RECONSTRUCTION PROCEDURE IN MODIFIED GRAVITY COSMOLOGICAL MODELS

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MODIFIED GRAVITY MODELS

Modified gravity cosmological models have been proposed in the hope of finding solutions to the important open problems of the standard cosmological model. There are lots of ways to deviate from Einstein's gravity:

- $F(R)$ gravity
- Addition of higher-derivative terms to the Einstein–Hilbert action
- Nonlocal gravity

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Formulation of Nonlocal Gravity via Scalar Fields

A modification that assumes the existence of a new dimensional parameter M_* can be of the form

$$S=\int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}R+\frac{1}{2}R\mathcal{F}\left(\frac{\Box}{M_*^2}\right)R-\Lambda\right) \qquad (1)$$

where M_* is the mass scale at which the higher derivative terms in the action become important, $8\pi G_N=1/M_P^2$.
An analytic function $\mathcal{F}(\Box/M_*^2)=\sum\limits_{n\geqslant 0} f_n\Box^n$.

Biswas T., Mazumdar A., and Siegel W., 2006, *JCAP* **0603** 009 (hep-th/0508194)
Biswas T., Koivisto T., and Mazumdar T., 2010, *JCAP* **1011** 008 (arXiv:1005.0590)
Biswas T., Koshelev A.S., Mazumdar T., Vernov S.Yu., *JCAP* **1208** (2012) 024 (arXiv:1206.6374)

By virtue of the field redefinition one can transform the non-local gravity action (1) as follows:

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2}{2}(1+\Phi)R + \frac{1}{2}\tau\mathcal{F}\left(\frac{\Box}{M_*^2}\right)\tau - \frac{M_P^2}{2}\Phi\tau - \Lambda\right) \tag{2}$$

with two new scalar fields Φ and τ.
Variation w.r.t. Φ gives $\tau = R$ and, therefore, the connection (2) with action (1) is obvious.

A modification that does not assume the existence of a new dimensional parameter in the action

$$S_2 = \int d^4x\sqrt{-g}\left\{\frac{1}{16\pi G_N}\left[R\left(1+f(\Box^{-1}R)\right)-2\Lambda\right]+\mathcal{L}_{\mathrm{m}}\right\}, \tag{3}$$

The term $\Box^{-1}R$ is dimensionless and it can appear as a prefactor for the Newtonian gravitational constant, and explain weakening of gravity at cosmological scales.
The action (3) can be rewritten by introducing two scalar fields ϕ and ξ in the following form:

$$\tilde{S}_2 = \int d^4x\frac{\sqrt{-g}}{16\pi G_N}\left\{\left[R\left(1+f(\eta)\right)+\xi\left(\Box\eta-R\right)-2\Lambda\right]+\mathcal{L}_{\mathrm{m}}\right\}. \tag{4}$$

By the variation over ξ, we obtain $\Box\phi = R$.
Substituting $\phi = \Box^{-1}R$ into (4), one reobtains (3).

For the model, describing by the initial nonlocal action, a technique for choosing the distortion function so as to fit an arbitrary expansion history has been derived in

C. Deffayet and R.P. Woodard, *JCAP* **0908** (2009) 023, [arXiv:0904.0961].

For the local model, contained a perfect fluid with a constant state parameter w_{m}, a reconstruction procedure has been made in

T.S. Koivisto, *Phys. Rev.* D **77** (2008) 123513, [arXiv:0803.3399]

and

E. Elizalde, E.O. Pozdeeva, and S.Yu. Vernov, *Class. Quantum Grav.* **30** (2013) 035002, [arXiv:1209.5957].

SUPERPOTENTIAL METHOD

The Hamilton–Jacobi formulation (superpotential method) has been proposed in the cosmological models with minimally coupling scalar field:

A.G. Muslimov, *Class. Quant. Grav.* **7** (1990) 231–237;

D.S. Salopek, J.R. Bond, *Phys. Rev.* D **42** (1990) 3936–3962;

and has been develop in:

