Forward-backward multiplicity correlations in *pp* collisions in ALICE at 0.9, 2.76 and 7 TeV

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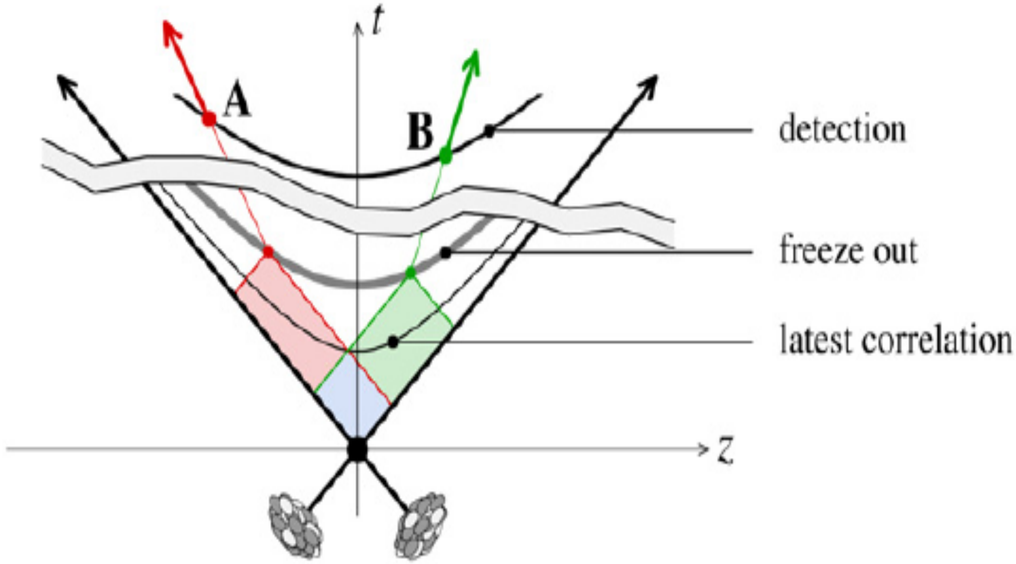
Outline

- *Introduction*: motivation of Forward-Backward (FB) correlation studies
- *Data analysis*: calculation of FB multiplicity correlations in ALICE
- *ALICE results*: FB correlations in separated pseudorapidity windows
- *Models*: FB correlations studies in separated pseudorapidity-azimuthal windows in models, interpretation of the results
- *Conclusions*

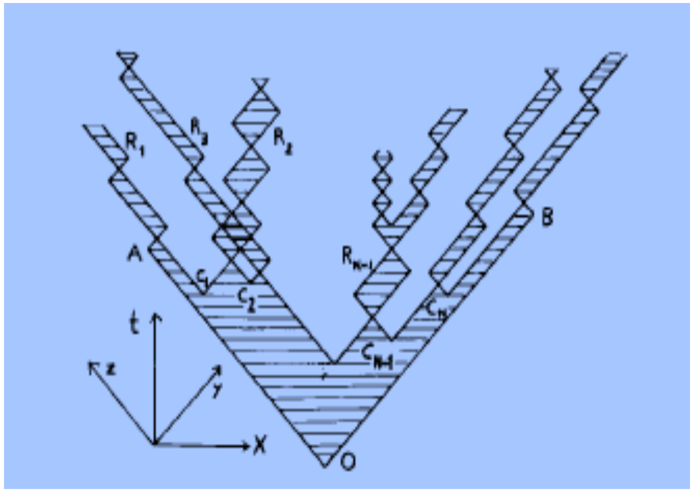
Introduction

Forward-Backward (*or* Long-Range) correlations

Causality requires appearance of long-range correlations – if they exist – at the very early stages between particles detected in separated rapidity intervals in any type of collisions (pp, pA, AA):



A.Dumitru et al./ Nuclear Physics A 810 (2008) 91-108

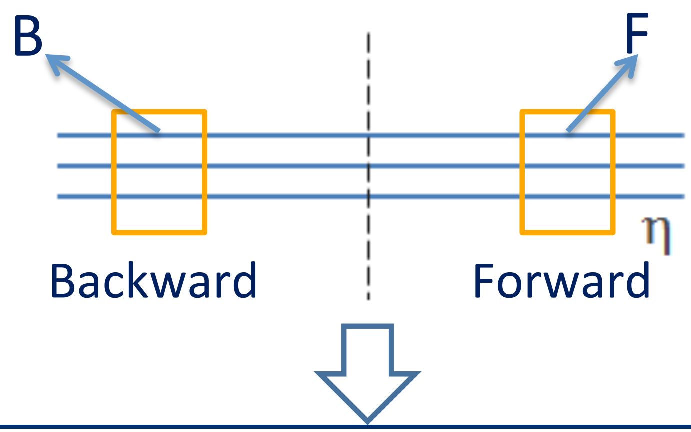


X. ARTRU and G. MENNESSIER,
"STRING MODEL AND MULTIPRODUCTION",
Nuclear Physics B70 (1974) 93-115

Forward-Backward (*or* Long-Range) correlations

Causality requires appearance of long-range correlations – if they exist – at the very early stages between particles detected in separated rapidity intervals in any type of collisions (pp, pA, AA):

Get some F and B quantities from separated η windows:



Types of correlations

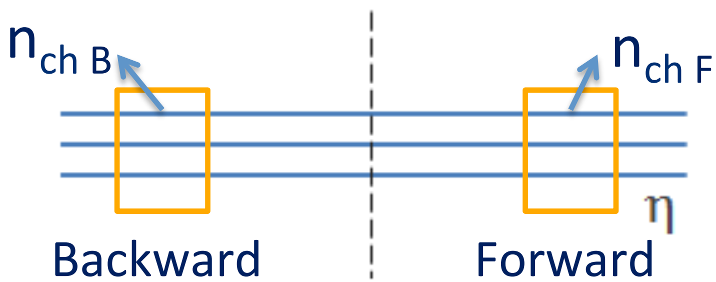
1. **n_B-n_F** - the correlation between charged particle multiplicities in backward (B) and forward (F) rapidity windows
2. **p_{tB}-n_F** - the correlation between the event mean transverse momentum in the backward rapidity window and the charged particle multiplicity in the forward window
3. **p_{tB}-p_{tF}** - the correlation between the event mean transverse momenta in backward and forward rapidity windows

The first early experimental indications of FB multiplicity correlations (*1988*)

Charged particle correlations in $\bar{p}p$ collisions at c.m. energies of 200, 546 and 900 GeV

UA5 Collaboration

Z. Phys. C – Particles and Fields 37, 191–213 (1988)



$$\langle n_B\rangle_{n_F} = a + \underline{b_{corr}} \cdot n_F$$

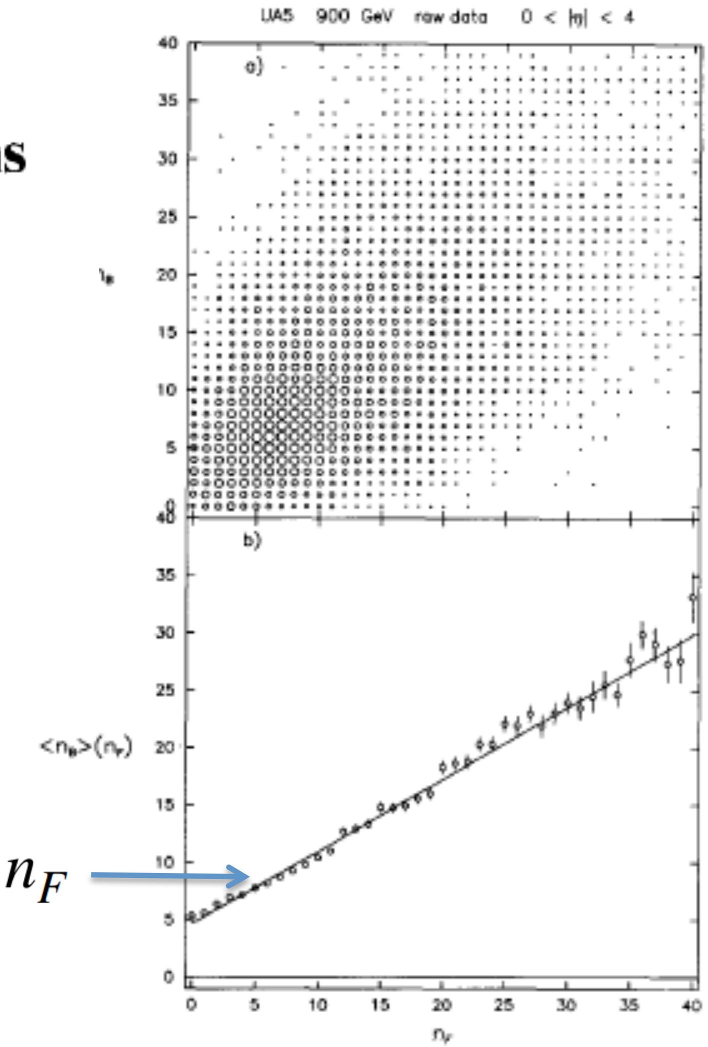


Fig. 12. a shows as a scatter plot (the circle area is proportional to the number of events) the backward multiplicity n_B versus the forward multiplicity n_F for raw data in the η range $0<|\eta|<4$ at 900 GeV. In **b** the average backward multiplicity $\langle n_B\rangle$ as function of the forward multiplicity n_F is plotted for the same data sample. The straight line shows the result of a least squares fit assuming a linear function

Motivation for FB correlations studies

The initial conditions for the QGP formation:

- **Color string fusion phenomenon (SFM)** M.A.Braun and C.Pajares (see Phys. Lett. B287 (1992) 154; Nucl. Phys. B390 (1993) 542, 549);

- **Color Glass Condensate (CGC) and Glasma flux tubes** (see e.g. L.McLerran, Nucl.Phys.A699,73c(2002))

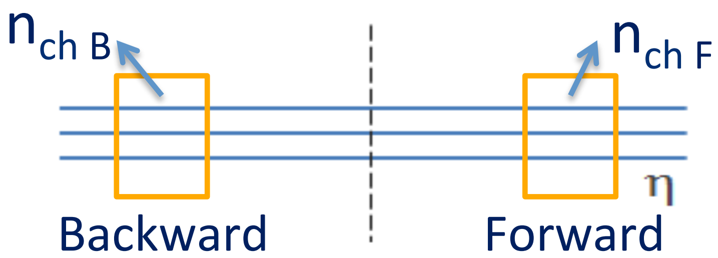
Both are defining the dynamics of AA collisions before the QGP formation predicting the appearance of long range correlations between observables measured in separated pseudorapidity windows:

→ *pp* collisions as a benchmark for Pb-Pb collisions analysis

Other motivations:

- string percolation picture of pp collisions at LHC energies
  (see P. Brogueira, J. Dias de Deus, and C. Pajares, Phys. Lett. B 675 (2009) 308)
- CGC predictions for pp collisions at the LHC
  (see E.Levin and Amir H. Rezaeia, "Gluon saturation and inclusive hadron production at LHC", PRD82, 014022 (2010), arXiv:1005.0631)
- "twisted correlations" as a tool to tune color reconnection and MPI in PYTHIA
  (see K. Wraight and P. Skands, Eur. Phys. J., C 71 (2011) 1628).

