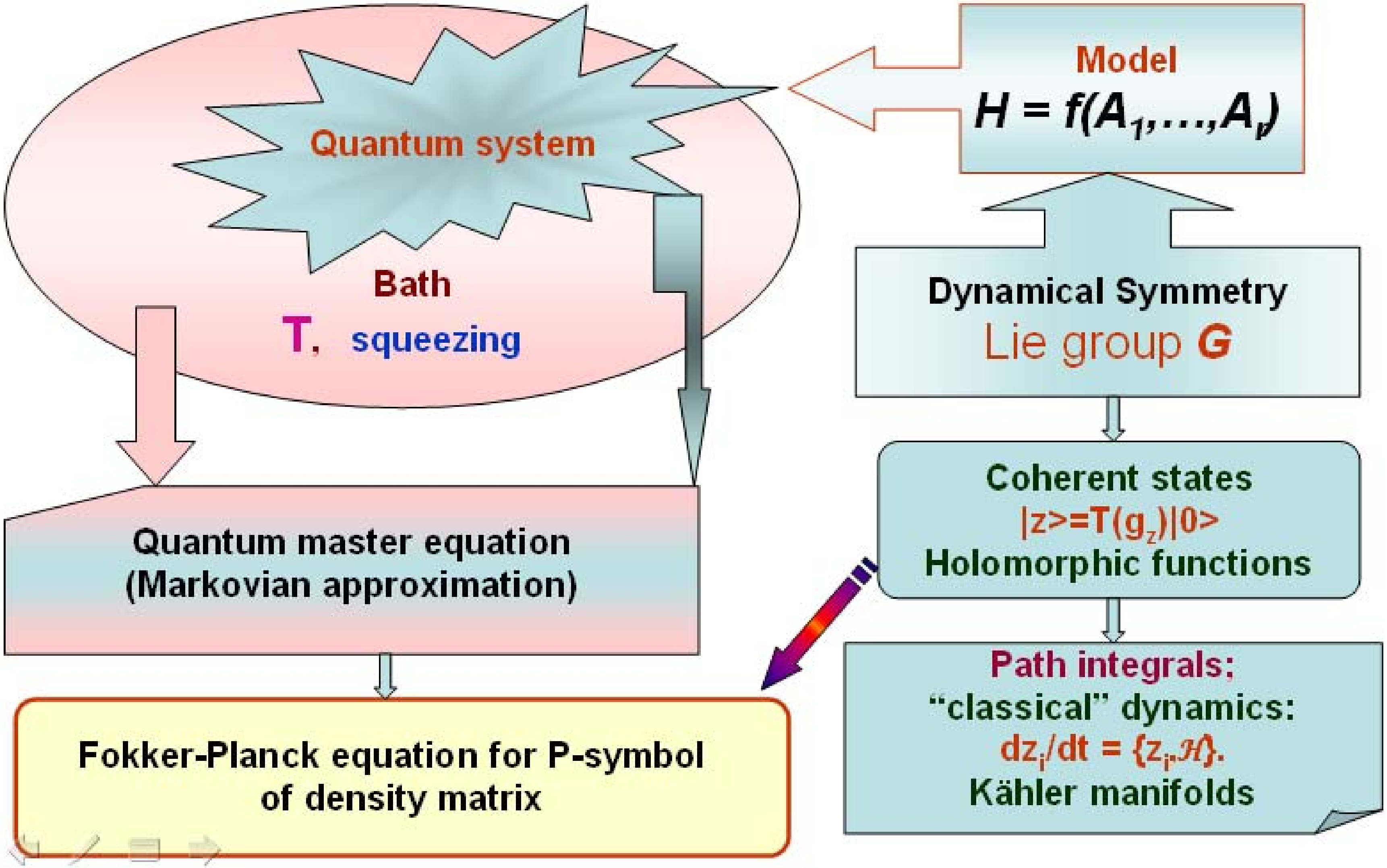
$$\mathrm{I}\left([m],[n']\right) = <[m] \,|\, [n']> = \left\langle [m] \right| \hat{\mathrm{V}} \left| [n] \right\rangle$$

$$\left\langle \vec{\beta} \right| \hat{\mathrm{V}} \left| \vec{\alpha} \right\rangle = \exp\left[-\frac{1}{2}\left(|\vec{\beta}|^2 + |\vec{\alpha}|^2 \right) \right] \times \sum_{[v],[n]} \frac{\vec{\beta}^{*[v]}\vec{\alpha}^{[n]}}{\sqrt{[v!]\cdot[n!]}} \left\langle [v] \right| \hat{\mathrm{V}} \left| [n] \right\rangle$$

$$I_{mn'} = \langle m | \hat{V} | n \rangle = \frac{I_{00}}{\sqrt{m!n!}} H_{mn}(\Delta, \Delta'), \;\; I_{00} = \sqrt{(\omega + \omega')/(2\sqrt{\omega\omega'})}\, \exp\left[-\frac{\omega\omega'}{2(\omega+\omega')}(\Delta x)^2 \right],$$

$$\Delta = \frac{\omega'\sqrt{2\omega}}{\omega + \omega'}\Delta x, \;\; \Delta' = -\frac{\omega\sqrt{2\omega'}}{\omega+\omega'}\Delta x; \;\; (\hbar = M = 1).$$

Group-theoretical approach



Model hamiltonians, dynamical groups and coherent states

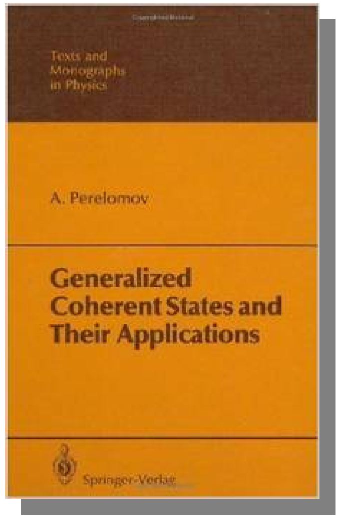
Klauder, Glauber, Sudarshan, Perelomov, Berezin, Gilmore, …

$$\hat{H} = \sum_{s_1,\ldots,s_r} \omega_{s_1\ldots s_r} \hat{A}_1^{s_1} \cdots \hat{A}_r^{s_r}$$

$$[\widehat{A}_k, \widehat{A}_l] = i\, C_{kl}^m\, \widehat{A}_m$$

$$\widehat{H} = \sum_k \omega_k \widehat{A}_k,$$

$$\widehat{U}(t, t_0) = \widehat{T}(g(t,t_0)) = \exp\left[-i \sum_k \lambda_k(t,t_0)\widehat{A}_k\right]$$



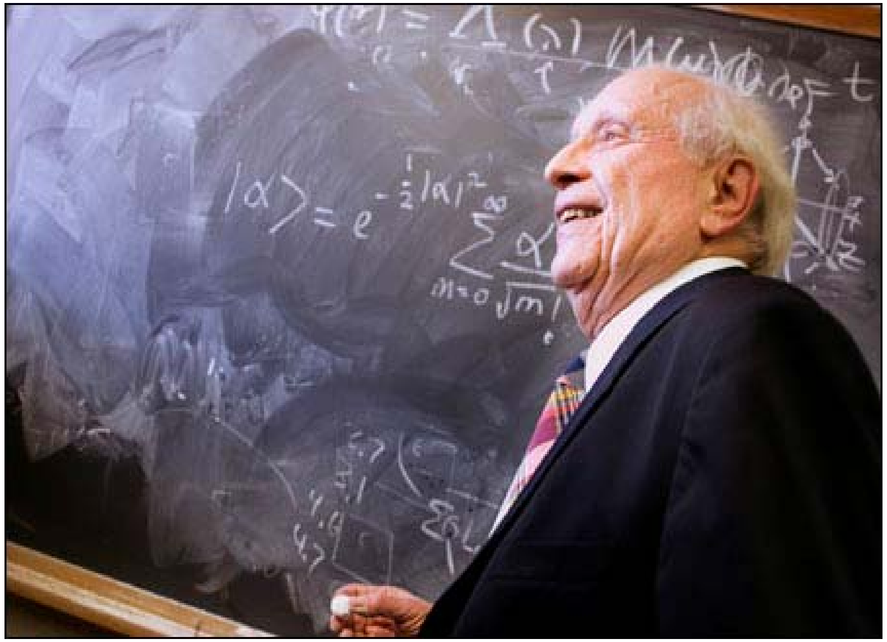
$$\forall g_\xi \in \mathcal{X} = G/G_0 \quad \longmapsto \quad |\xi >= \hat{T}(g_\xi)|0 >$$

$$\hat{T}(h)|\Psi_0\rangle = e^{i\alpha(h)}|\Psi_0\rangle$$

1972

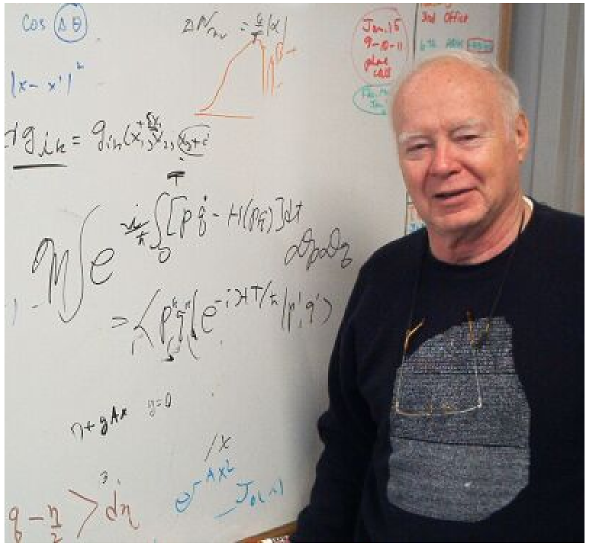
$$\xi \longmapsto z \equiv (z^1, \ldots, z^n), \qquad 2n = dim(G/G_0).$$

Heisenberg – Weyl group W_1 and coherent states



R. Glauber, 2005, October 5

1963



J.R. Klauder

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\beta|^2+|\alpha|^2-2\beta^*\alpha)} \neq \delta(\alpha-\beta)$$

Coherence properties of quantum electromagnetic fields, lasers

Holomorphic functions representation

$$|\Psi> \longmapsto \Psi(z) =< z|\Psi > / < z|0 >, \quad |z >\equiv |\xi(z^1, \ldots, z^n) >,$$

$$\Psi(z) = \int_{\mathcal{X}} K(z, \bar{w}) \, \Psi(w) \, \exp\left[-\rho(w, \bar{w})\right] \, d\mu(w, \bar{w}),$$

$$K(z, \bar{w}) =< z|w > / < z|0 >< 0|w >$$

$$\omega^2 = i \sum_{\alpha,\beta} \frac{\partial^2 \ln K(z, \bar{z})}{\partial z^\alpha \, \partial \bar{z}^\beta} \, dz^\alpha \wedge d\bar{z}^\beta$$

