Embeddings of the black holes in a flat space

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- What is an isometrical embedding
- Applications
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- Thermodynamical properties of the black holes

Idea of embedding

Janet-Cartan theorem (1916)

An arbitrary n-dimensional Riemannian manifold can be locally isometrically embedded in N-dimensional flat space with

$$N \geq \frac{n(n+1)}{2}. \quad (1)$$

For our $4D$ manifold $N = 10$; if the manifold has symmetries, N may be smaller. Metric of this manifold can be expressed in terms of embedding function:

$$g_{\mu\nu} = \partial_\mu y^a(x) \partial_\nu y^b(x) \eta_{ab}, \quad (2)$$

where $y^a(x)$ – embedding function, η_{ab} – metric of flat ambient space.

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Embedding-based theory of gravity

Change of variables in action:

$$g_{\mu\nu}(x^\mu) \to y^a(x^\mu) \tag{3}$$

$$\delta S = \frac{1}{2\varkappa}\int d^4x\sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu})\delta g_{\mu\nu} =$$
$$= \frac{1}{2\varkappa}\int d^4x\sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu})\partial_\mu y^a(x)\partial_\nu \delta y^b(x)\eta_{ab} =$$
$$= \partial_\mu(\sqrt{-g}(G^{\mu\nu} - \varkappa T^{\mu\nu})\partial_\nu y^a) = (G^{\mu\nu} - \varkappa T^{\mu\nu})D_\mu\partial_\nu y^a = 0. \tag{4}$$

(Regge, Teitelboim 1975) Natural appearance of the flat spacetime in this approach can be useful in the quantization of gravity:

- Preferred «time slicing»
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Applications of isometrical embedding

Quantization

The Wheeler-de Witt equation for RT approach (Davidson, PRD 2003)

$$\frac{8\pi G}{2\sqrt{h}}\left[\left(\frac{\sqrt{h}}{8\pi G}\right)^2(\hat{\lambda}+\mathcal{R}^{(3)})(x)- \right.$$
$$\left. -\hbar^2\left((\Psi-\hat{\lambda}I)^{-1}\right)^{AB}(x)\frac{\delta^2}{\delta y^A(x)\delta y^B(x)}\right]\Phi[y]=0\,. \quad (5)$$

Classification of exact solutions of Einstein equations

Embedding class $p = N - d = \frac{d(d-1)}{2}$ of a metric is invariant.

Geometrical properties of Riemannian manifolds

Fronsdal embedding of the Schwarzchild black hole is closely related to the Kruskal coordinates:

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Geometrical properties of Riemannian manifolds

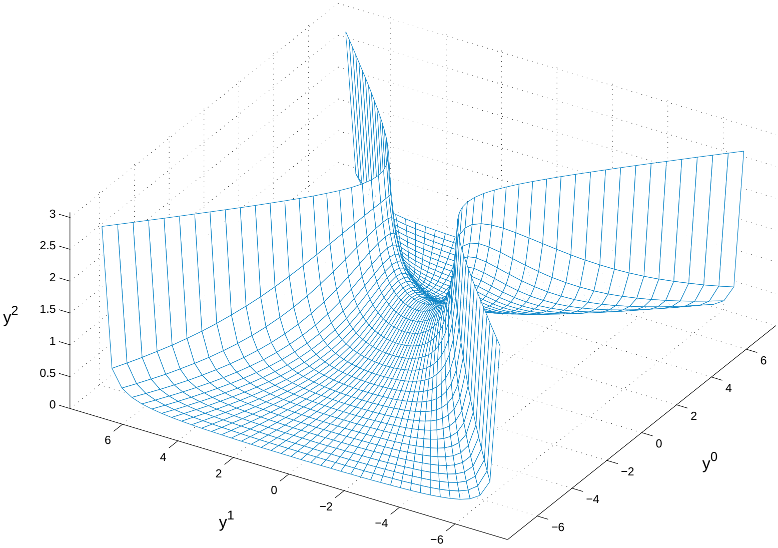
Fronsdal embedding of the Schwarzchild black hole is closely related to the Kruskal coordinates:

Fronsdal embedding (1959)

$$
\begin{array}{cc}
r > R: & r < R: \\
y^0 = 2R\sqrt{1-\frac{R}{r}}\,\mathrm{sh}\left(\frac{t}{2R}\right), & y^0 = \pm 2R\sqrt{\frac{R}{r}-1}\,\mathrm{ch}\left(\frac{t}{2R}\right), \\
y^1 = \pm 2R\sqrt{1-\frac{R}{r}}\,\mathrm{ch}\left(\frac{t}{2R}\right), & y^1 = 2R\sqrt{\frac{R}{r}-1}\,\mathrm{sh}\left(\frac{t}{2R}\right),
\end{array}
$$

$$
\begin{aligned}
y^2 &= r\cos(\theta), \\
y^3 &= r\sin(\theta)\cos(\phi), \\
y^4 &= r\sin(\theta)\sin(\phi), \\
y^5 &= g(r).
\end{aligned} \tag{6}
$$

Ambient space metric is $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1,-1,-1)$.