Definition of the FB correlation coefficient



Extract correlation strength by:

1) linear regression [1]

$$\langle n_B \rangle_{n_F} = a + b_{corr} \cdot n_F$$

2) correlator [2]

$$b_{corr} = \frac{\langle n_B n_F \rangle - \langle n_B \rangle \langle n_F \rangle}{\langle n_F^2 \rangle - \langle n_F \rangle^2}$$

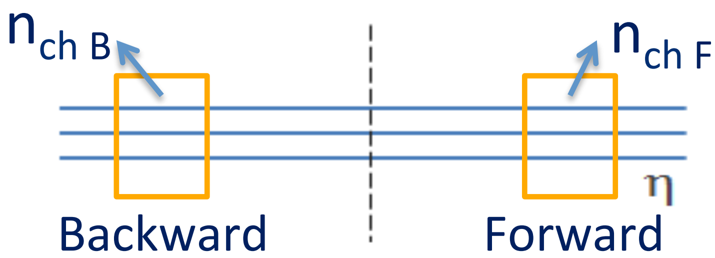
(is used by RHIC and ATLAS)

Following:

[1] UA5 Collaboration, Z.Phys,C-Particles and Fields 37,191-213 (1988)

[2] A.Capella et al.,Phys.Rep. 236,225(1994)

Definition of the FB correlation coefficient



$$\nu_F = n_F/\langle n_F\rangle$$
$$\nu_B = n_B/\langle n_B\rangle$$

Extract correlation strength by:

1) linear regression [1]

$$\langle n_B\rangle_{n_F} = a + b_{corr}\cdot n_F \quad\Rightarrow\quad \langle \nu_B\rangle_{\nu_F} = a_{rel} + b_{corr}^{rel}\nu_F$$

2) correlator [2]

$$b_{corr} = \frac{\langle n_B n_F\rangle - \langle n_B\rangle\langle n_F\rangle}{\langle n_F^2\rangle - \langle n_F\rangle^2} \quad\Rightarrow\quad b_{corr}^{rel} = \frac{\langle \nu_F\nu_B\rangle - 1}{\langle \nu_F^2\rangle - 1} = \frac{\langle n_F\rangle}{\langle n_B\rangle} b_{corr}$$

(is used by RHIC and ATLAS)

For symmetrical windows $b_{corr}^{rel} = b_{corr}$

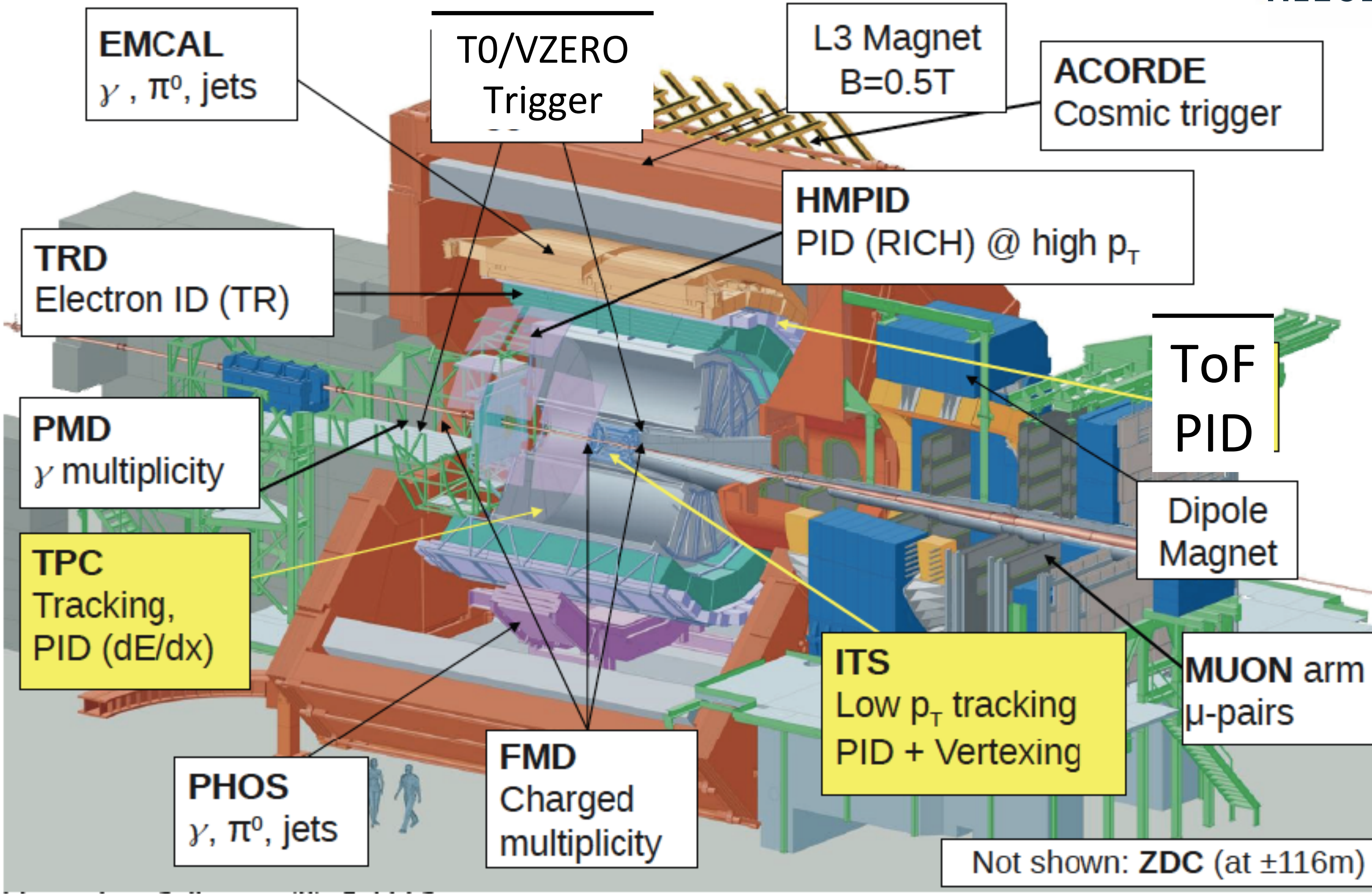
Following:

[1] UA5 Collaboration, Z.Phys,C-Particles and Fields 37,191-213 (1988)
[2] A.Capella et al.,Phys.Rep. 236,225(1994)

Relative variables are being used in the present study.

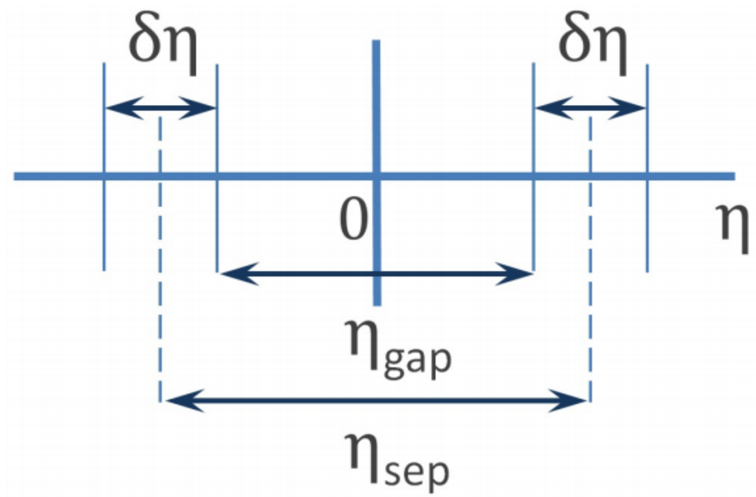
Data Analysis

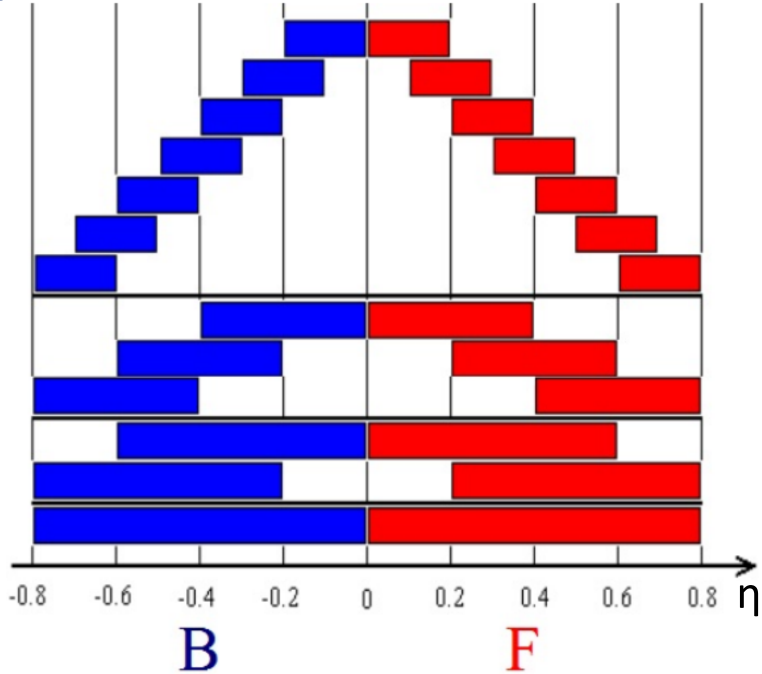
A Large Ion Collider Experiment



K. Aamodt et al. (ALICE), JINST, 3 , S08002 (2008)

η-window pairs chosen for the analysis in ALICE





η gap – distance between windows (η sep – between middles)

δη – windows width

Correlation coefficient b_{corr} is calculated for every configuration of η-windows pair.

$|\eta|<0.8$

p_T range 0.3-1.5 GeV/c

<u>Three *pp* energies:</u>

900 GeV (2 Mln events)

2.76 TeV (10 Mln events)

7TeV Runs (6.5 Mln events)

Calculation of b_{corr} and correction procedures

- Calculation:
  1) by linear regression
  2) using correlator formula

Calculation and corrections of coefficient b_{corr} were done for every configuration of η-windows pair.

- Corrections (three alternative procedures)
  1) by correcting b_{corr} *raw* value
  2) by correcting $\langle n_B n_F \rangle$, $\langle n_B \rangle$, $\langle n_F \rangle$ and $\langle n_F^2 \rangle$
  3) by extrapolation of b_{corr} to corrected $\langle n_F \rangle$ values

Calculations from all these procedures were combined to obtain resulting values and errors.