I.Ya. Aref'eva, A.S. Koshelev, S.Yu. Vernov, Phys. Rev. D **72** (2005) 064017, astro-ph/0507067;

D. Bazeia, C.B. Gomes, L. Losano, R. Menezes, *Phys. Lett.* B **633** (2006) 415–419; astro-ph/0512197;

K. Skenderis, P.K. Townsend, *Phys. Rev.* D **74** (2006) 125008, hep-th/0609056;

A.A. Andrianov, F. Cannata, A.Yu. Kamenshchik, and D. Regoli, *JCAP* **0802** (2008) 015, arXiv:0711.4300

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The key point in this method is that the Hubble parameter is considered as a function of the scalar field.
For models with non-minimally coupling scalar field this method has been delevoped: A.Yu. Kamenshchik, A. Tronconi, G. Venturi, and S.Yu. Vernov, Phys. Rev. D **87** (2013) 063503, arXiv:1211.6272

The superpotential method is actively used in models with extra spatial dimensions:
A. Brandhuber, K. Sfetsos, *J. High Energy Phys.* **9910** (1999) 013; hep-th/9908116;
O. DeWolfe, D.Z. Freedman, S.S. Gubser, A. Karch, *Phys. Rev.* D **62** (2000) 046008; hep-th/9909134;
A.S. Mikhailov, Yu.S. Mikhailov, M.N. Smolyakov, I.P. Volobuev, *Class. Quant. Grav.* **24** (2007) 231–242, hep-th/0602143.

MODELS WITH NON-MINIMALLY COUPLING SCALAR FIELDS

Let us consider the model with the following action

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} - V(\sigma)\right],$$

In FLRW metric:

$$ds^2 = \; - dt^2 + a^2(t)\left(dx_1^2 + dx_2^2 + dx_3^2\right),$$

we get the following equations:

$$6UH^2 + 6\dot{U}H = \frac{1}{2}\dot{\sigma}^2 + V, \qquad (5)$$

$$2U\left(2\dot{H} + 3H^2\right) + 4\dot{U}H + 2\ddot{U} = \; - \frac{1}{2}\dot{\sigma}^2 + V. \qquad (6)$$

$$\ddot{\sigma} + 3H\dot{\sigma} + V_{,\sigma} = 6\left(\dot{H} + 2H^2\right)U_{,\sigma}\,.$$

Combining Eqs. (5) and (6) we obtain:

$$4U\dot{H} - 2\dot{U}H + 2\ddot{U} + \dot{\sigma}^2 = 0. \tag{7}$$

This equation plays a key role in the reconstruction procedure.
Let $H = Y(\sigma)$, and $\dot{\sigma} = F(\sigma)$.
Substituting H, $\dot{\sigma}$ and $\ddot{\sigma} = F_{,\sigma}F$ into (7), we obtain:

$$4UY_{,\sigma} + 2(F_{,\sigma} - Y)U_{,\sigma} + (2U_{,\sigma\sigma} + 1)F = 0. \tag{8}$$

Equation (8) contains three functions. If two of them are given, then the third one can be found as the solution of a linear differential equation.
The potential $V(\sigma)$ can then be obtained from (5):

$$V(\sigma) = 6UY^2 + 6U_{,\sigma}FY - \frac{1}{2}F^2. \tag{9}$$

If $U(\sigma)$ and $F(\sigma)$ are given, then

$$Y(\sigma) = -\left(\int^{\sigma} \frac{2F_{,\tilde{\sigma}}U_{,\tilde{\sigma}} + (2U_{,\tilde{\sigma}\tilde{\sigma}} + 1)F}{4U^{3/2}} d\tilde{\sigma} + c_0\right)\sqrt{U} \quad (10)$$

For given $Y(\sigma)$ and $U(\sigma)$, we obtain

$$F(\sigma) = \left[\int^{\sigma} \frac{U_{,\tilde{\sigma}}Y - 2UY_{,\tilde{\sigma}}}{U_{,\tilde{\sigma}}} e^{\Upsilon} d\tilde{\sigma} + \tilde{c}_0\right] e^{-\Upsilon(\sigma)}, \qquad (11)$$

where

$$\Upsilon(\sigma) \equiv \frac{1}{2}\int^{\sigma} \frac{2U_{,\tilde{\sigma}\tilde{\sigma}} + 1}{U_{,\tilde{\sigma}}} d\tilde{\sigma}.$$