3
2.5
2
1.5
1
0.5
0
y^2
6
4
2
0
−2
−4
−6
y^1
6
4
2
0
−2
−4
−6
y^0

Known embeddings of the Schwarzshild black hole

Kasner (1921)

$$y^0 = f(r), \quad y^1 = R\sqrt{1-\frac{R}{r}}\,\sin(t/R), \quad y^2 = R\sqrt{1-\frac{R}{r}}\,\cos(t/R). \tag{7}$$

Fujitani et al. (1961)

$$y^0 = t\sqrt{1-\frac{R}{r}}, \quad y^{1,2} = \frac{1}{\sqrt{2}\gamma}\left(\frac{\gamma^2 t^2}{2} \mp 1\right)\sqrt{1-\frac{R}{r}} + \frac{u(r)}{\sqrt{2}}. \tag{8}$$

Davidson and Paz (1999)

$$y^{0,1} = \frac{R}{2\beta\sqrt{r_c r}}\left(e^{\beta t + u(r)} \mp \frac{r-r_c}{R}\, e^{-\beta t - u(r)}\right), \quad y^2 = kt. \tag{9}$$

$$y^3 = r\,\cos\theta, \qquad y^4 = r\,\sin\theta\,\cos\phi, \qquad y^5 = r\,\sin\theta\,\sin\phi.$$

New embeddings of the Schwarzshild black hole

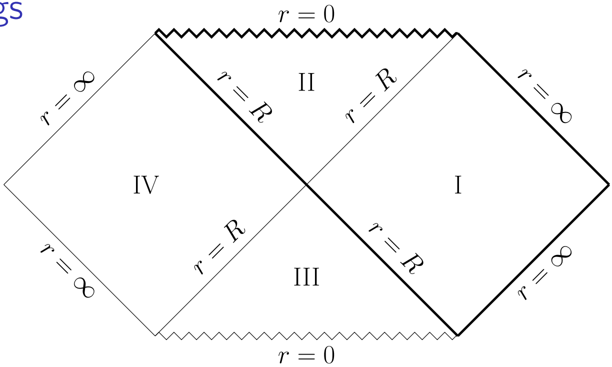
Asymptotically flat embedding (S.Paston, A.S., CQG, 2012)

$$y^0 = t, \quad y^1 = \sqrt{\frac{27R^3}{r}} \sin\left(\frac{t}{\sqrt{27}R} - \sqrt{\frac{(r+3R)^3}{27R^2 r}}\right), \qquad (10)$$
$$y^2 = \sqrt{\frac{27R^3}{r}} \cos\left(\frac{t}{\sqrt{27}R} - \sqrt{\frac{(r+3R)^3}{27R^2 r}}\right).$$

Cubic embedding (S.P., A.S., TMPh, 2013)

$$\begin{aligned} y^0 &= \frac{\xi^2}{6} t^3 + \left(1 - \frac{R}{2r}\right) t + u(r), \\ y^1 &= \frac{\xi^2}{6} t^3 - \frac{R}{2r} t + u(r), \\ y^2 &= \frac{\xi}{2} t^2 + \frac{1}{2\xi}\left(1 - \frac{R}{r}\right). \end{aligned} \qquad (11)$$

The global structure of the Schwarzchild black hole embeddings



I – our universe; II – black hole; III – white hole; IV – parallel universe.
Fronsdal – I,II,III,IV; Kasner, Fujitani – I; Davidson, As. flat, Cubic – I,II.

New global embeddings of the charged black hole

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)} - r^2 d\Omega^2.$$

Spiral

$$\begin{aligned} y^0 &= \frac{\sqrt{(mr - q^2)^2 + b^2 r^2}}{\alpha q r} \sin\left(\alpha t + u(r)\right), \\ y^1 &= \frac{\sqrt{(mr - q^2)^2 + b^2 r^2}}{\alpha q r} \cos\left(\alpha t + u(r)\right), \\ y^2 &= \frac{\sqrt{b^2 + m^2 - q^2}}{q}\, t. \end{aligned} \tag{12}$$