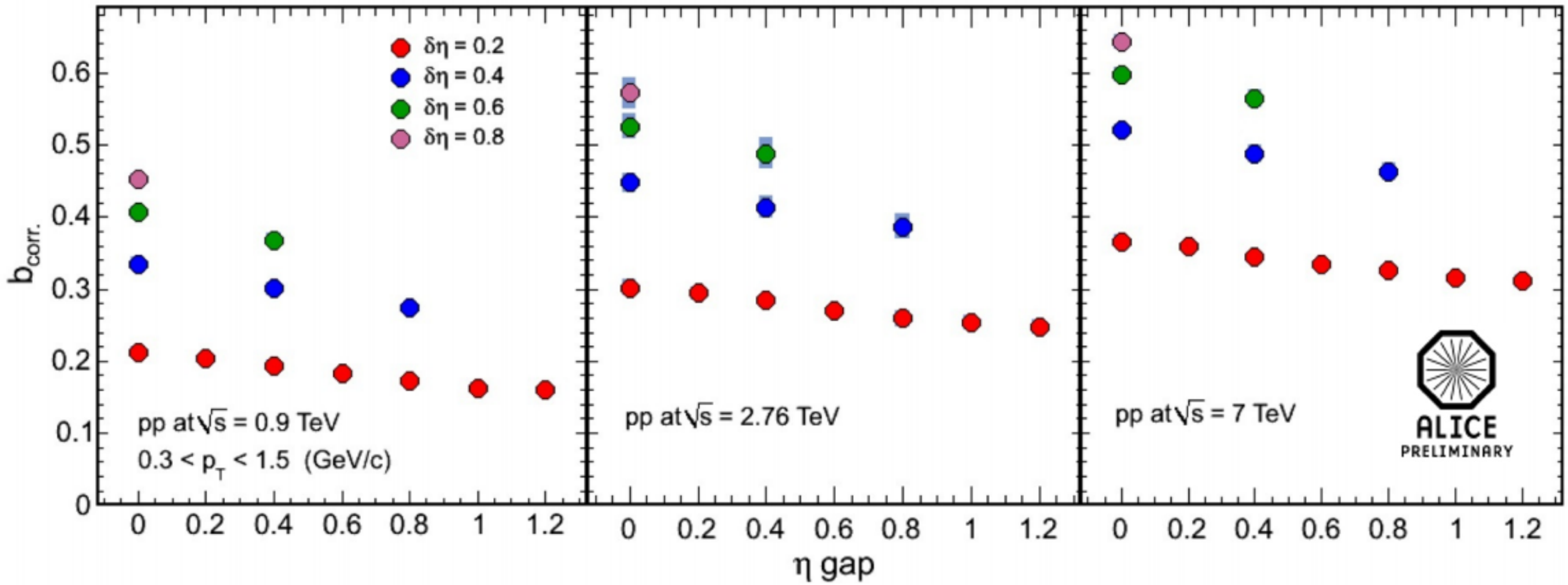
Correction factors are found to be of the order of *5-10%*

Systematic uncertainties are of the order *2-5%*

Experimental results:

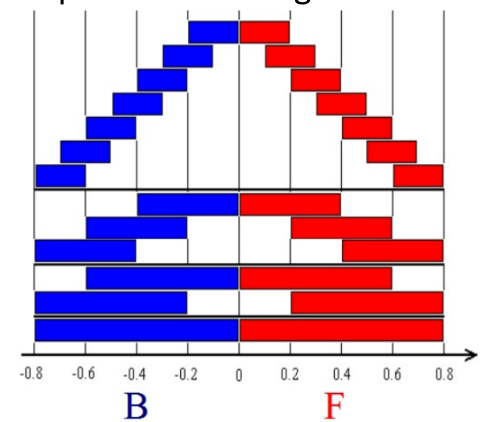
FB multiplicity correlations in separated η windows

1) FB multiplicity correlations in separated η windows



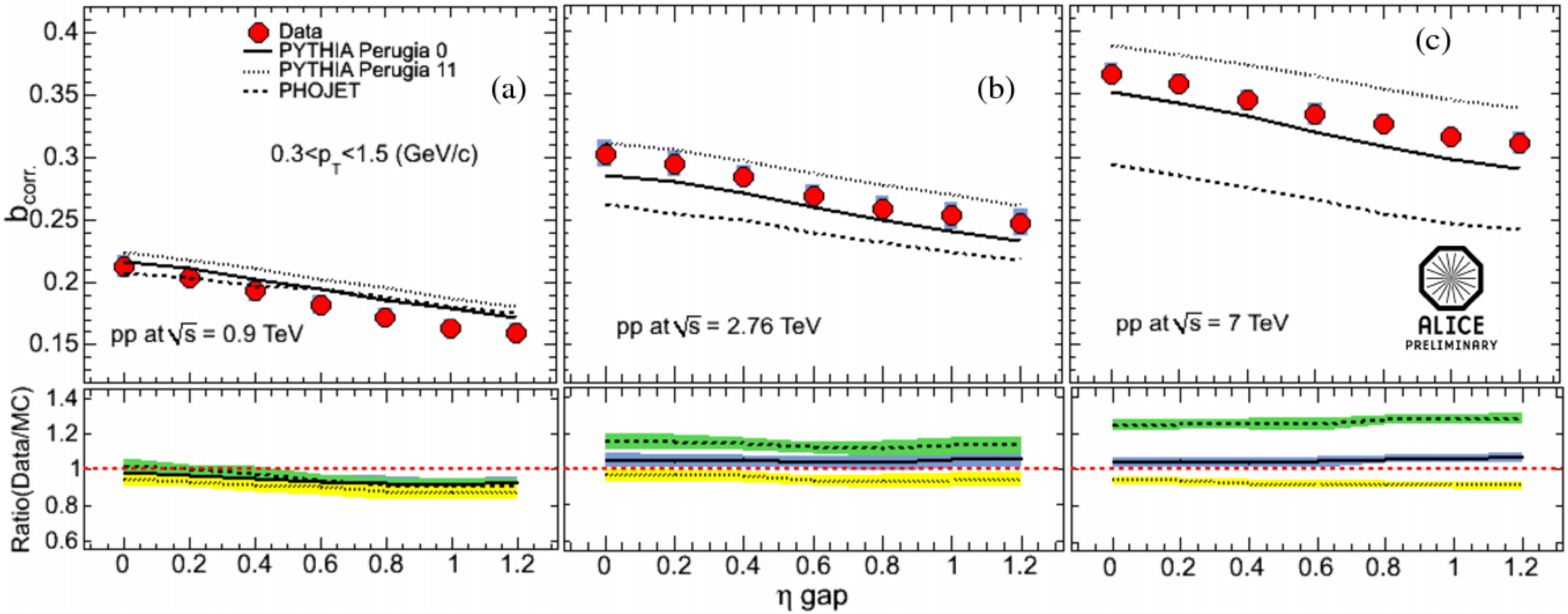
Multiplicity forward-backward correlation strength vs. η gap and for different $\delta\eta$ bin-widths in pp collisions at $\sqrt{s}$=0.9, 2.76 and 7 TeV.

η windows configurations



- correlation coefficient b_{corr} drops with η gap
- the slope is constant with the collision energy
- wider windows give higher b_{corr} values
- values of b_{corr} increase with the collision energy

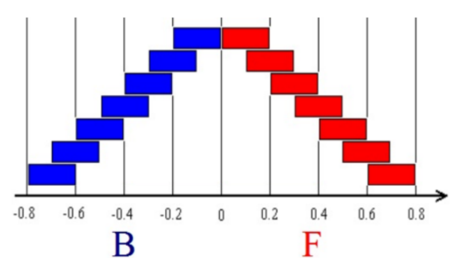
2) Comparison with generators:
FB multiplicity correlations in separated η windows



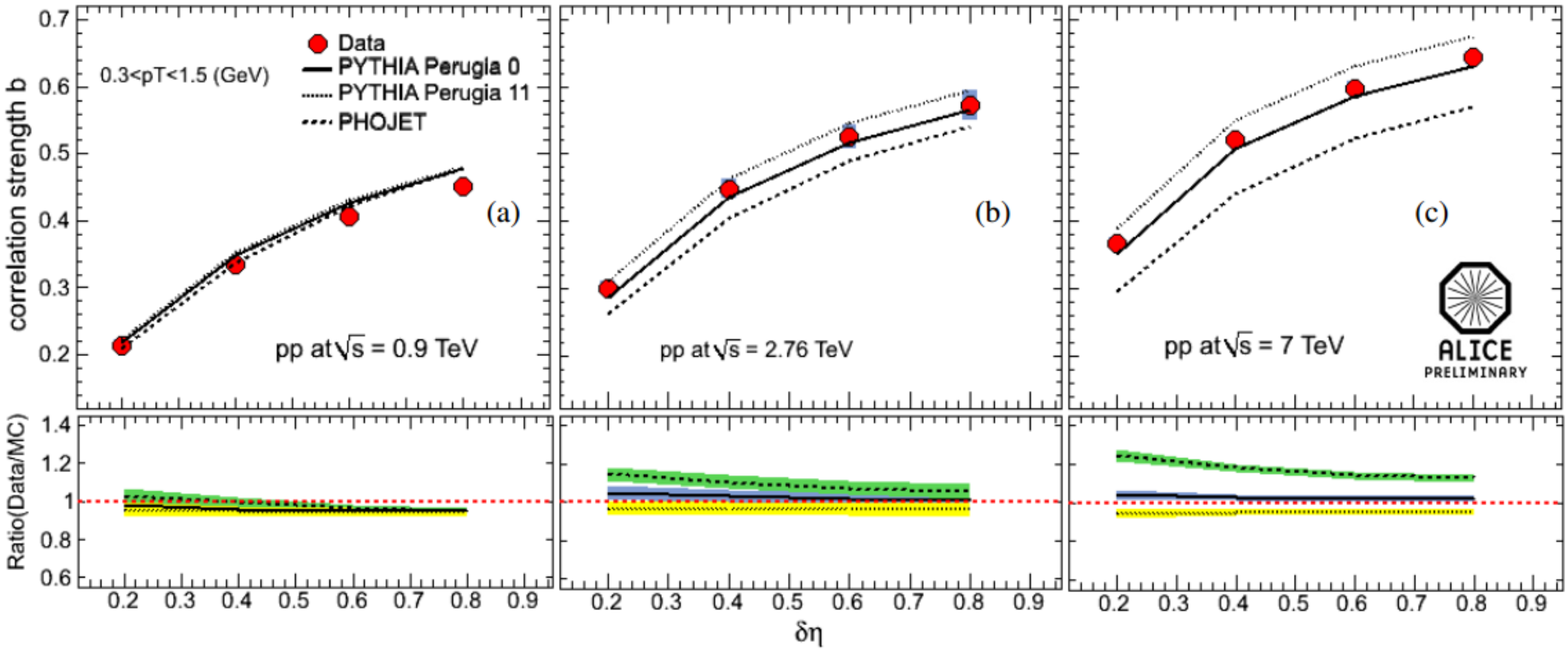
Comparison with generators:

- trends are reproduced
- PYTHIA reproduce the numbers better than PHOJET
- generators give similar estimations at 900 GeV but different at higher energies