For the case of induced gravity $U(\sigma) = \xi\sigma^2$ the reconstruction procedure has been proposed in
A.Yu. Kamenshchik, A. Tronconi, G. Venturi, *Reconstruction of scalar potentials in induced gravity and cosmology*, Phys. Lett. B **702** (2011) 191–196, arXiv:1104.2125.
They have not used the superpotential method and got a lot of potential for different types of the Hubble behaviors.
There are two main reasons to use the superpotential method:

- $U(\sigma)$ can be arbitrary function.
- $H(t)$ can be more complicated than $H = Y(\sigma)$.

A.Yu. Kamenshchik, A. Tronconi, G. Venturi, and S.Yu. Vernov, Phys. Rev. D **87** (2013) 063503, arXiv:1211.6272.
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The two methods supplement each other and together allow one to construct different cosmological models with some required properties.

Models with non-minimally coupled scalar fields are interesting because of their connection with particle physics.
The are models of inflation, where the role of the inflaton is played by the Higgs field non-minimally coupled to gravity. (F.L. Bezrukov and M. Shaposhnikov, *Phys. Lett.* B **659** (2008) 703–706, arXiv:0710.3755).
For such models

$$U(\sigma) = \xi\sigma^2 + J. \tag{12}$$

$$\begin{aligned} Y(\sigma) &= -\left[\int^{\sigma} \frac{4\xi\tilde{\sigma}F_{,\tilde{\sigma}} + (4\xi+1)F}{4(\xi\tilde{\sigma}^2 + J)^{3/2}}\, d\tilde{\sigma} + c_0\right]\sqrt{\xi\sigma^2 + J}\,, \\ F(\sigma) &= \left\{\int^{\sigma}\left[\tilde{\sigma}Y - \left(\tilde{\sigma}^2 + \frac{J}{\xi}\right)Y_{,\tilde{\sigma}}\right]\tilde{\sigma}^{\frac{1}{4\xi}}\, d\tilde{\sigma} + \tilde{c}_0\right\}\sigma^{-\frac{1+4\xi}{4\xi}}\,. \end{aligned}$$

Models with de Sitter solutions

Let us consider the general form of the potential $V(\sigma)$, which leads to the existence of the de Sitter solution

$$H = Y(\sigma) = H_0 = const. \tag{13}$$

Using (11), we obtain (for $U = \xi\sigma^2 + J$)

$$F(\sigma) = \left[\int^{\sigma} H_0 e^{\Upsilon} d\tilde{\sigma} + \tilde{c}_0\right] e^{-\Upsilon} = \frac{4\xi H_0}{8\xi + 1}\sigma + \tilde{c}_0 \sigma^{-\frac{1+4\xi}{4\xi}}.$$

$$\sigma(t) = \left[\sigma_0 e^{H_0 t} + \frac{\tilde{c}_0(8\xi + 1)}{H_0 \xi}\right]^{\frac{4\xi}{8\xi+1}}, \tag{14}$$

where σ_0 is an arbitrary constant.

$$
\begin{aligned}
V(\sigma) &= 2H_0^2\left[3J + \frac{(3+32\xi)(1+12\xi)\xi}{(8\xi+1)^2}\sigma^2\right] - \\
&- \frac{\tilde{c}_0^2}{2}\sigma^{-(4\xi+1)/(2\xi)} + \frac{8(12\xi+1)\xi}{8\xi+1}H_0\tilde{c}_0\sigma^{-1/(4\xi)}.
\end{aligned}
$$

For $\tilde{c}_0 = 0$, $F(\sigma)$ is a linear function,

$$
V = 2H_0^2\left[3J + \frac{(3+32\xi)(1+12\xi)\xi}{(8\xi+1)^2}\sigma^2\right].
$$

At $\xi = -1/4$, $\quad V = -5H_0^2\sigma^2 - 4H_0\tilde{c}_0\sigma + 6H_0^2 J_0 - \frac{\tilde{c}_0^2}{2}.$

The same result has been obtained by the method proposed in A.Yu. Kamenshchik, A. Tronconi, G. Venturi, Phys. Lett. B **702** (2011) 191.