Exponential

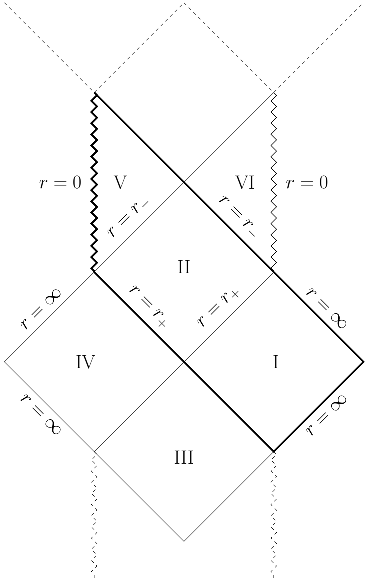
$$
\begin{aligned}
y^0 &= \gamma t, \\
y^{1,2} &= \frac{e^{-\beta t - v(r)}}{2\beta} \mp \left(1 - \frac{2m}{r} + \frac{q^2}{r^2} - \gamma^2\right) \frac{e^{\beta t + v(r)}}{2\beta}.
\end{aligned}
$$

Cubic

$$
\begin{aligned}
y^0 &= -\frac{q^4}{8m^3}\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) + \frac{2m^3}{q^4} t^2, \\
y^{1,2} &= w(r) - \frac{1}{4}\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) t + \frac{4m^6}{3q^8} t'^3 \mp t.
\end{aligned}
$$

Ambient space metric is $\eta_{ab} = \mathrm{diag}(1, 1, -1, -1, -1, -1)$.

The global structure of the charged black hole embeddings



Hawking and Unruh effect

Hawking effect

A black hole has a radiation with a thermal spectrum $T_0 = \dfrac{1}{4\pi R}$.

Tolman law

In thermal equilibrium a temperature is not constant when space is curved. For the Schwarzchild black hole

$$T = T_0/\sqrt{1 - R/r}. \tag{13}$$

Unruh effect

Uniformly accelerated observer, when coupled to the quantum fields, detects a radiation with a thermal spectrum and a temperature proportional to his acceleration: $T = \dfrac{a}{2\pi}$.

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Connection between Hawking and Unruh effect in ambient space

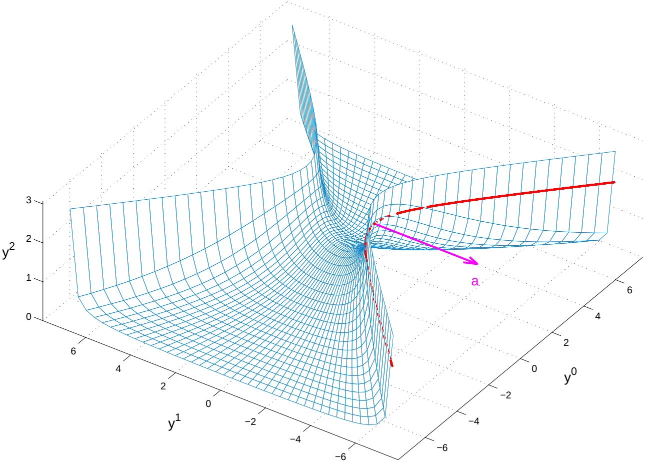
Deser & Levin, PRD 1999

Unruh radiation detected by observer moving on the embedding surface in the ambient space has the same spectrum and temperature as the Hawking radiation from the horizon in the corresponding manifold: $T = \frac{a_{emb}}{2\pi}$. This hypothesis was tested for Fronsdal embedding.

Further development

- Hong, GRG 2003 – charged black holes,
- Chen et al., JHEP 2004 – stationary motions,
- Santos et al., PRD 2004 – D-dimensional black holes,..

$$T = a/2\pi = \frac{1}{4\pi R\sqrt{1 - R/r}}, \quad T_0 = T\sqrt{1 - R/r} = const. \qquad (14)$$



Accelerations for new embeddings

- Asymptotically flat: $a = \dfrac{1}{\sqrt{27Rr}\left(1 - \dfrac{R}{r}\right)}$
- Davidson-Paz: $a = \dfrac{\beta}{1 - \dfrac{R}{r}}\sqrt{1 - \dfrac{R}{r} + k^2}$
- Cubic $a = \dfrac{\xi}{\left(1 - \dfrac{R}{r}\right)}$

Corresponding temperatures (if we assume that the temperature and acceleration are always related as in the Unruh effect) violate the Tolman law and do not coincide with BH temperature!

For moving on the arbitrary trajectories the spectrum of Unruh radiation is not exactly thermal, but at sufficiently slowly varying accelerations ($\dot{a}/a^2 << 1$) the Unruh formula works good (Barbado & Visser, PRD, 2012).

Exact spectrum was found to be non-thermal for the trajectories corresponding to lines of time in cubic (Letaw, PRD 1981) and Davidson-Paz (Abdolrahimi, arXiv:1304:4237) embeddings.

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Summary

- Isometrical embedding is a powerful tool for studying of Riemannian manifolds.
- Some features of embeddings possibly can help to quantize gravity.
- Exact embeddings of black holes might be related to their thermodynamical properties.
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