↑

G-invariant Kähler's 2-form

$$\left(\hat{F} \, \Psi\right)(z) = \int_{\mathcal{X}} \mathcal{F}(z, \bar{w}) \, \frac{K(z, \bar{w})}{K(w, \bar{w})} \, \Psi(w) d\mu(w, \bar{w}),$$

$$\mathcal{F}(z, \bar{w}) = \frac{< z|F|w >}{< z|w >}.$$

$$\hat{F} = \hat{F}_1 \cdot \hat{F}_2, \longrightarrow \mathcal{F}(z, \bar{z}) = \int_{\mathcal{X}} \mathcal{F}_1(z, \bar{w}) \, \mathcal{F}_2(w, \bar{z}) \, \frac{K(z, \bar{w}) \, K(w, \bar{z})}{K(z, \bar{z}) \, K(w, \bar{w})} \, d\mu(w, \bar{w}).$$

$$\{\mathcal{F}_1, \, \mathcal{F}_2\}(z, \bar{z}) = \frac{i}{\hbar} \sum_{\alpha, \bar{\beta}} g^{\alpha\bar{\beta}} \, \left[\frac{\partial \mathcal{F}_1}{\partial z^\alpha} \frac{\partial \mathcal{F}_2}{\partial \bar{z}^\beta} - \frac{\partial \mathcal{F}_1}{\partial \bar{z}^\beta} \frac{\partial \mathcal{F}_2}{\partial z^\alpha} \right]$$

Path Integrals in CS - representation

$$\hat{U}(t,t_0) \equiv \hat{U}(t_N,t_0) = \hat{U}(t_N,t_{N-1})\hat{U}(t_{N-1},t_{N-2})\cdots\hat{U}(t_1,t_0)$$

$$\hat{U}(t_k,t_{k-1}) = T_D \exp\left\{-\frac{i}{\hbar}\int_{t_{k-1}}^{t_k}\hat{H}(\tau)\,d\tau\right\} \approx \hat{I} - \frac{i}{\hbar}\int_{t_{k-1}}^{t_k}\hat{H}(\tau)\,d\tau + O(|\Delta t|)$$

$$\mathcal{U}(z,\bar{z}|t,t_0) = \int \exp\left(\frac{i}{\hbar}\mathcal{S}\right)\prod_{t_0<\tau<t} d\mu\left(z(\tau),\bar{z}(\tau)\right),$$

$$\mathcal{S} = \int_{t_0}^{t}\left\{i\hbar\sum_{\alpha=1}^{n}\left[\bar{\mathcal{Z}}_\alpha(\tau+0)\,\dot{z}^\alpha(\tau) - \mathcal{Z}_\alpha(\tau)\,\dot{\bar{z}}^\alpha(\tau)\right] - \mathcal{H}\left(z(\tau),\bar{z}(\tau+0)|\tau\right)\right\}d\tau -$$

$$-\frac{i\hbar}{2}\,ln\left[\frac{K(z(t),\bar{z}(t))}{K(z(t_0),\bar{z}(t_0))}\right].$$

$$\boxed{\mathcal{Z}_\alpha = \frac{1}{2}\frac{\partial}{\partial\bar{w}^\alpha}\,ln\,K(w,\bar{w})\Big|_{\bar{w}=\bar{z}(\tau)}^{w=z(\tau)}}$$

$$\boxed{\mathcal{U}(z,\bar{z}|t,t_0) = \lim_{\epsilon\to 0}\int \exp\left(\frac{i}{\hbar}\mathcal{S}_\epsilon\right)\prod_{t_0<\tau<t} d\mu\left(z(\tau),\bar{z}(\tau)\right),}$$

$$\mathcal{S}_\epsilon = \mathcal{S} + \Delta\mathcal{S}(\epsilon)$$

"Classical" Equations

$$\mathcal{U}_{cl}(z,\bar{z}|t,t_0) = \widetilde{\mathcal{U}}\, \exp\left(\frac{i}{\hbar}\mathcal{S}_{cl}\right),$$

$$\frac{i}{\hbar}\frac{\partial \mathcal{H}(z,\overline{z})}{\partial z^{\alpha}} = \sum_{\beta=1}^{n} \frac{\partial^2 \ln K(z,\overline{z})}{\partial z^{\alpha}\partial \overline{z}^{\beta}} \dot{\overline{z}}^{\beta},$$

$$-\frac{i}{\hbar}\frac{\partial \mathcal{H}(z,\overline{z})}{\partial \overline{z}^{\alpha}} = \sum_{\beta=1}^{n} \frac{\partial^2 \ln K(z,\overline{z})}{\partial \overline{z}^{\alpha}\partial z^{\beta}} \dot{z}^{\beta}.$$

$$\dot{z}^{\alpha} = \{z^{\alpha}, \mathcal{H}\}, \quad \dot{\bar{z}}^{\alpha} = \{\bar{z}^{\alpha}, \mathcal{H}\}; \quad \alpha = 1, \dots, n.$$

$$z(t_0) = z, \quad \bar{z}(t) = \bar{z}.$$

$$\{\mathcal{F}_1, \mathcal{F}_2\}(z,\bar{z}) = \frac{i}{\hbar}\sum_{\alpha,\bar{\beta}} g^{\alpha\bar{\beta}} \left[\frac{\partial \mathcal{F}_1}{\partial z^{\alpha}}\frac{\partial \mathcal{F}_2}{\partial \bar{z}^{\beta}} - \frac{\partial \mathcal{F}_1}{\partial \bar{z}^{\beta}}\frac{\partial \mathcal{F}_2}{\partial z^{\alpha}}\right]$$

$$g_{\alpha\overline{\beta}} = \frac{\partial^2 \ln K(z,\overline{z})}{\partial z^{\alpha}\partial \overline{z}^{\beta}}, \quad g^{\alpha\overline{\beta}} \cdot g_{\kappa\overline{\beta}} = \delta^{\alpha}_{\kappa}, \; g^{\alpha\overline{\beta}} \cdot g_{\alpha\overline{\gamma}} = \delta^{\overline{\beta}}_{\overline{\gamma}}.$$

Quantum oscillator in a field of external force

$$\hat{H}(t) = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_0^2\,\hat{x}^2 - \hat{x}\,F(t)$$

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t),\;\; \hat{H}_0 = \omega_0\left(\hat{a}^+\hat{a} + \frac{1}{2}\right),\;\; \hat{H}_1(t) = -F(t)\sqrt{\frac{1}{2\omega_0}}\,(\hat{a} + \hat{a}^+)$$

Oscillator coherent states, **R. Glauber**, 1963

$$|\alpha\rangle = \hat{D}|0\rangle = e^{\alpha\hat{a}^+ - \bar{\alpha}\hat{a}}|0\rangle,\;\; \hat{D}(\alpha) = e^{\alpha\hat{a}^+ - \bar{\alpha}\hat{a}},\; \hat{a}|0\rangle = 0,\; \langle 0|0\rangle = 1$$

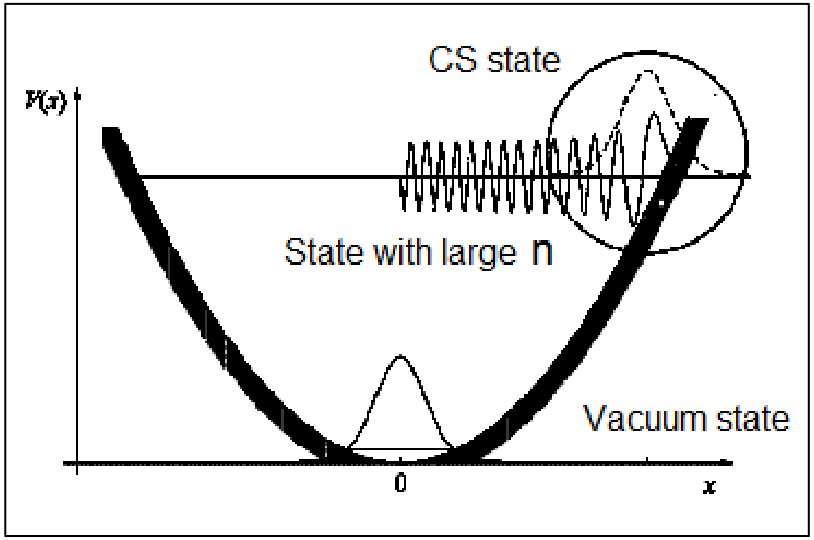
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,\;\; |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n}}|n\rangle$$

$$\hat{H}(t) = \frac{1}{2M}\hat{P}^2 + \frac{M\omega_0^{\,2}}{2}x^2 - \alpha \mathrm{E}(t)x,$$

$$\hat{a} = \sqrt{\frac{M\omega_0}{2}}\left(\hat{x} + i\frac{\hat{p}}{M\omega_0}\right), \quad \hat{a}^+ = \sqrt{\frac{M\omega_0}{2}}\left(\hat{x} - i\frac{\hat{p}}{M\omega_0}\right)$$

$$\hat{H}(t) = \omega_0\left(\hat{a}^+\hat{a} + 1/2\right) - \frac{\alpha \mathrm{E}(t)}{\sqrt{2\omega_0 M}}\left(\hat{a}^+ + \hat{a}\right)$$

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = \hat{H}(t)|\Psi(t)\rangle, \quad |\Psi(t)\rangle = e^{i\phi(t)}|z(t)\rangle$$



$$\hat{a}|z\rangle = z|z\rangle$$

$$\dot{z} + i\omega_0 z = -i\,f(t)$$

$$z(t) = e^{-i\omega_0 t}z(0) - i\int_0^t e^{-i\omega_0(t-t')}f(t')\,dt'$$

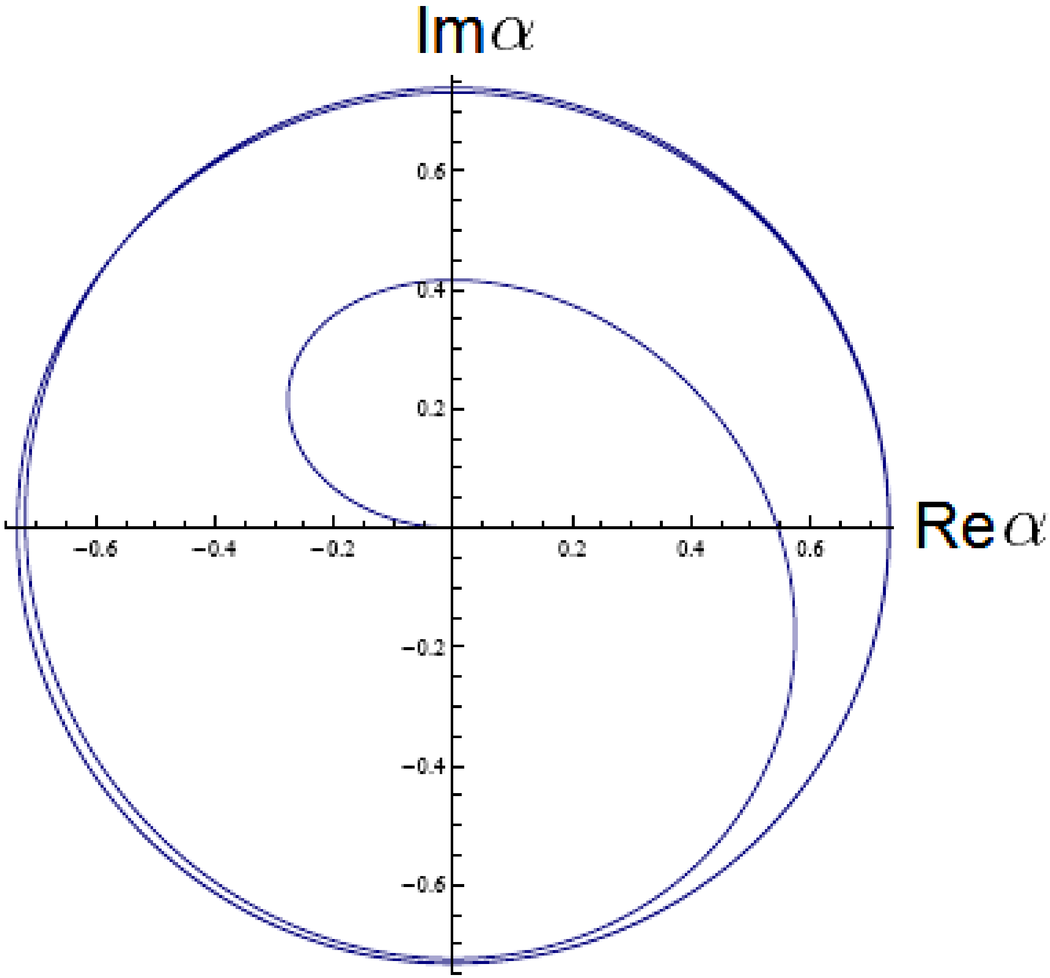
$$f(t) = -\frac{\alpha}{\sqrt{2\omega_0 M}}\mathrm{E}(t)$$