The case $\xi = -1/8$

In the case $\xi = -1/8$, we get:

$$F(\sigma) = \sigma H_0 \ln\left(\frac{\sigma}{\sigma_0}\right),$$

where σ_0 is an integration constant, and the corresponding potential has the following form:

$$V = \frac{H_0^2}{4}\left[24J - \frac{\sigma^2}{2}\left(\ln^2\left(\frac{\sigma}{\sigma_0}\right) + 3 + \sqrt{3}\right)\left(\ln^2\left(\frac{\sigma}{\sigma_0}\right) + 3 - \sqrt{3}\right)\right].$$

The scalar field evolution is given by

$$\sigma(t) = \sigma_0 \exp\left[e^{H_0(t-t_0)}\right].$$

Solutions with the hyperbolic tangent

Let us construct cosmological models, when the Hubble parameter is a function of the hyperbolic tangent.

$$\sigma(t) = A \tanh\left[\omega(t - t_0)\right], \tag{15}$$

where A, ω and t_0 are constants.
Note that t_0 can be complex, so the parametrization (15) includes the functions $\sigma(t) = A \coth\left[\omega(t - t_0)\right]$ as well.
For such functions

$$\dot{\sigma} = \omega\left(A - \frac{1}{A}\sigma^2\right) = F(\sigma). \tag{16}$$

To get the desired Hubble parameter evolution (with constant H attractors in the past and in the future), we assume

$$H = Y(\sigma) = B - C\sigma,$$

where B and C are constants.
Equation (8) becomes the following equation for $U(\sigma)$:

$$2\Omega(A^2 - \sigma^2)U_{,\sigma\sigma} + 2\left[(C - 2\Omega)\sigma - B\right]U_{,\sigma} - 4CU + (A^2 - \sigma^2)\Omega = 0,$$

where $\omega = \Omega A$.
A particular solution of this equation is the second degree polynomial

$$U(\sigma) = -\frac{1}{12}\sigma^2 + \frac{B}{6(2\Omega + C)}\sigma + \frac{2A^2C\Omega + 4A^2\Omega^2 - B^2}{12(2\Omega + C)C},$$

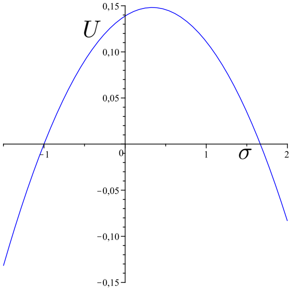
Let us consider, for example, $A = B = C = 1$. We get

$$U(\sigma) = -\frac{1}{12}\sigma^2 + \frac{1}{6(2\Omega + 1)}\sigma + \frac{(2\Omega + 4\Omega^2 - 1)}{12(2\Omega + 1)},$$

hence, $U(\sigma) = 0$ at

$$\sigma_{1,2} = \frac{1 \pm 2\sqrt{2\Omega^2 + 2\Omega^3}}{2\Omega + 1}.$$

$U(1) > 0$ for all $\Omega > 0$, so, if we choose as a solution $\sigma(t) = \tanh(\Omega t)$, then $U(t)$ is positive at late times. When $\Omega = 1$, $U(\sigma(t)) \geq 0$ at any time because $-1 \leqslant \sigma(t) \leqslant 1$.



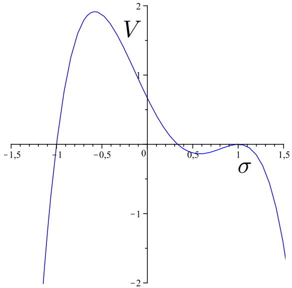


Figure : $U(\sigma)$ and $V(\sigma)$ at $A = +B = C = \Omega = 1$.