η windows configurations

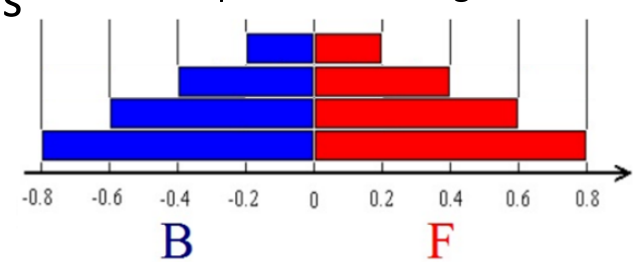


3) FB multiplicity correlations in separated η windows as function of windows width (+comparison to the generators)



- Correlations grow with windows η-width for all energies
- generators give similar estimations at 900 GeV but different at higher energies

η windows configurations



Conclusions on analysis of FB correlations in separated η-windows

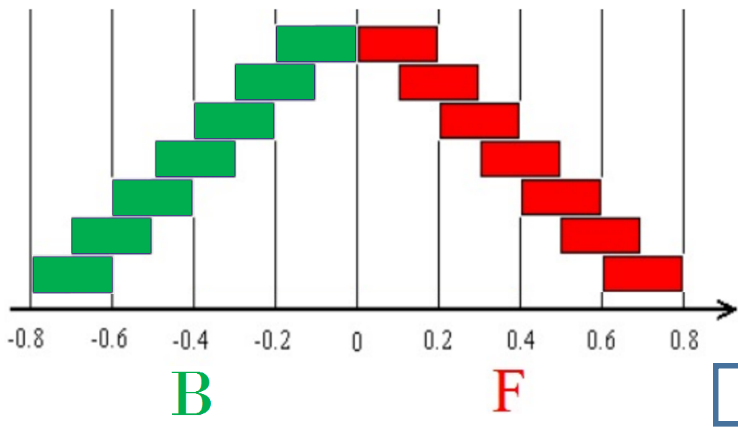
- Forward-backward multiplicity correlations were measured for minimum bias *pp* events in separated η windows for charged particles with transverse momenta 0.3-1.5 GeV/c.
- A considerable increase of the FB correlation strength b_{corr} with the growth of the collision energy is observed.
- b_{corr} increases with the width of pseudorapidity windows but only slightly decreases with the growth of the gap between the windows.

Models: FB multiplicity correlations in separated η and φ windows

Extension of FB multiplicity correlations study into *azimuthal* dimension: study correlations in η and φ separated window pairs

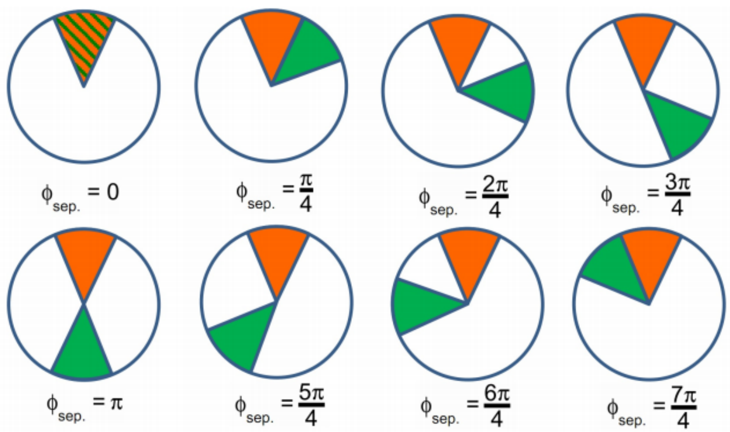
η and φ windows pairs chosen for the analysis:

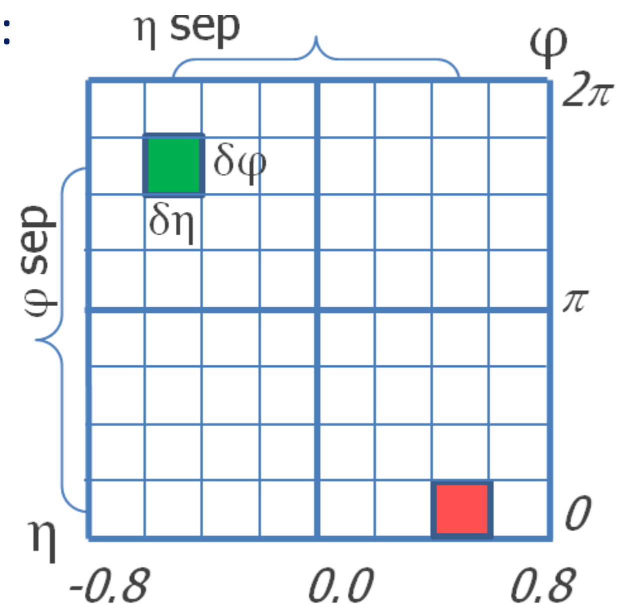
η windows

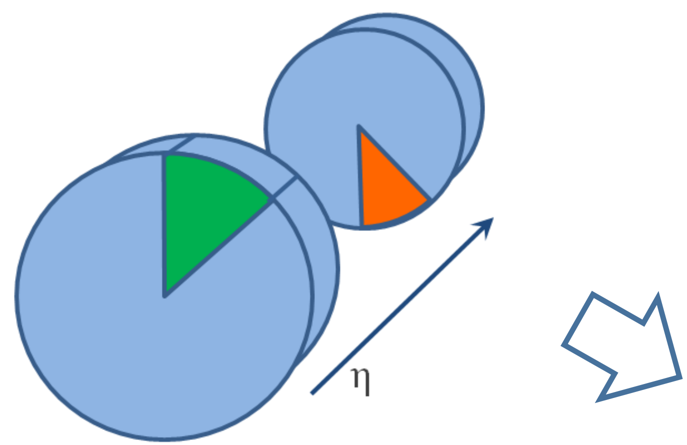


+

φ windows

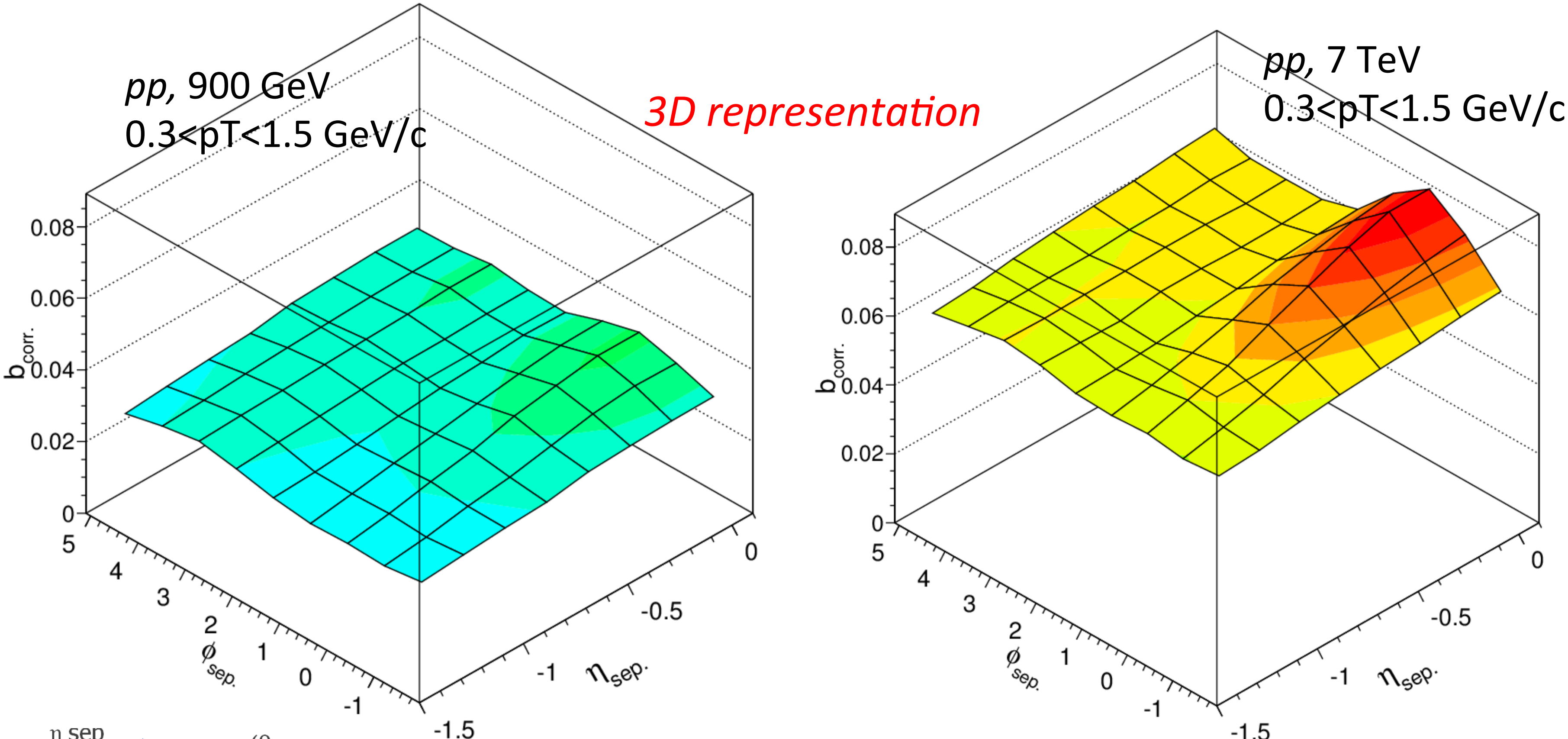


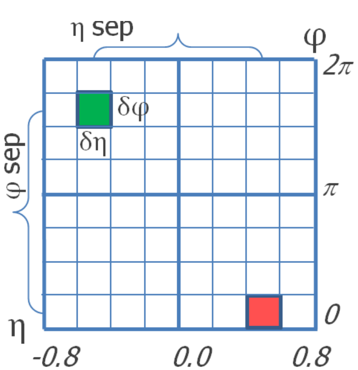




FB multiplicity correlations in separated η and φ windows

PYTHIA6, Perugia-2011





- short-range and long-range contributions are distinguishable
- non-zero plateau is observed and increases with the energy
- first look in data shows same behavior like in PYTHIA

Which processes have influence on multiplicity correlations?

try to look into generator:

Forward-Backward multiplicity correlations in *PYTHIA 8*

FB multiplicity correlations: *pp*, look at Pythia8

samples:
Pythia8 (8170)
Tune 4Cx
Beams:eCM = 7000 ! CM energy of collision

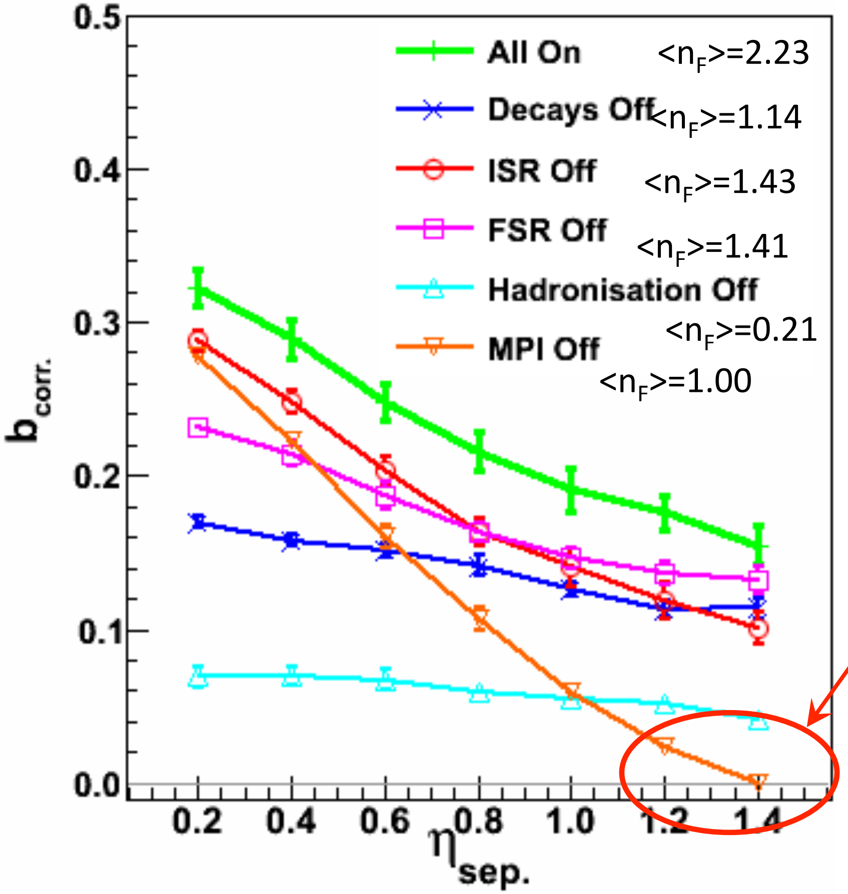
Can switch on/off the key event generation steps:

#PartonLevel:MPI = off	! no multiparton interactions
#PartonLevel:ISR = off	! no initial-state radiation
#PartonLevel:FSR = off	! no final-state radiation
#HadronLevel:Hadronize = off	! no hadronization
#HadronLevel:Decay = off	! no decays

⇨ 100k events were generated for each option set "off"

FB multiplicity correlations, *pp*, 7 TeV: look at Pythia8

Take separated in η windows (φ size is 2π):



($<n_F>$ is the mean multiplicity in Forward rapidity window)

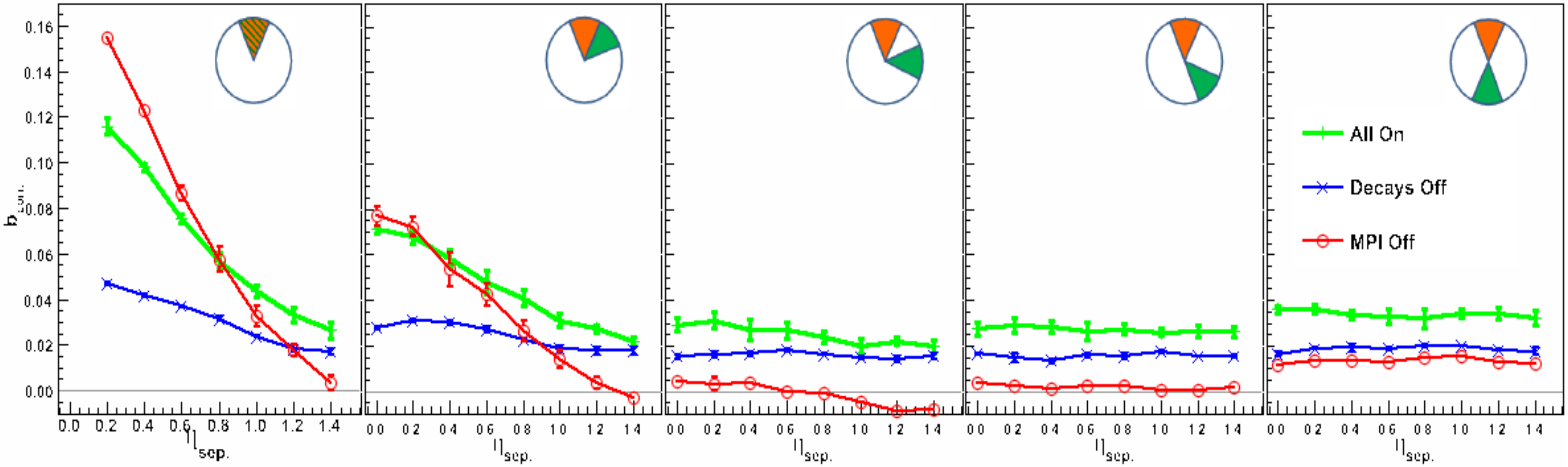
Switching off MPI removes fluctuations in number of emitting sources (in Pythia way of implementation).

That causes absence of correlations in distant η windows

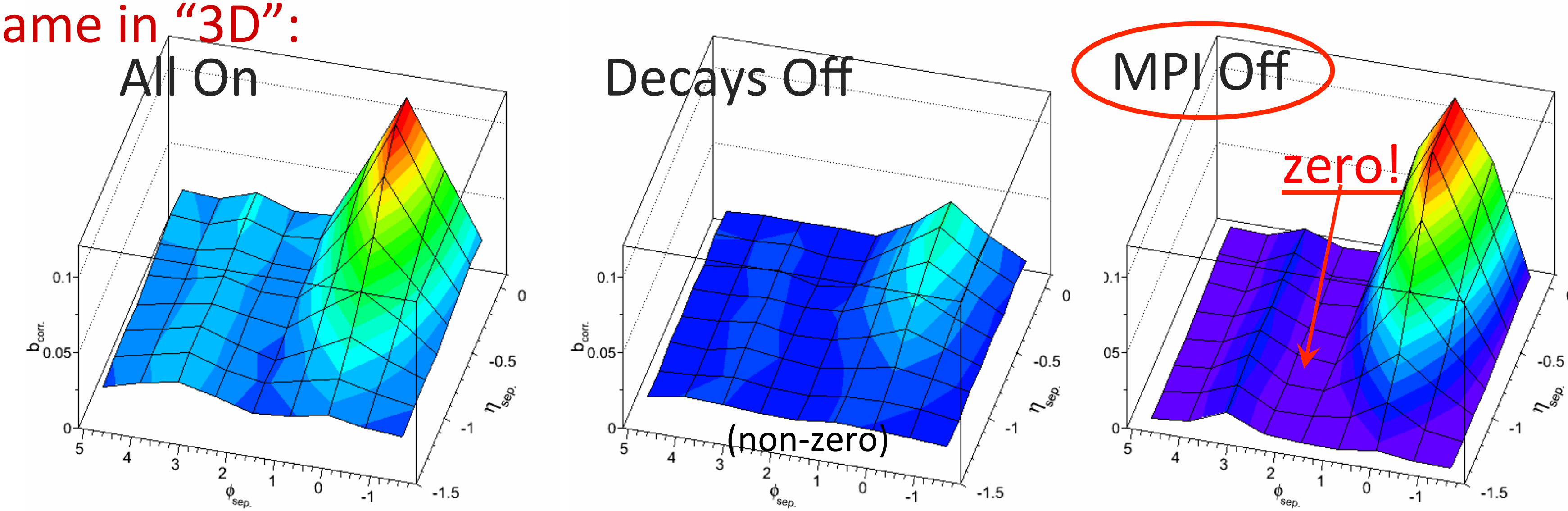
, while switching off the other processes leads mainly to multiplicity reduction.

FB multiplicity correlations : *pp*, 7 TeV, look at Pythia8

Take separated windows in both η and φ:



Same in "3D":



Model with strings as independent emitters [1,2]

Introduce $\delta a = \delta\eta\,\delta\phi/2\pi$ - *acceptance* of the forward and backward windows.

For windows with small acceptances in rapidity and azimuth situated in a mid rapidity region:

ω_N is the event-by-event scaled variance of the number of strings.

μ_0 is the average rapidity density of the charged particles produced by one string.

$\Lambda(\eta,\phi)$ is the pair correlation function of a single string

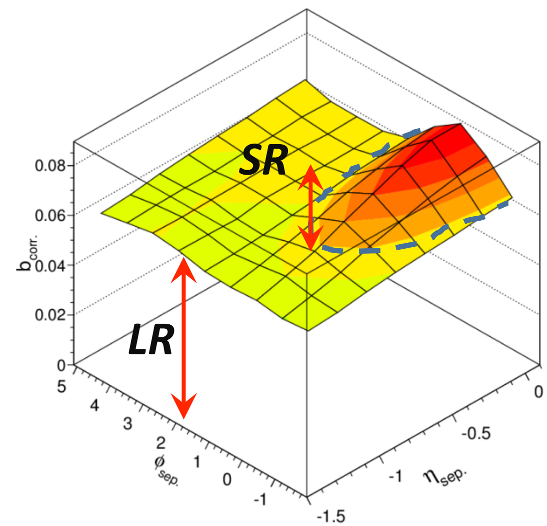
$$b_{corr} = b^{LR} + b^{SR}$$

$$b^{LR} = \frac{\omega_N \mu_0 \delta a}{1 + [\omega_N + \Lambda(0,0)]\mu_0 \delta a}$$

$$b^{SR} = \frac{\mu_0 \delta a}{1 + [\omega_N + \Lambda(0,0)]\mu_0 \delta a}\Lambda(\eta_{sep},\phi_{sep})$$

At $\Lambda(\eta,\phi) = 0$ we have

$$b^{SR}_{\Lambda=0} = 0, \text{ but } b^{LR}_{\Lambda=0} = \frac{\omega_N \mu_0 \delta a}{1 + \omega_N \mu_0 \delta a} \neq 0$$



b^{LR} (through the ω_N)
is sensible to the fluctuations in number of emitters

1) V.V. Vechernin, arXiv: 1305.0857, 2013
2) M.A. Braun, R.S. Kolevatov, C. Pajares, V.V. Vechernin, Eur. Phys. J. C32, 535 (2004).

Conclusions on FB correlations in separated η-φ windows in models

- Analysis of b_{corr} for various configurations of azimuthal sectors enables to separate the short-range (SR) and long-range (LR) effects.
  - the LR part arises due to event-by-event fluctuation of the number of emitters,
  - the SR part is due to pair correlation between particles produced by the same emitter.
- The LR part reveals itself as a common pedestal increasing with collision energy, which is eventually missed in di-hadron analysis.
- PYTHIA and PHOJET MC event generators and the model based on the string picture of hadronic interactions indicate that the behavior of b_{corr} in azimuth and rapidity is compatible with the multiparticle production by independent string emitters.
- Experimental results for η-φ windows are upcoming.

Summary

Experimental results:

- Forward-backward multiplicity correlations were measured for minimum bias *pp* events in separated η windows for charged particles with transverse momenta 0.3-1.5 GeV/c.
- A considerable increase of the FB correlation strength b_{corr} with the growth of the collision energy is observed.
- b_{corr} increases with the width of pseudorapidity windows but only slightly decreases with the growth of the gap between the windows.