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$$

$$\langle x|z(t)\rangle = \psi_z(x,t) = \left(\frac{M\omega_0}{\pi}\right)^{\frac{1}{4}}\exp\left[-\left(\sqrt{\frac{M\omega_0}{2}}x - z(t)\right)^2 - \frac{|z(t)|^2}{2} + \frac{z^2(t)}{2}\right]$$

$$P(x,t) = |\langle \mathbf{x}|z(t)\rangle|^2$$

$$\dot{\alpha} = -i\omega_0\alpha - F(t)/\sqrt{2\omega_0}\ ,\ F(t) = F_0 e^{-\left(\frac{t}{\tau}\right)^2}\cos(\Omega t)$$



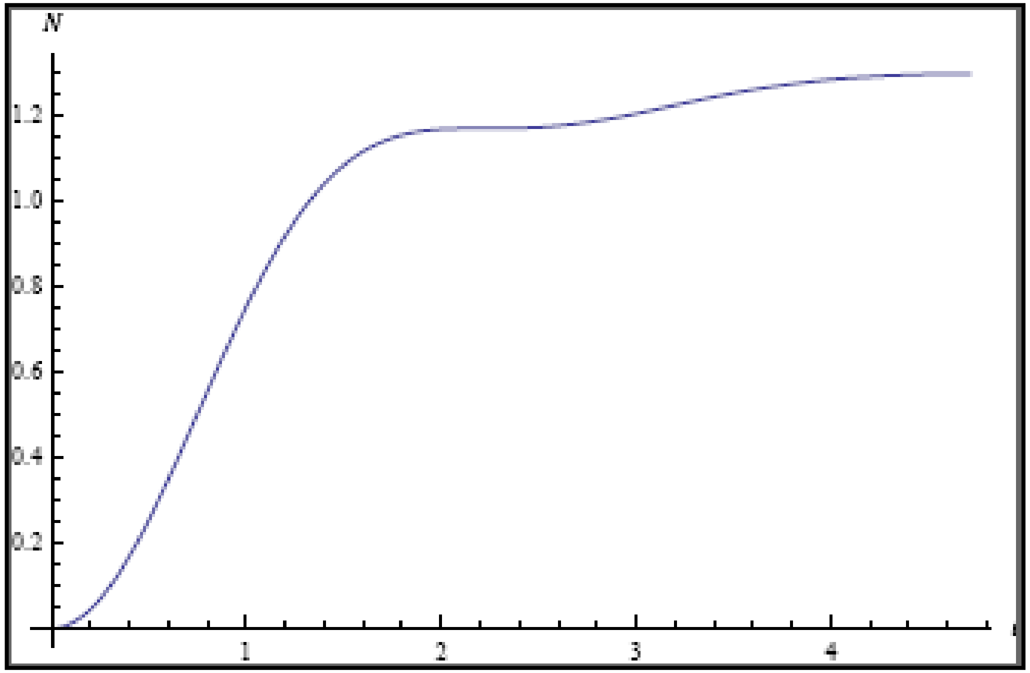
$\alpha(0) = 0,$

$\omega_0 = 1,$

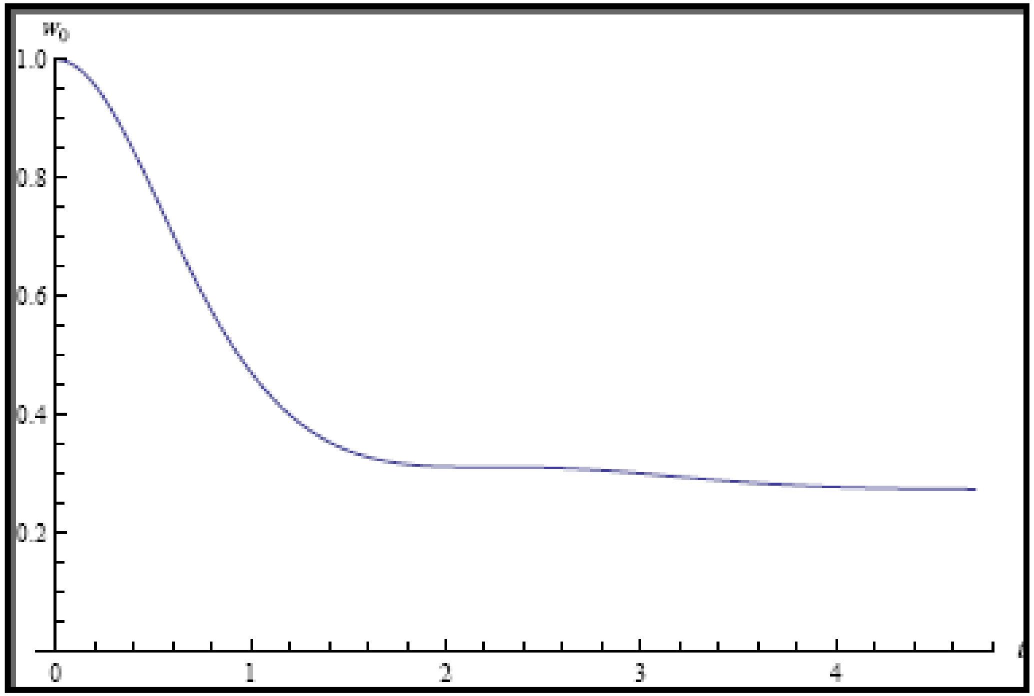
$F_0 = 2,$

$\tau = 5,$

$\Omega = 7/5$

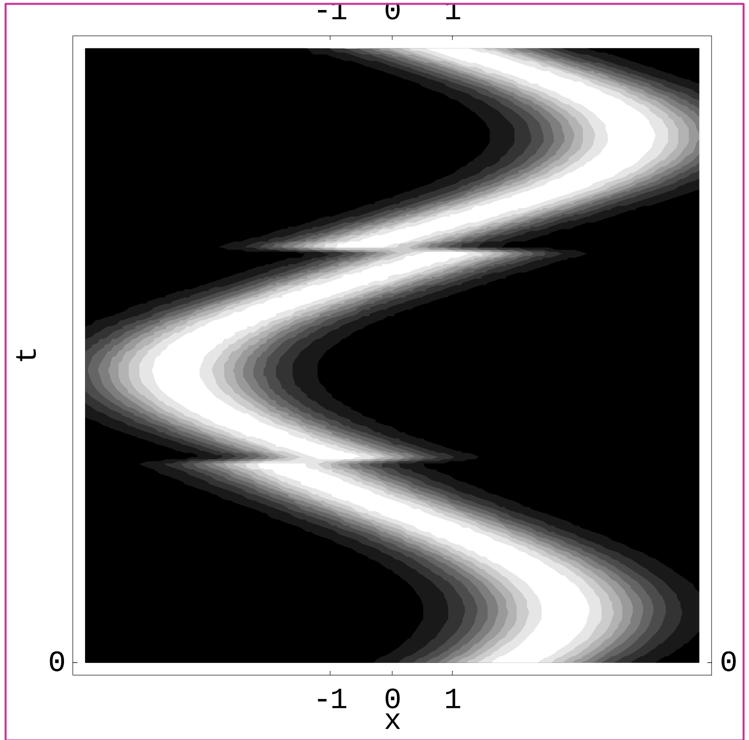


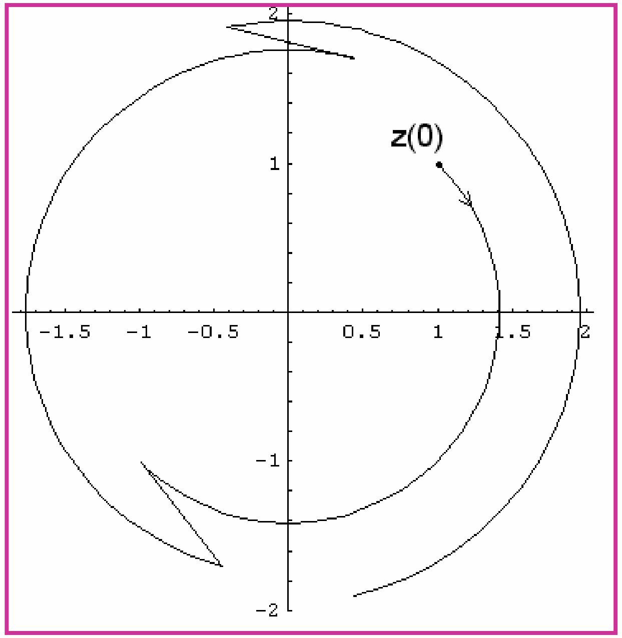
$\langle n(t) \rangle$

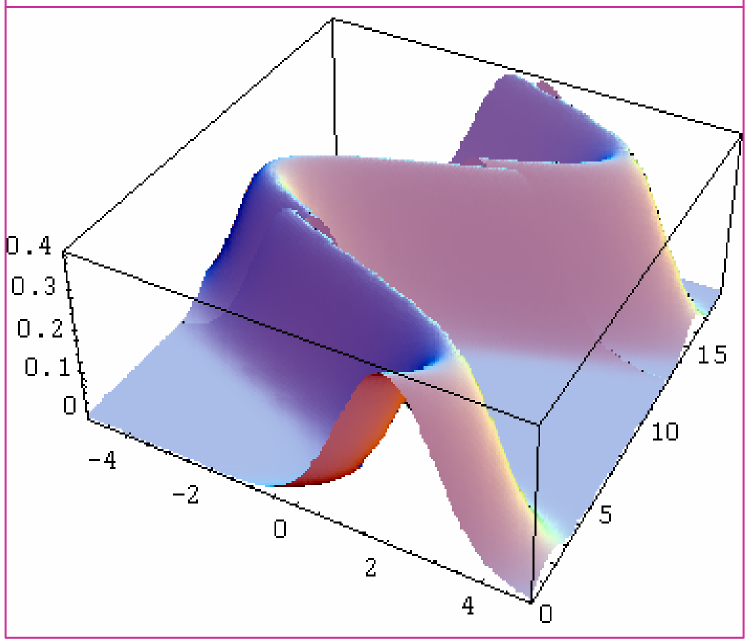


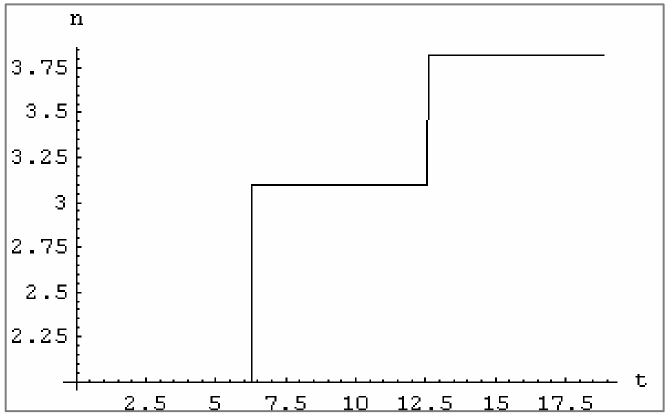
$\langle W_0(t) \rangle$

$$E(t) = E_0 \sum_{l=1}^{\infty} \delta(t - lT), \quad T \neq T_0 = \frac{2\pi}{\omega_0}$$