By the change of variable $\tilde{\sigma} = \sigma - 1/3$, we get

$$U(\tilde{\sigma}) = \; -\frac{1}{12}\tilde{\sigma}^2 + \frac{4}{27}, \tag{17}$$

In terms of $\tilde{\sigma}$, we finally obtain

$$Y(\tilde{\sigma}) = \frac{2}{3} - \tilde{\sigma},$$

$$F(\tilde{\sigma}) = \frac{(2 - 3\tilde{\sigma})\,(4 + 3\tilde{\sigma})}{9}.$$

So, we found a model with exact solutions and $U(\tilde{\sigma})$ in the form

$$U(\tilde{\sigma}) = \xi\tilde{\sigma}^2 + J.$$

Non-monotonic behavior of the Hubble Parameter in the Case of Induced Gravity

We put $U(\sigma) = \xi\sigma^2$.
Let us consider $Y(\sigma)$ as a quadratic polynomial:

$$Y(\sigma) = A_2\sigma^2 + A_1\sigma + A_0, \tag{18}$$

where A_k are constants.
We obtain that $F(\sigma)$ does not depend on A_1:

$$F(\sigma) = \frac{4\xi\left((16\xi+1)A_0 - (8\xi+1)A_2\sigma^2\right)\sigma}{(8\xi+1)(16\xi+1)} + \tilde{c}_0\sigma^{-\frac{1+4\xi}{4\xi}}. \tag{19}$$

We assume that $\xi \neq -1/8$ and $\xi \neq -1/16$.

When $\tilde{c}_0 = 0$, $F(\sigma)$ is a cubic polynomial and the equation $\dot{\sigma} = F(\sigma)$ has the following general solution:

$$\sigma(t) = \pm \frac{\sqrt{(16\xi + 1)A_0}}{\sqrt{(16\xi + 1)A_0 c_2 e^{-\omega t} + (8\xi + 1)A_2}}\,, \qquad (20)$$

where $\omega = 8\xi A_0/(8\xi + 1)$, c_2 is an arbitrary integration constant.

The corresponding potential, $V(\sigma)$, is the sixth degree polynomial which, for example, when $\xi = 1$ has the following form:

$$\begin{aligned} V(\sigma) &= \frac{910}{289}A_2^2\sigma^6 + \frac{156}{17}A_1A_2\sigma^5 + \\ &+ \left(6A_1^2 + \frac{2236}{153}A_0A_2\right)\sigma^4 + \frac{52}{3}A_0A_1\sigma^3 + \frac{910}{81}A_0^2\sigma^2\,. \end{aligned}$$

The cosmological consequences.

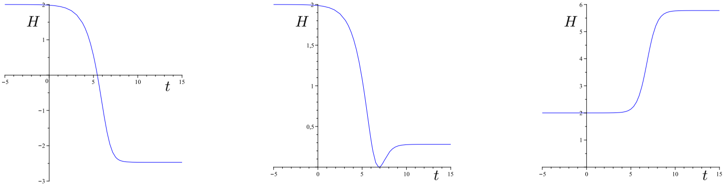


Figure : **The function $H(t)$ with $A_1 = -6$, $A_1 = -4$, and $A_1 = 0$ (from left to right). At all pictures we use $A_2 = 1$, $A_0 = 2$, and $c_2 = 100000$.**

The same functions $\sigma(t)$ is associated with different behaviors of the Hubble parameter.
At $A_1 = -4$ we get a non-monotonic behavior of $H(t)$.

Conclusion

- A gravity model with a non-minimally coupling scalar field and $U(\sigma) = \xi\sigma^2 + J$ has been considered.
- The superpotential method has been used for the reconstruction procedure.
- We do not need the expression of the Hubble parameter in terms of the cosmic time or of the scale factor.
- We have found the potentials and the corresponding evolutions of the associated scalar field leading to de Sitter solutions.
- We have investigated a few models having a different de Sitter asymptotic behaviour in the past and in the future. Non-monotonic behaviour have been found.