Model analysis and interpretation

- Analysis of b_{corr} for various configurations of azimuthal sectors enables to separate the short-range (SR) and long-range (LR) effects:
  - the LR part arises due to event-by-event fluctuation of the number of emitters,
  - the SR part is due to pair correlation between particles produced by the same emitter.
- The LR part reveals itself as a common pedestal increasing with collision energy, which is eventually missed in di-hadron analysis.
- PYTHIA and PHOJET MC event generators and the model based on the string picture of hadronic interactions indicate that the behavior of b_{corr} collisions in azimuth and rapidity is compatible with the multiparticle production by independent string emitters.
- Experimental results for η-φ windows are upcoming.

Backup

Connection of the FB correlation coefficient with two-particle correlation function - 1

By definition the two-particle correlation function C_2 is defined through the inclusive ρ_1 and double inclusive ρ_2 distributions:

$$C_2(\eta_F,\phi_F;\eta_B,\phi_B) = \frac{\rho_2(\eta_F,\phi_F;\eta_B,\phi_B)}{\rho_1(\eta_F,\phi_F)\rho_1(\eta_B,\phi_B)} - 1 \qquad (1)$$

$$\rho_1(\eta,\phi) = \frac{d^2N}{d\eta\, d\phi}\,, \qquad \rho_2(\eta_F,\phi_F;\eta_B,\phi_B) = \frac{d^4N}{d\eta_F\, d\phi_F\, d\eta_B\, d\phi_B} \qquad (2)$$

To measure the ρ_1 one has by definition to take a small window $\delta\eta\,\delta\phi$ around η, ϕ, then

$$\rho_1(\eta,\phi) \equiv \frac{\langle n\rangle}{\delta\eta\,\delta\phi}\,, \qquad (3)$$

here $\langle n\rangle$ is the mean multiplicity in the acceptance $\delta\eta\,\delta\phi$.
One has to reduce the acceptance until the ratio (3) becomes constant.

Connection of the FB correlation coefficient with two-particle correlation function - 2

To measure the ρ_2 one has by definition to take TWO small windows: $\delta\eta_F\,\delta\phi_F$ around η_F, ϕ_F and $\delta\eta_B\,\delta\phi_B$ around η_B, ϕ_B, then

$$\rho_2(\eta_F,\phi_F;\eta_B,\phi_B) \equiv \frac{\langle n_F n_B\rangle}{\delta\eta_F\,\delta\phi_F\,\delta\eta_B\,\delta\phi_B} \ . \tag{4}$$

One has to reduce the acceptances of the observation windows until the ratio (4) becomes constant.
So by (3) and (4) the definition (1) means the following experimental procedure of the determination of the correlation function C_2:

$$C_2(\eta_F,\phi_F;\eta_B,\phi_B) \equiv \frac{\langle n_F n_B\rangle - \langle n_F\rangle\langle n_B\rangle}{\langle n_F\rangle\langle n_B\rangle} \ , \tag{5}$$

where n_F and n_B are the event multiplicities in TWO small windows: $\delta\eta_F\,\delta\phi_F$ around η_F, ϕ_F and $\delta\eta_B\,\delta\phi_B$ around η_B, ϕ_B.

Connection of the FB correlation coefficient with two-particle correlation function - 3

Traditionally one uses the following definition of the FB correlation coefficient:

$$b_{abs} \equiv \frac{\langle n_F n_B \rangle - \langle n_F \rangle \langle n_B \rangle}{\langle {n_F}^2 \rangle - \langle n_F \rangle^2} \qquad \text{or} \qquad b_{rel} \equiv \frac{\langle n_F \rangle}{\langle n_B \rangle} b_{abs} \tag{6}$$

For small FB windows by (5) we have

$$b_{abs} = \frac{\langle n_F \rangle \langle n_B \rangle}{D_{n_F}} C_2(\eta_F, \phi_F; \eta_B, \phi_B) \; , \qquad b_{rel} = \frac{\langle n_F \rangle^2}{D_{n_F}} C_2(\eta_F, \phi_F; \eta_B, \phi_B) \tag{7}$$

Note that for small forward window: $D_{n_F} \to \langle n_F \rangle$.
So by (7) we see that the traditional definition of the FB correlation coefficient in the case of TWO small observation windows coincides with the standard definition of two-particle correlation function C_2 upto some common factor $\langle n_B \rangle$ or $\langle n_F \rangle$, which depends on the width of windows.

Connection of the FB correlation coefficient with two-particle correlation function - 4

Note that one can go in C_2 to the variables:

$$\Delta\eta = \eta_F - \eta_B \ , \qquad \eta_C = (\eta_F + \eta_B)/2 \tag{8}$$

$$\Delta\phi = \phi_F - \phi_B \ , \qquad \phi_C = (\phi_F + \phi_B)/2 \tag{9}$$

and EXPERIMENTALLY check up the dependence of the two-particle correlation function C_2 on η_C for the different configurations and separations between FB observation windows.
Summing up, we see that by the standard definition (1) the experimental determination of the two-particle correlation function $C_2(\eta_F, \phi_F; \eta_B, \phi_B)$ requires (5) the measurements of the event multiplicities n_F and n_B in TWO SMALL windows: $\delta\eta_F\,\delta\phi_F$ around η_F, ϕ_F, and $\delta\eta_B\,\delta\phi_B$ around η_B, ϕ_B, which is performed in our approach.

Connection of the FB correlation coefficient with two-particle correlation function - 5

Note that the so-called di-hadron correlation function

$$C(\Delta\eta, \Delta\phi) \equiv S/B - 1 \ , \tag{10}$$

which takes into account all possible pair combinations of particles produced in given event in some ONE LARGE pseudorapidity window, where

$$S = \frac{d^2N}{d\Delta\eta \, d\Delta\phi} \tag{11}$$

and the B is the same but in the case of uncorrelated particle production, has only indirect connection with the standard definition (1) of the two-particle correlation function $C_2(\eta_F, \phi_F; \eta_B, \phi_B)$ and can coincide with it only in the case when the pseudorapidity translation invariance (the independence C_2 on η_C) takes place. This indirect method can also lead to the loss in $C(\Delta\eta, \Delta\phi)$ the common "pedestal", which takes place in $C_2(\Delta\eta, \Delta\phi)$ (see arXiv:1305.0857 for details).

Connection of the FB correlation coefficient with two-particle correlation function - 6

This common "pedestal" in $C_2(\Delta\eta, \Delta\phi)$ is physically important as, for example, in the model with fluctuating number N of the independent identical emitters (strings) one has the following expression for the $C_2(\Delta\eta, \Delta\phi)$:

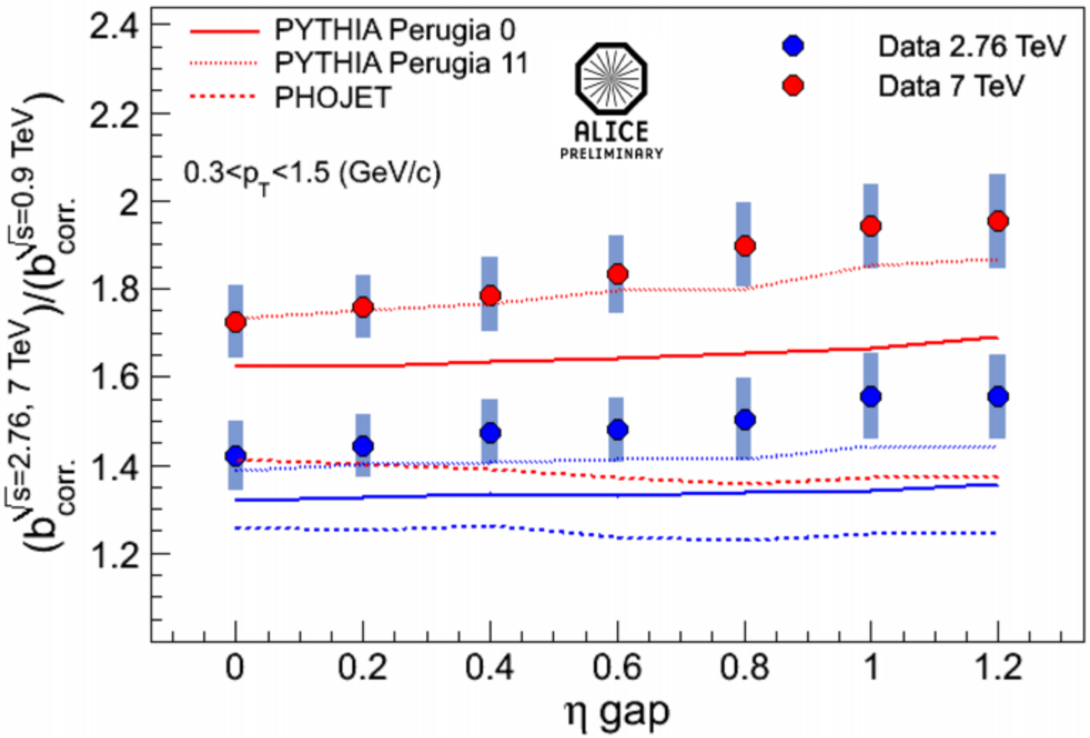
$$C_2(\Delta\eta, \Delta\phi) = \frac{\omega_N + \Lambda(\Delta\eta, \Delta\phi)}{\langle N \rangle} \ , \tag{12}$$

where the $\Lambda(\Delta\eta, \Delta\phi)$ is the pair correlation function of a single string and the $\omega_N = D_N/\langle N \rangle$ is the scaled variance of the event-by-event fluctuation of the number N of strings.

Hence by (12) we see that from the height of the "pedestal" ($\omega_N/\langle N \rangle$) one can obtain the important physical information on the magnitude of the fluctuation of N at different energies and centrality fixation.

Backup: Supplementary studies

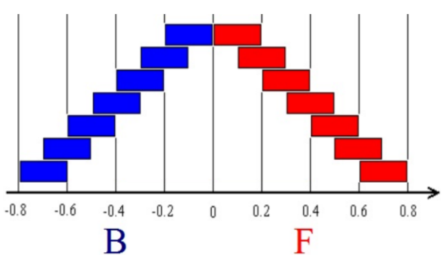
Result 4: Comparison with generators: b_{corr} ratios 2.76/0.9 and 7/0.9



b_{corr} ratio for 2.76 (blue circle) and 7 TeV (red circle) with respect to 0.9 TeV vs. η gap. The calculation from generators is also shown for 2.76 TeV (blue lines) and 7 TeV (red lines), PYTHIA Perugia-0 corresponds to solid lines, Perugia-2011 to dotted lines and PHOJET – to dashed lines.

- rate of increasing of b_{corr} in data is higher than that from MC generators.
- PYTHIA Perugia-2011 better describes the relative increase

η windows configurations



investigations with the
Toy model

The fluctuations in number of emitting sources could be seen in multiplicity-multiplicity correlation approach.