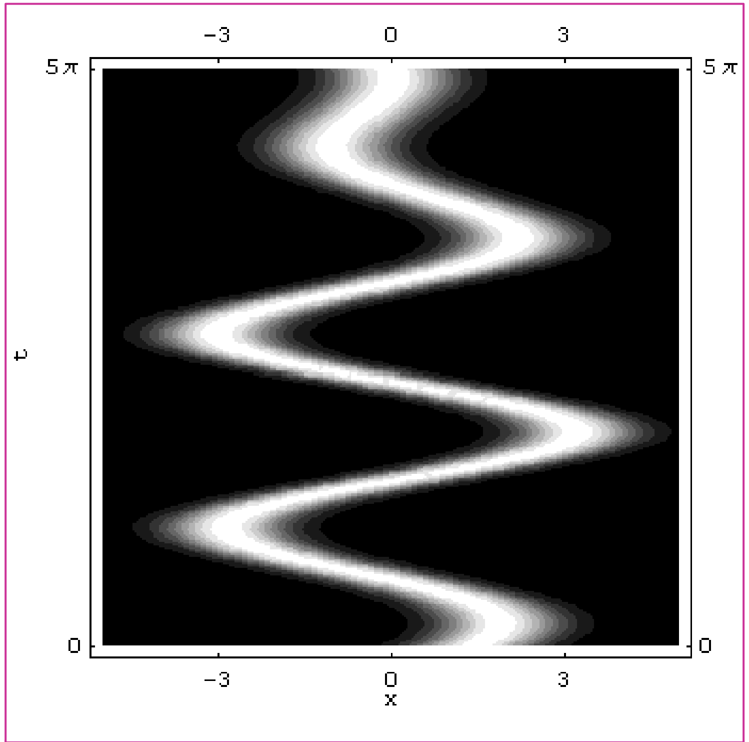


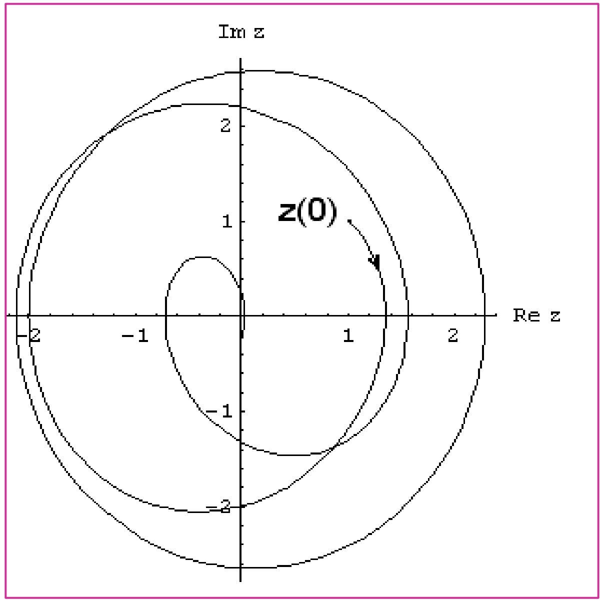


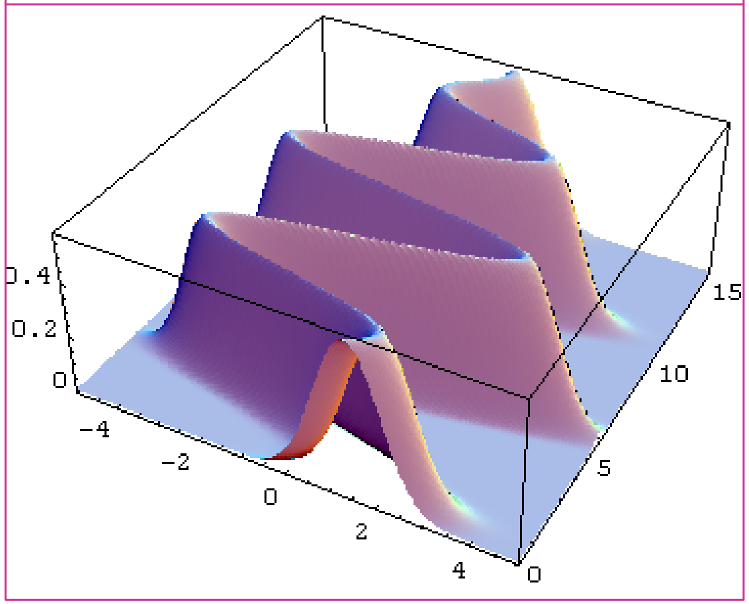


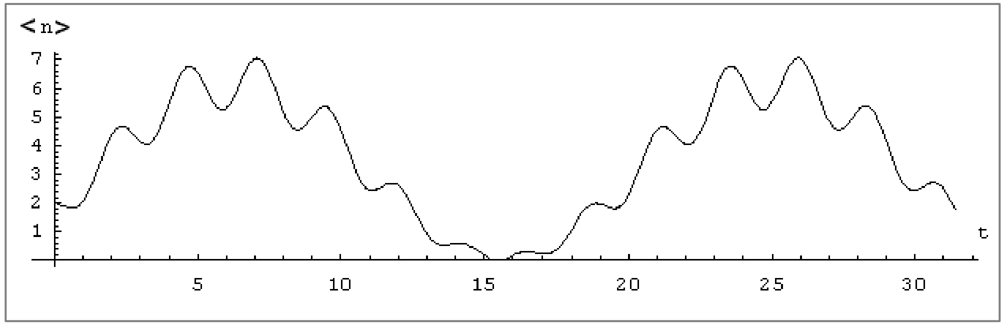
$|\Psi(x,t)|^2 \leftarrow$ a quantum carpets

$$f(t) = A \sin(\omega t)$$









N-level atoms in classical fields. $G = SU(N)$, ($|k\rangle\langle l| \in u(N)$)

$$|z\rangle = \left(1 + \sum_{\alpha=1}^{n-1} z^\alpha \bar{z}^\alpha\right)^{-p} \prod_{\beta=1}^{n-1} \exp(z^\beta \hat{E}_\beta^+)|0\rangle$$

$$z = (z^1, \ldots z^{n-1}) \in SU(n)/U(n-1) \approx CP^{n-1}$$

$$K(z, \bar{w}) = \left(1 + \sum_{\alpha=1}^{n-1} z^\alpha \bar{w}^\alpha\right)^{2p}$$

$$i\dot{z}^\alpha = H_{\alpha n}(t) + \sum_{\beta=0}^{n-1} \left[H_{\alpha\beta}(t) - H_{nn}(t)\delta_{\alpha\beta}\right] z^\beta - \sum_{\beta=0}^{n-1} H_{n\beta}(t) z^\alpha z^\beta$$

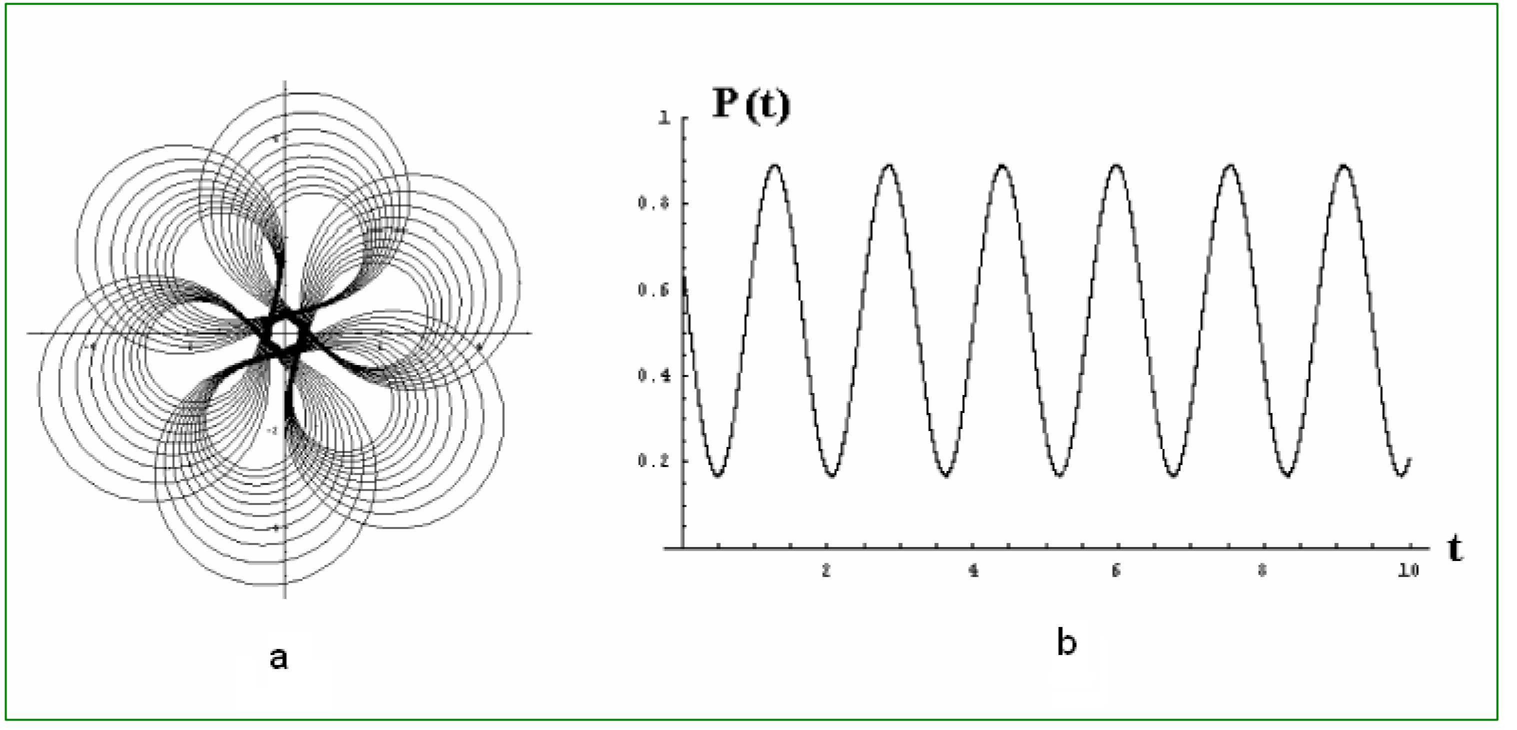
$$z^\alpha(t) = \frac{\sum_{\beta=1}^{n-1} \theta_{\alpha\beta}(t, t_0) z^\beta(t_0) + \theta_{n\alpha}(t, t_0)}{\sum_{\beta=1}^{n-1} \theta_{n\beta}(t, t_0) z^\beta(t_0) + \theta_{nn}(t, t_0)}.$$

G= SU(2), (2j+1)-level atom

$$|z\rangle = (1 + z\bar{z})^{-j} e^{z\hat{J}_+}|0\rangle$$

$$i\dot{z} = A(t) + \omega_0 z - \bar{A}(t) z^2,$$

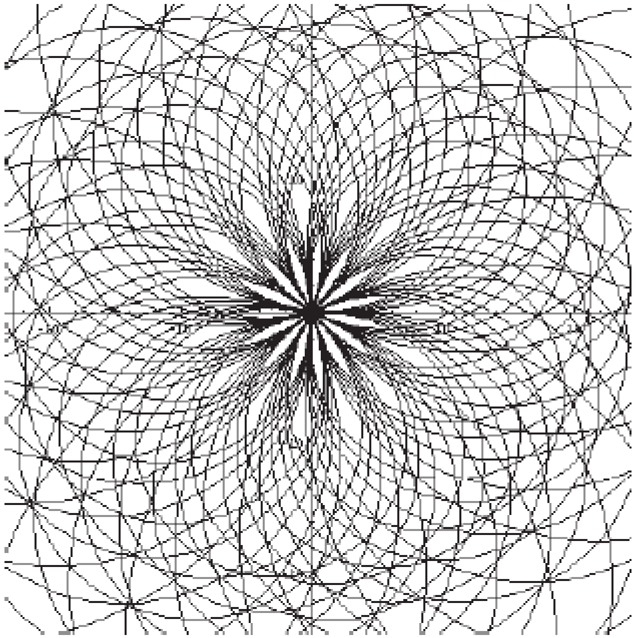
$$P(t) = \frac{A^2 \sin^2 \Omega t}{(\omega - \omega_0)^2 + A^2}$$ ← ***j=1/2***

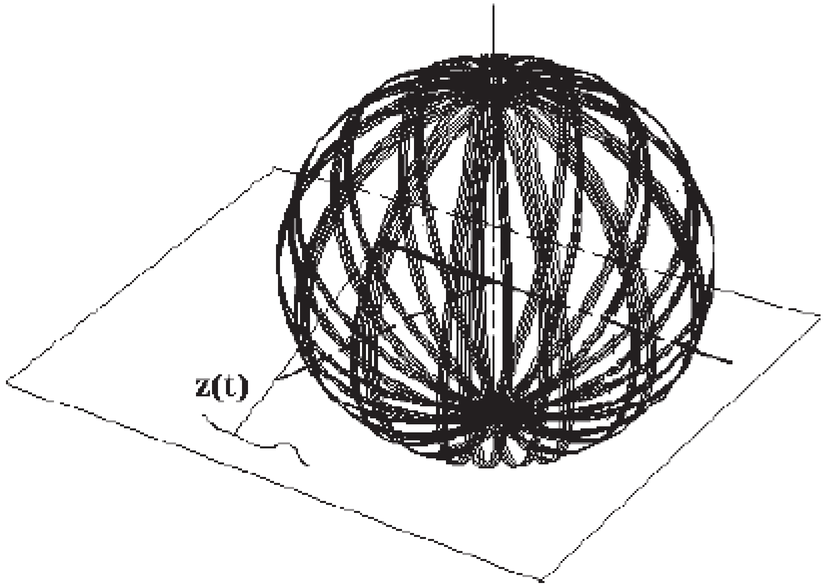


Coherent state dynamics. (a) – trajectory, (b) – Upper level probability P(t).

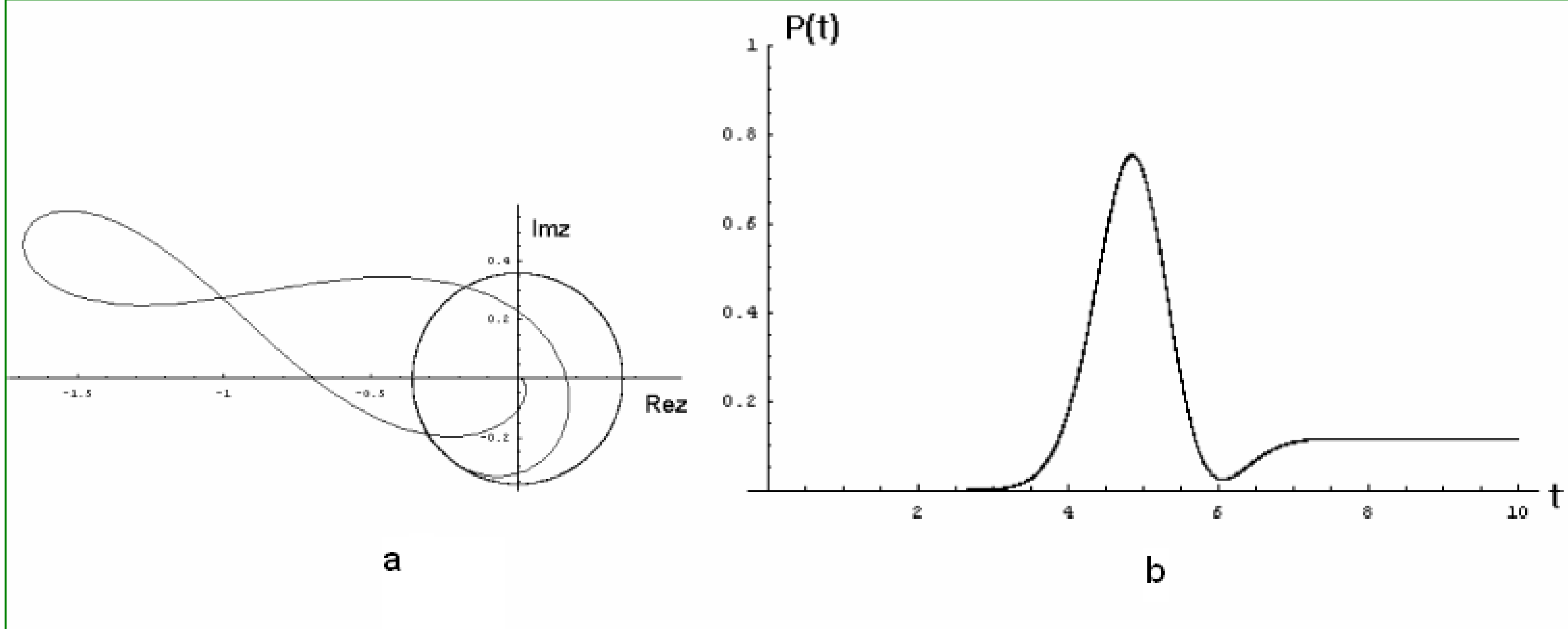
(*z(0) = 1+i, ω_0 = 1. ω = 2/3, A = 2*)