Question and aim: to understand (using toy model) could fluctuations in number of emitting sources be sensed in:

1) two-particle correlations approach
2) "per-trigger yield" approach

MC toy: realization of independent emitters model

Inspiration:

to clearly see observable effects when introduce some mechanisms in “physics”

The model is MC simulations of

- fluctuating number of strings and
- fluctuating number of particles from each string

Some other features:

- conservation of p_T for each string (sum_pT of particles=0)
- possible formation of “jets” and “decays” (gauss cone)

Toy model: part 1

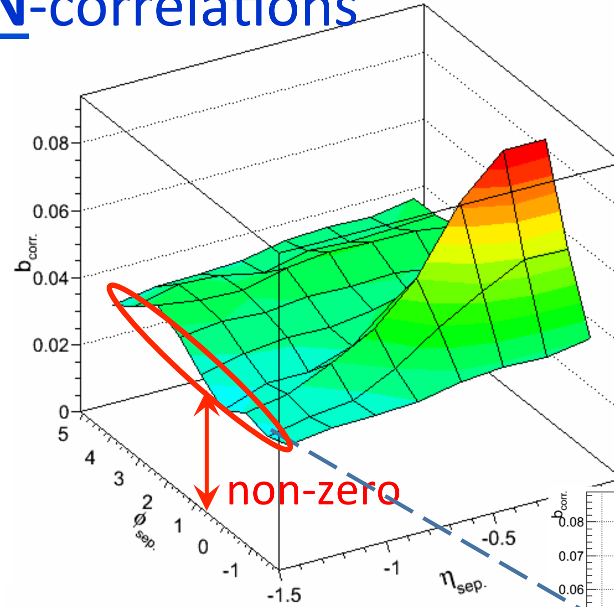
→ Compare multiplicity-multiplicity and di-hadron correlations approaches (using toy model)

MC string toy model results for **NN**-correlations

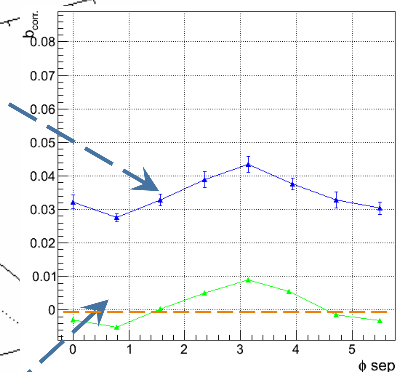
2Mln events

fluctuating number of strings:

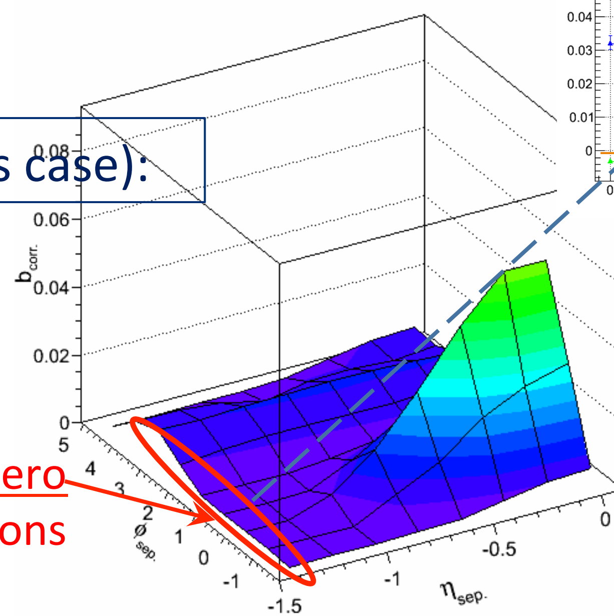
NN correlations



p_T 0.3-1.5



some **fixed** number of strings (=3 in this case):

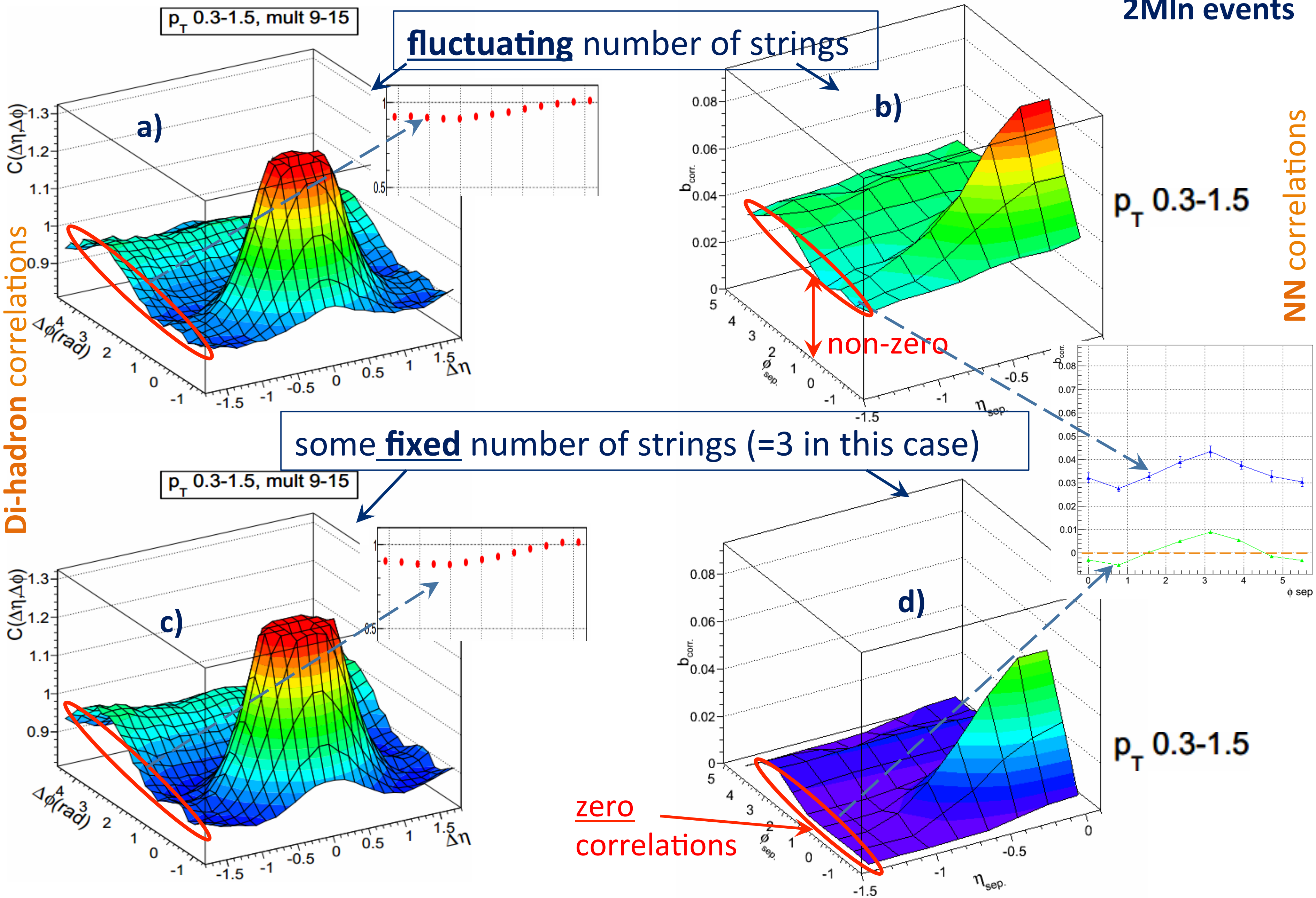


p_T 0.3-1.5

NN correlations are sensitive to fluctuations in number of emitting sources

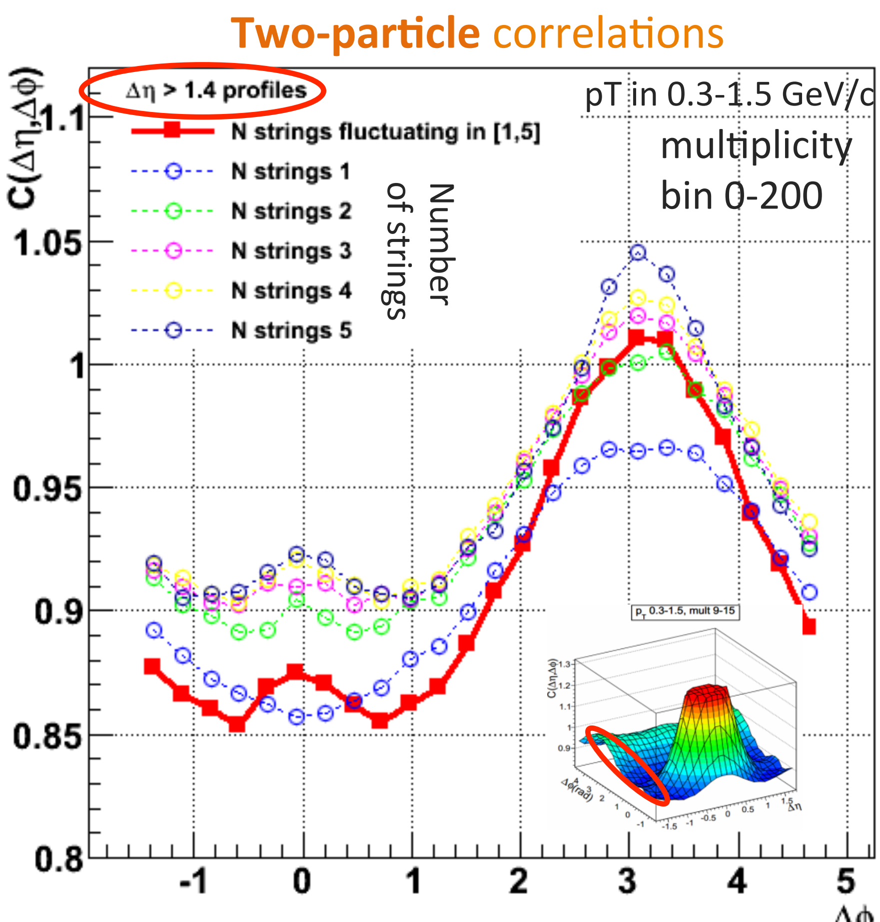
Now go to compare NN correlation results with 2-particle correlations approach