Complex plane & Bloch's sphere. Qubits





$$z = e^{i\phi} \tan\left(\frac{\vartheta}{2}\right)$$

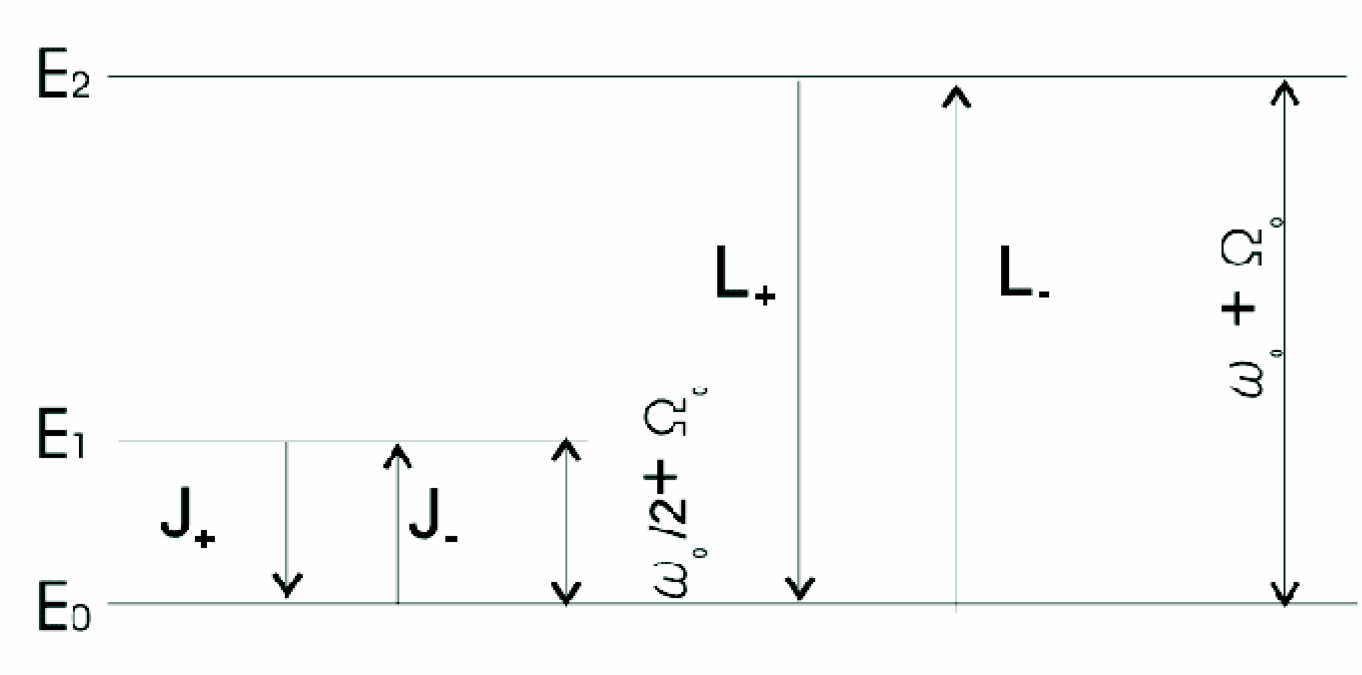


$$A(t) = A \exp\left[-i\omega t - (t - t_0)^2/\tau^2\right]$$

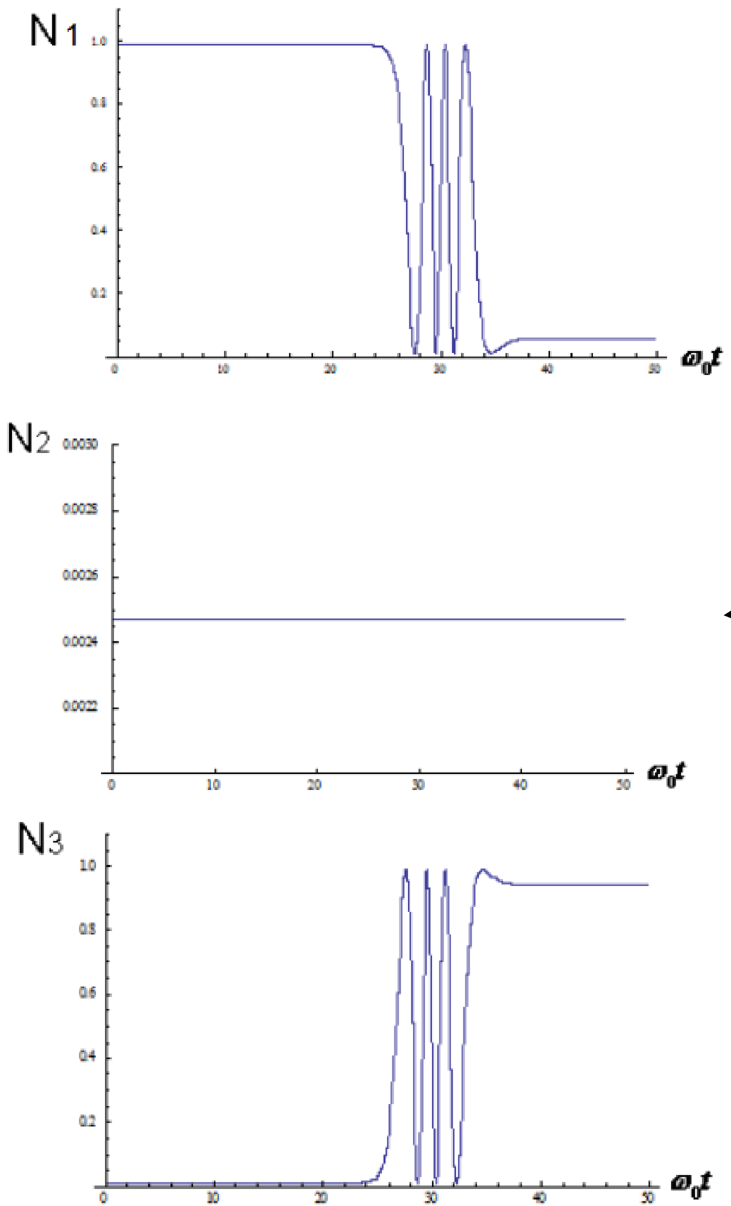
SU(2) CS generation: (a) – trajectory, (b) Upper level probability *P(t).*

($z(0) = 0$, $\omega_0 = 1$. $\omega = 2$, $A = 1.5$, $t_0 = 5$, $\tau = (3/5)^{1/2}$)

Three-level atom, G = SU(3). Qutrits



$$\hat{H} = \hat{H}_0 + \hat{H}_{int} = \hbar\omega_0 \hat{H}_1 + \hbar\Omega_0 \hat{H}_2 + (A\hat{J}_+ + B\hat{K}_+ + D\hat{L}_+ + h.c.)$$



We have not here a funny pictures for CS trajectories as in *SU(2)* case…

coherent population trapping

$$A(t) = |\Omega_{12}| \exp\left[-i\omega_1 t - \frac{(t-t_{01})^2}{\sigma_1^2}\right]$$

$$D(t) = |\Omega_{13}| \exp\left[-i\omega_3 t - \frac{(t-t_{03})^2}{\sigma_3^2}\right]$$

Case of V – atom transitions and dynamics of the level populations.

$\omega_0/2 + \Omega_0 = 0.8;\ \omega_0 + \Omega_0 = 1.3;\ \Omega_{12} = 0;\ \Omega_{13} = 2;\ \ \omega_1 = 0\,;\ \omega_3 = 1.3;\ \tau = 20;\ \sigma_3^2 = 10$

Q-corrections

$$\left|\Psi(t)\right\rangle = \iint d^2\alpha \, d\mu(\varsigma) f_{\mathrm{E}}(\alpha,\zeta;\alpha_0,\zeta_0 \,|\, t)\left|\alpha,\zeta\right\rangle.$$

$$i\hbar\frac{\partial}{\partial t}\left|\Psi(t)\right\rangle = \hat{H}\left|\Psi(t)\right\rangle$$

$$d^2\alpha = \frac{d\,\mathrm{Re}\,\alpha \, d\,\mathrm{Im}\,\alpha}{\pi}, \quad d\mu(\varsigma) = \frac{2j+1}{\pi}\frac{d\,\mathrm{Re}\,\varsigma \, d\,\mathrm{Im}\,\varsigma}{\left(1+|\varsigma|^2\right)^2}.$$

$$\left|\Psi(0)\right\rangle = |\alpha(0)> \otimes |\zeta(0)>. \qquad \mathrm{E} = \left\langle E\right\rangle = \left\langle\Psi(0)\right|\hat{H}\left|\Psi(0)\right\rangle = \left\langle\Psi(t)\right|\hat{H}\left|\Psi(t)\right\rangle;$$

$$f_E = \sum_{m=-j}^{j} f_m(\alpha;t)\psi_m^j(\zeta); \;\; f_m(\alpha;t) \sim \exp\left[-\kappa_m(t)\cdot\left|\alpha-\alpha_{cl}(t)\right|^2\right]; \; \psi_m^j(\zeta) = \left\langle\zeta\,|\,j,m\right\rangle$$

The initial condition: $\lim_{t\to 0} f_{\mathrm{E}}(\alpha,\zeta\,|\,\alpha_0,\zeta_0;t) = \delta_2(\alpha-\alpha_0)\delta_2(\zeta-\zeta_0)$

$$\hat{a}\,|\,\alpha >= \alpha\,|\,\alpha >; \; \hat{a}^+\,|\,\alpha >= \left(\partial/\partial\alpha + \bar{\alpha}/2\right)|\,\alpha >;$$

$$\hat{a}^+\hat{a}\,|\,\alpha >= \left(\alpha\cdot\partial/\partial\alpha + \alpha\bar{\alpha}/2\right)|\,\alpha >.$$

$$\hat{J}_+\,|\,\zeta >= \left(\partial/\partial\zeta + j\bar{\zeta}/(1+\zeta\bar{\zeta})\right)|\,\zeta >; \; \hat{J}_-\,|\,\zeta >= \left(-\zeta^2\cdot\partial/\partial\zeta + j\zeta/(1+\zeta\bar{\zeta})\right)|\,\zeta >;$$

$$\hat{J}_0\,|\,\zeta >= \left(\zeta\cdot\partial/\partial\zeta + \tfrac{j}{2}(1-\zeta\bar{\zeta})/(1+\zeta\bar{\zeta})\right)|\,\zeta >.$$

Open systems

Coherent relaxation of N-level systems, (the Markovian approximation)

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{int}$$

$$\hat{\mathcal{R}}(t) = \hat{\rho}_A(t) \otimes \hat{\rho}_B(0)$$

$$\hat{\rho}(t) = tr_B[\hat{\mathcal{R}}(t)]$$

$$\frac{\partial \hat{\rho}}{\partial t} = \sum_{a=1}^{M} \frac{1}{2}\gamma_a [(\mathcal{N}_a + 1)(2\hat{E}_a^- \hat{\rho} \hat{E}_a^+ - \hat{E}_a^+ \hat{E}_a^- \hat{\rho} - \hat{\rho} \hat{E}_a^+ \hat{E}_a^-) +$$
$$+\mathcal{N}_a (2\hat{E}_a^+ \hat{\rho} \hat{E}_a^- - \hat{E}_a^- \hat{E}_a^+ \hat{\rho} - \hat{\rho} \hat{E}_a^- \hat{E}_a^+) -$$
$$-\mathcal{S}_a (2\hat{E}_a^+ \hat{\rho} \hat{E}_a^+ - \hat{E}_a^+ \hat{E}_a^+ \hat{\rho} - \hat{\rho} \hat{E}_a^+ \hat{E}_a^+) -$$
$$-\bar{\mathcal{S}}_a (2\hat{E}_a^- \hat{\rho} \hat{E}_a^- - \hat{E}_a^- \hat{E}_a^- \hat{\rho} - \hat{\rho} \hat{E}_a^- \hat{E}_a^-)].$$

$$\mathcal{N}_a = [(< n > + \frac{1}{2}) ch(2 r_j) - \frac{1}{2}]|_{\omega_j = \omega_a}, \quad \mathcal{S}_a = [(< n > + \frac{1}{2}) e^{i\theta_j} sh(2 r_j)]|_{\omega_j = \omega_a}$$

$$|0>_{sq} = \exp[(\bar{\zeta}\hat{b}^2 - \zeta \hat{b}^{+2})/2]\,|0>, \quad \zeta = r\, e^{i\theta}.$$

Glauber – Sudarshan P-representation for density operator

$$\hat{\rho}(t) = \int_{\mathcal{X}} d\mu(z,\bar{z})\,\mathcal{P}(z,\bar{z},t)|z><z|,$$

$$\frac{\partial}{\partial t}\mathcal{P}(z,\bar{z},t) = \hat{\mathbf{L}}\,\mathcal{P}(z,\bar{z},t)$$

$$\mathcal{P}(z,\,\bar{z};t) = \int_{\mathcal{X}} d\mu(z',\bar{z}')\,\mathcal{K}(z,\bar{z};t|z',\bar{z}';t_0)\,\mathcal{P}_0(z',\,\bar{z}';\,t_0)$$

$$\lim_{t\to 0}\mathcal{K}(z,\,\bar{z};\,t\,|\,z',\,\bar{z}';\,0) = \delta(z,\,\bar{z};\,z',\,\bar{z}')$$

$$<\hat{A}(t)>= \int <z|\,\hat{A}^0(t)\,|z>\,\mathcal{P}(z,\,\bar{z};\,t)\,d\mu(z,\,\bar{z})$$

$$\langle\hat{A}(t)\,\hat{B}(0)\rangle = \int\int <z|\hat{A}^0\,|z>\,\mathcal{K}(z,\bar{z};t|z',\bar{z}';0)\,\mathcal{P}_{\hat{B}\cdot\hat{\rho}(0)}(z',\bar{z}';0)\,d\mu(z,\bar{z})\,d\mu(z',\bar{z}'),$$

$$\hat{A}^0(t) = \hat{U}_0^{-1}(t)\,\hat{A}\,\hat{U}_0(t), \qquad \mathcal{P}_{\hat{B}\cdot\hat{\rho}(t)} \longleftarrow \hat{B}\cdot\hat{\rho}(0)$$

$$g_a(\omega) \sim Re\int_0^\infty e^{-i\omega t}\,\langle\hat{E}_a^+(t)\hat{E}_a^-(0)\rangle\,dt.$$

G=SU(2), (2j+1)-level atom.(j=1/2, 1; j $\longrightarrow$ ∞)

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\gamma}{2}\Big[(<\nu> +1)(2\hat{J}_-\hat{\rho}\hat{J}_+ - \hat{J}_+\hat{J}_-\hat{\rho} - \hat{\rho}\hat{J}_+\hat{J}_-)$$
$$+ <\nu> (2\hat{J}_+\hat{\rho}\hat{J}_- - \hat{J}_-\hat{J}_+\hat{\rho} - \hat{\rho}\hat{J}_-\hat{J}_+)\Big]$$

$$\hat{J}_+|z\rangle\langle z| = \left(\frac{\partial}{\partial z} + \frac{2j\bar{z}}{1+z\bar{z}}\right)|z\rangle\langle z|,$$
$$\hat{J}_-|z\rangle\langle z| = \left(-z^2\frac{\partial}{\partial z} + \frac{2jz}{1+z\bar{z}}\right)|z\rangle\langle z|,$$
$$\hat{J}_0|z\rangle\langle z| = \left(z\frac{\partial}{\partial z} - j\,\frac{1-z\bar{z}}{1+z\bar{z}}\right)|z\rangle\langle z|.$$

The Fokker-Planck (FP) equation:

$$\frac{\partial f}{\partial t} = \frac{\gamma}{2}\left(\frac{\partial}{\partial z}\Big[(<\nu> +1)\left(2jz + \frac{\partial}{\partial z}z^2 + \frac{\partial}{\partial \bar{z}}z^2\bar{z}^2\right)\right.$$
$$\left. + <\nu> \left(-2jz + \frac{\partial}{\partial \bar{z}} + z^2\frac{\partial}{\partial z}\right)\Big] + c.c.\right) f\ , \quad f(z,\bar{z};t) = \frac{\mathcal{P}(z,\bar{z};t)}{(1+z\bar{z})^2}.$$

FP-equation in the case of squeezed bath

$$\frac{\partial f}{\partial t} = \frac{\gamma}{2}\{\frac{\partial}{\partial z}[(\mathcal{N}+1)(2\,j\,z + \frac{\partial}{\partial z}\,z^2 + \frac{\partial}{\partial \bar{z}}\,z^2\,\bar{z}^2) +$$
$$+\mathcal{N}(-2\,j\,z + \frac{\partial}{\partial \bar{z}} + z^2\,\frac{\partial}{\partial z}) + \mathcal{S}(\frac{\partial}{\partial z} + 2\,\frac{\partial}{\partial\,\bar{z}}\,\bar{z}^2) + \bar{\mathcal{S}}\,z^2\frac{\partial}{\partial z}\,z^2\,] + \text{к.с.}\,\}\,f$$

Method on spherical functions expansion:

$$f(z,\bar{z};t) = \sum_{l=0}^{2j}\sum_{m=-l}^{l} F_{lm}(t)Y_{lm}(z,\bar{z})$$

$$\left[(1+z\bar{z})^2\frac{\partial^2}{\partial z\partial\bar{z}} + l(l+1)\right]Y_{lm}(z,\bar{z}) = 0,$$
$$\left(\bar{z}\frac{\partial}{\partial\bar{z}} - z\frac{\partial}{\partial z}\right)Y_{lm}(z,\bar{z}) = mY_{lm}(z,\bar{z}),$$

$$\bar{C} = SU(2)/U(1)$$

$$\mathcal{P}(z,\bar{z};t) = \int_{\mathcal{X}} d\mu(z^{'},\bar{z}^{'})\mathcal{K}(z,\bar{z};t|z^{'},\bar{z}^{'};0)\mathcal{P}_0(z^{'},\bar{z}^{'};\,0)$$

$$\lim_{t\to 0}\mathcal{K}(z,\bar{z};t|z^{'},\bar{z}^{'};0) = \delta(z,\bar{z};z^{'},\bar{z}^{'}), \qquad \delta(z,\bar{z};z^{'},\bar{z}^{'}) = \sum_{l=0}^{2j} Y_{lm}(z,\bar{z})\bar{Y}_{lm}(z',\bar{z}')$$

FP-equation propagator for a qubit in a squeezed bath

$$\mathcal{K}(z,\bar{z};t|z',\bar{z}';0)=\frac{1}{\pi}+\frac{6}{\pi}\frac{z}{1+z\,\bar{z}}\cdot\frac{\bar{z}'}{1+z'\bar{z}'}e^{-\frac{\Gamma}{2}t}ch(\gamma|\mathcal{S}|t)+$$

$$+\frac{6}{\pi}\frac{z}{(1+z\bar{z}}\cdot\frac{z'}{1+z'\bar{z}'}e^{-\frac{\Gamma}{2}t-i\Psi}sh(\gamma|\mathcal{S}|t)+\frac{3}{\pi}\frac{1-z\bar{z}}{(1+z\bar{z}}e^{-\Gamma t}+\frac{3}{\pi}\frac{1-z\bar{z}}{(1+z\bar{z}}\frac{\gamma}{\Gamma}(1-e^{-\Gamma t})+$$

$$+\frac{6}{\pi}\frac{\bar{z}}{1+z\bar{z}}\cdot\frac{z'}{1+z'\bar{z}'}e^{-\frac{\Gamma}{2}t}ch(\gamma|\mathcal{S}|t)+\frac{6}{\pi}\frac{\bar{z}}{1+z\bar{z}}\cdot\frac{\bar{z}'}{1+z'\bar{z}'}e^{-\frac{\Gamma}{2}t+i\Psi}sh(\gamma|\mathcal{S}|t),$$

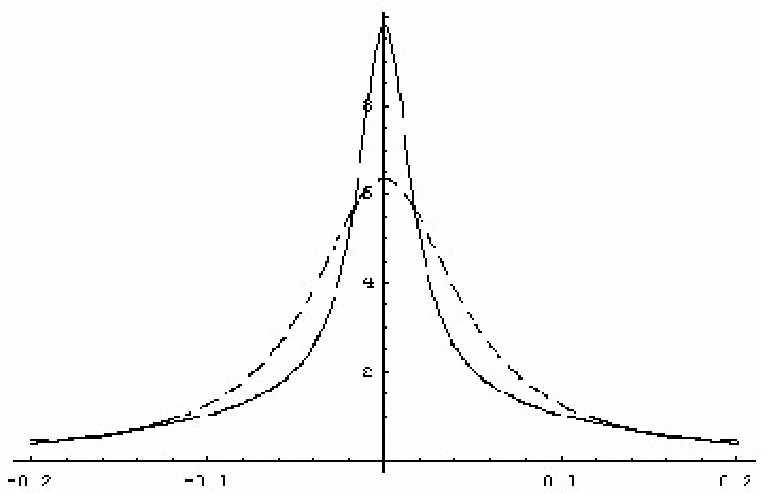
$$e^{i\Psi}=\frac{\mathcal{S}}{|\mathcal{S}|},\ \Gamma=\gamma(2\mathcal{N}+1).$$

$$t\to\infty$$

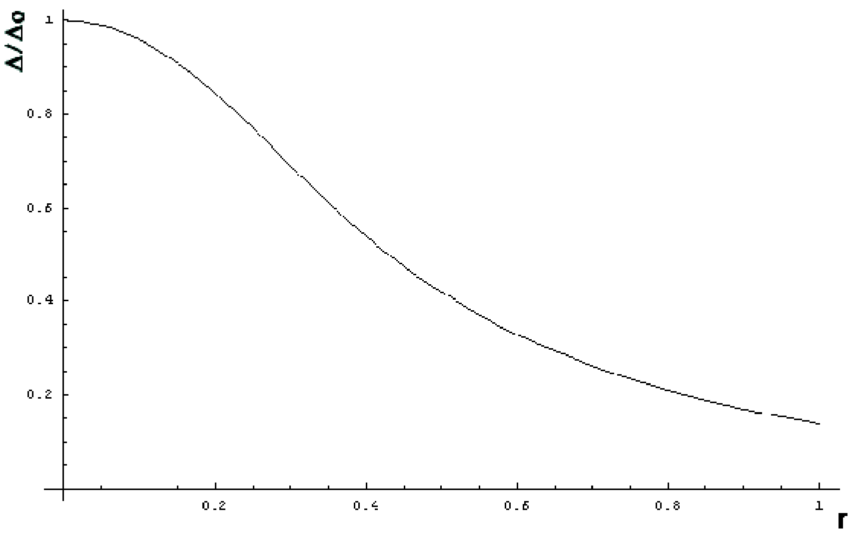
$$\mathcal{P}_{eq}(z,\bar{z})=\frac{1}{\pi}+\frac{3}{\pi}\frac{\gamma}{\Gamma}\frac{1-z\bar{z}}{1+z\bar{z}}=\frac{1}{\pi}\left(1+\frac{3}{2\mathcal{N}+1}\cdot\frac{1-z\bar{z}}{1+z\bar{z}}\right)$$

Contour of the emission line for j=1/2 “atom” in a squeezed bath

$$g(\omega) = \frac{1}{2\pi}\left[\frac{\frac{\Gamma}{2}+\gamma|\mathcal{S}|}{(\omega-\omega_0)^2+\left(\frac{\Gamma}{2}+\gamma|\mathcal{S}|\right)^2}+\frac{\frac{\Gamma}{2}-\gamma|\mathcal{S}|}{(\omega-\omega_0)^2+\left(\frac{\Gamma}{2}-\gamma|\mathcal{S}|\right)^2}\right].$$



$\gamma/\omega_0 = 0.01, <\nu>=5, r = 0.3$



- - - - - - - - No squeezing

——— squeezing

The ratio of radiation lines is independent on the bath temperature, **r** is a squeezing parameter

Parametric Amplifier in a thermal bath

$$\hat{H}(t)=\omega_0(\hat{a}^+\hat{a}+\frac{1}{2})+g(\hat{a}\hat{a}e^{2i\omega t}+\hat{a}^+\hat{a}^+e^{-2i\omega t})$$

$$\hat{\mathrm{U}}_I(t+\Delta t,t)\approx\exp\left(\frac{1}{2}\xi\hat{a}^+\hat{a}^+-\frac{1}{2}\bar{\xi}\hat{a}\hat{a}\right)\qquad \xi=-2ig\Delta t\exp[-2i(\omega-\omega_0)t]$$

$$\frac{\partial \mathrm{P}}{\partial t}=\left\{g[\mathrm{e}^{-2i(\omega_0-\omega)t}\left(2z\frac{\partial}{\partial\bar{z}}+\frac{\partial^2}{\partial\bar{z}^2}\right)+\mathrm{e}^{2i(\omega_0-\omega)t}\left(2\bar{z}\frac{\partial}{\partial z}+\frac{\partial^2}{\partial z^2}\right)]+\frac{1}{2}\gamma\left[\frac{\partial}{\partial z}z+\frac{\partial}{\partial\bar{z}}\bar{z}+\mathrm{N}\frac{\partial^2}{\partial z\partial\bar{z}}\right]\right\}P\equiv\hat{L}P$$

$$P(t)=\hat{U}(t,t_0)P(t_0);\qquad \frac{\partial\hat{U}(t,t_0)}{\partial t}=\hat{L}\hat{U},\ \ \hat{U}(t,t_0)=\hat{I}.$$

$$\dot{a}+2a\dot{d}+be^{\bar{d}-d}\dot{r}=ge^{2i(\omega_0-\omega)t},\ \ \dot{\bar{a}}+2\bar{a}\dot{\bar{d}}+be^{d-\bar{d}}\dot{\bar{r}}=ge^{-2i(\omega_0-\omega)t},$$

$$\dot{b}+\left(\dot{d}+\dot{\bar{d}}\right)b+2\bar{a}e^{\bar{d}-d}\dot{r}+2ae^{d-\bar{d}}\dot{\bar{r}}=\frac{1}{2}\gamma N,$$

$$\dot{r}e^{d-\bar{d}}=2ge^{2i(\omega_0-\omega)\mathrm{t}},\ \dot{\bar{r}}e^{d-\bar{d}}=2ge^{-2i(\omega_0-\omega)\mathrm{t}},\ \dot{d}=\dot{\bar{d}}=\frac{1}{2}\gamma.$$

$$P(z,\bar{z},t)=\int K(z,\bar{z},t\,|\,z',\bar{z}',0)\,P_0(z',\bar{z}')\,d^2z',$$

$$K(z,\bar{z},t\,|\,z_0,\bar{z},0)=\left(|\alpha|^2-|\beta|^2\right)^{-1}\exp\left[-\frac{|\alpha f+\beta\bar{f}|^2}{\left(|\alpha|^2-|\beta|^2\right)^2}\right],$$

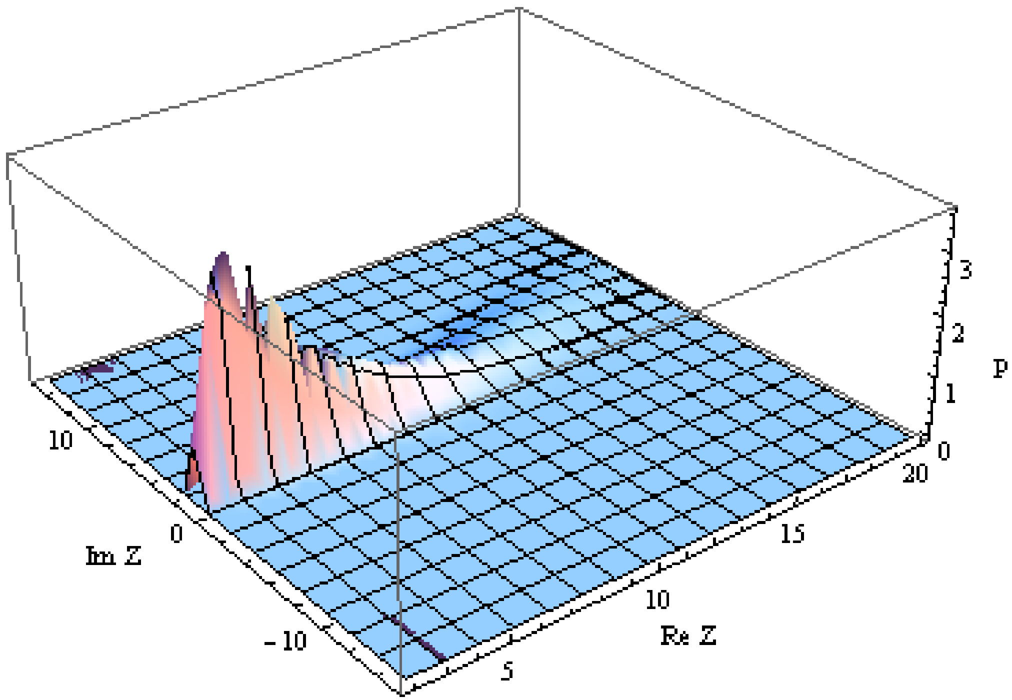
where $\alpha=\sqrt{\frac{1}{2}b+\sqrt{\left(\frac{1}{2}b\right)^2-|\alpha|^2}},\ \beta=-\frac{a}{\alpha},\ f=z-z_0e^{-d}ch|r|-\bar{z}_0e^{-d\frac{|r|}{r}}sh|r|.$

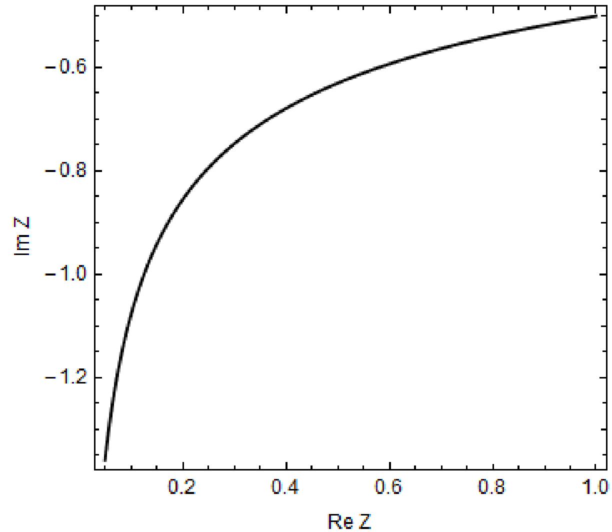
If $\omega_0-\omega=0$

$$a(t)=\left(\gamma^2-16g^2\right)^{-1}\{(\gamma g-\gamma gN)\left(1-e^{-\gamma t}ch4gt\right)+\left(\frac{1}{4}\gamma^2N-4g^2\right)e^{-\gamma t}sh4gt\},$$

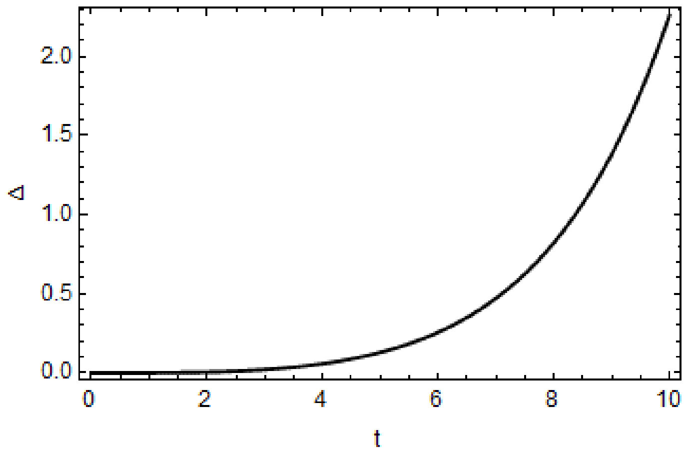
$$b(t)=\left(\gamma^2-16g^2\right)^{-1}\left\{\left(\frac{1}{2}\gamma^2N-8g^2\right)\left(1-e^{-\gamma t}ch4gt\right)+\left(4\gamma g-2\gamma gN\right)e^{-\gamma t}sh4gt\right\},$$

The case of zero detuning



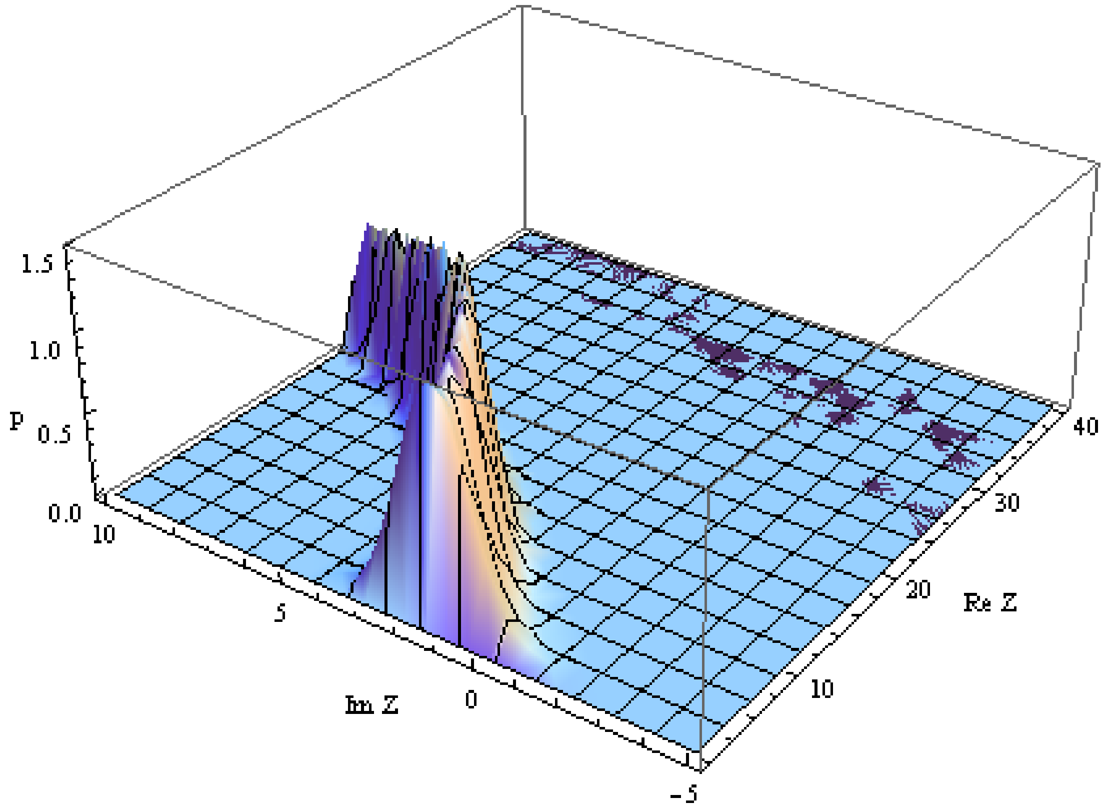


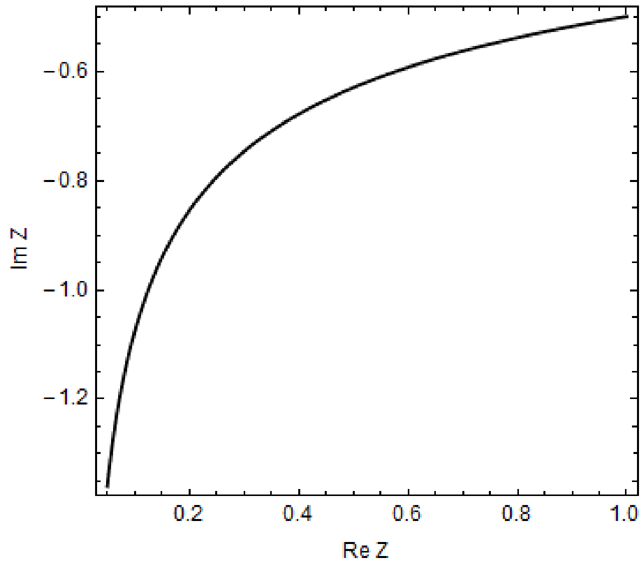
The trajectory of the package center



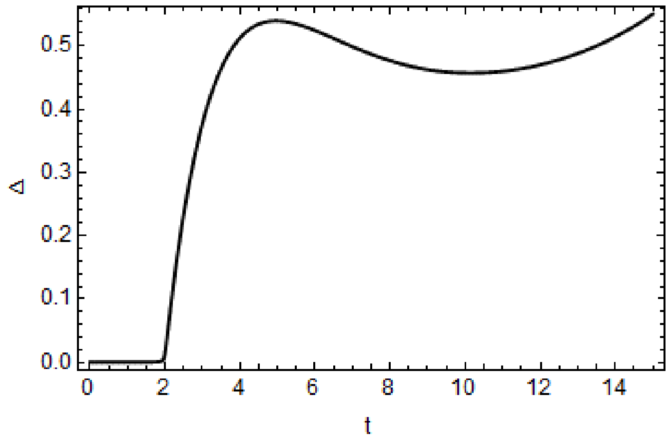
Time dependence of the package width

Nonzero detuning





The trajectory of the package center



Time dependence of the package width

Summary and some problems

- We have presented a mathematical formalism for describing the dynamics and relaxation of quantum systems.
- Group theoretical method and the CS technique are naturally used in quantum optics, quantum information theory, condensed matter and so on.
- Search for quantum corrections to the semi-classical dynamics CS in this approach in the general case has not been solved to date.
- One of the main problems here is the inclusion of non-Markovian effects into consideration.
- Possible generalizations of the concept of dynamical symmetries (super-algebras, associative algebras, …) to more complicated and realistic systems also a worthy of special consideration.


Thank You!
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