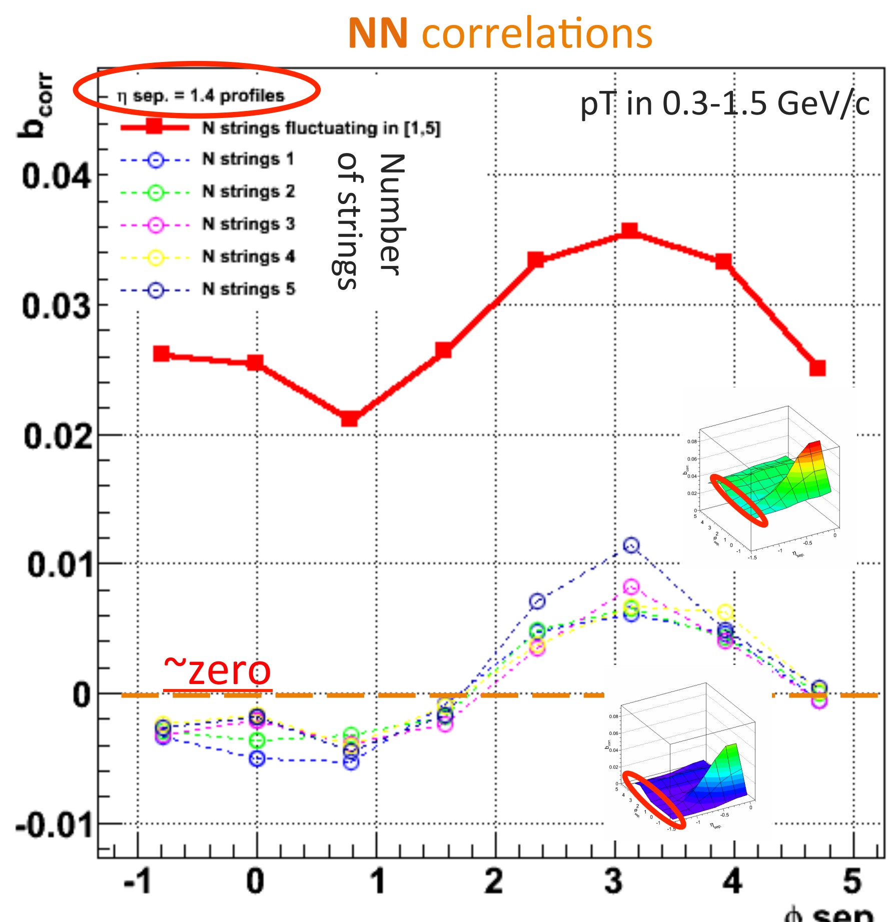
MC string toy model results: **di-hadron** vs **NN** correlations



Comparison of **di-hadron** and **NN**-correlations

(samples with fixed number of strings VS sample with fluctuating number)





FB multiplicity-multiplicity correlations enable to measure the correlations produced by fluctuations in number of sources. Di-hadron correlations are unable to distinguish such fluctuations.

Toy model: part 2

→ Compare multiplicity-multiplicity correlations and "per-trigger yield" approaches (using toy model)

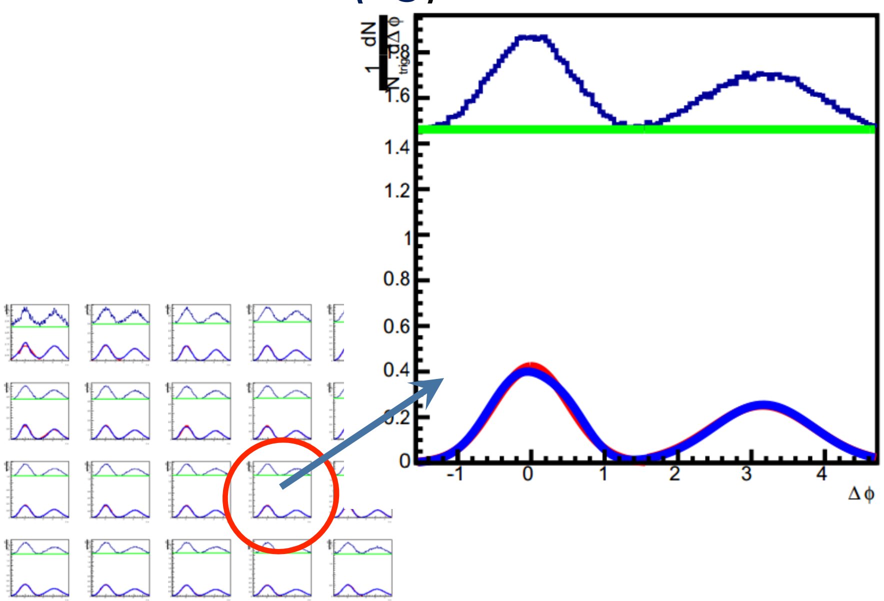
Move to **MC toy model**:

Aim (again): could we sense the fluctuations in number of emitting sources?

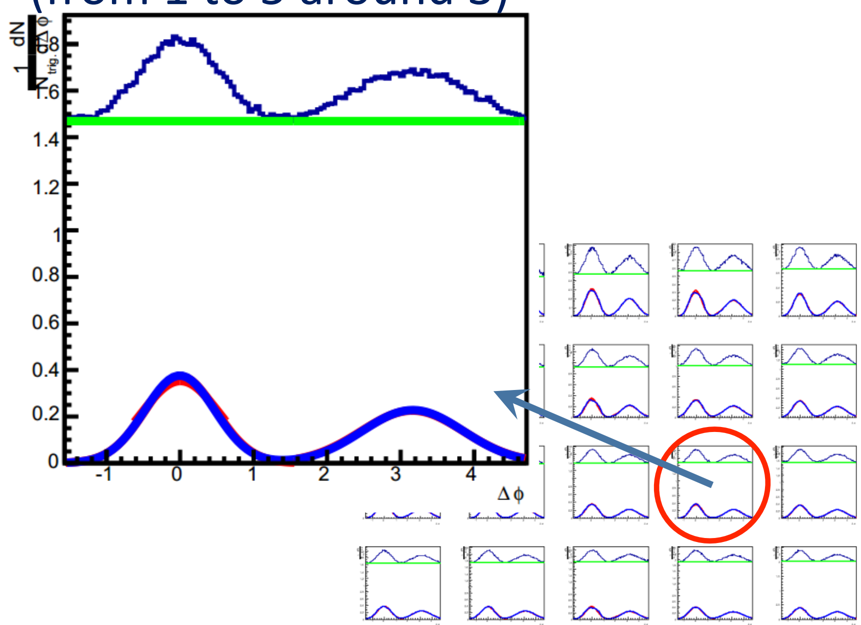
⇨ Let's see what per-trigger yields give on that in cases of:

2Mln events

fixed number of sources (=3)



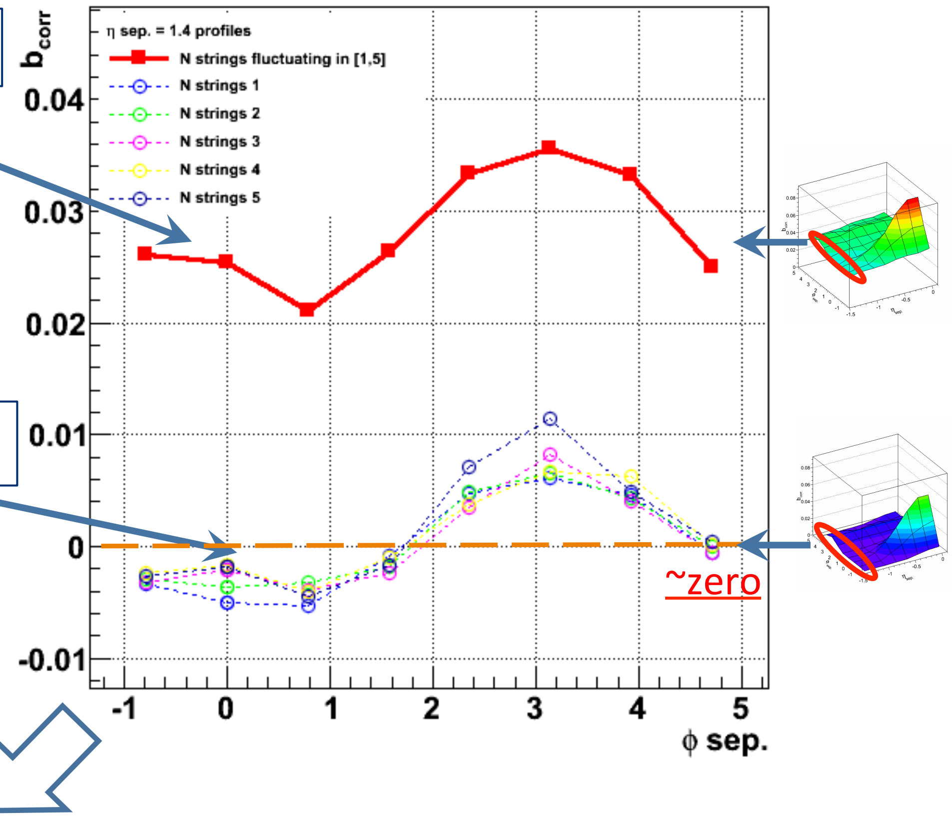
fluctuating number of sources (from 1 to 5 around 3)



…Finally (again back to NN-correlation approach): MC string toy model results for **NN**-correlations

events with **fluctuating** number of strings:

events with **fixed** number of strings



NN correlations are sensitive to fluctuations in number of emitting sources

MC toy: realization of independent string model

Aim:

- to clearly see observable effects when introduce some mechanisms in “physics”

Toy mechanics:

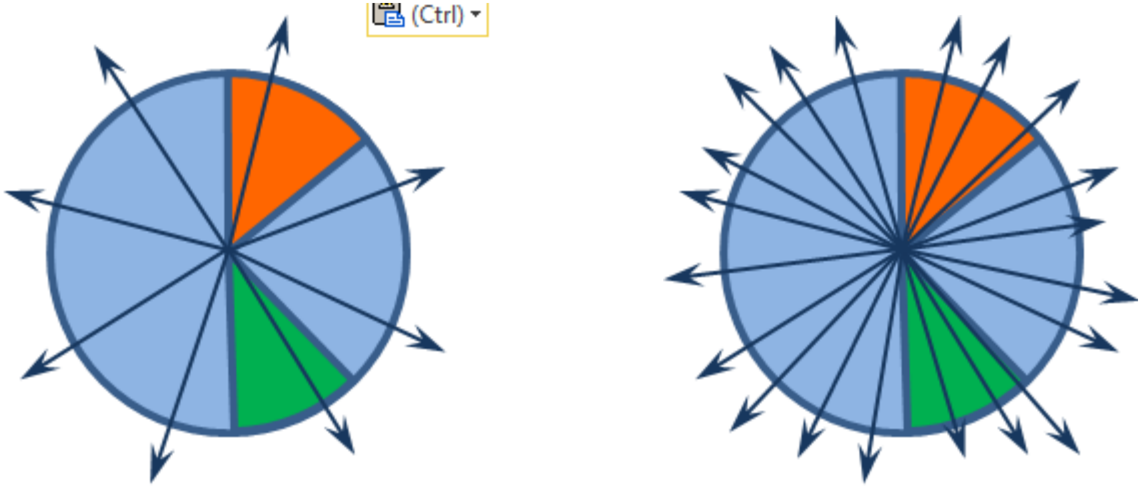
- number of strings: Poisson or another
- particles from each string: Poisson or another
- conservation of pT for each string (sum_pT of particles=0)
- possible formation of “jets” and “decays” (gauss cone)
- string fusion could be switched on

Main mechanism (I)

(to be updated)

E-by-e fluctuations of the number and properties of emitters (no η_{gap} and φ_{gap} dependence)

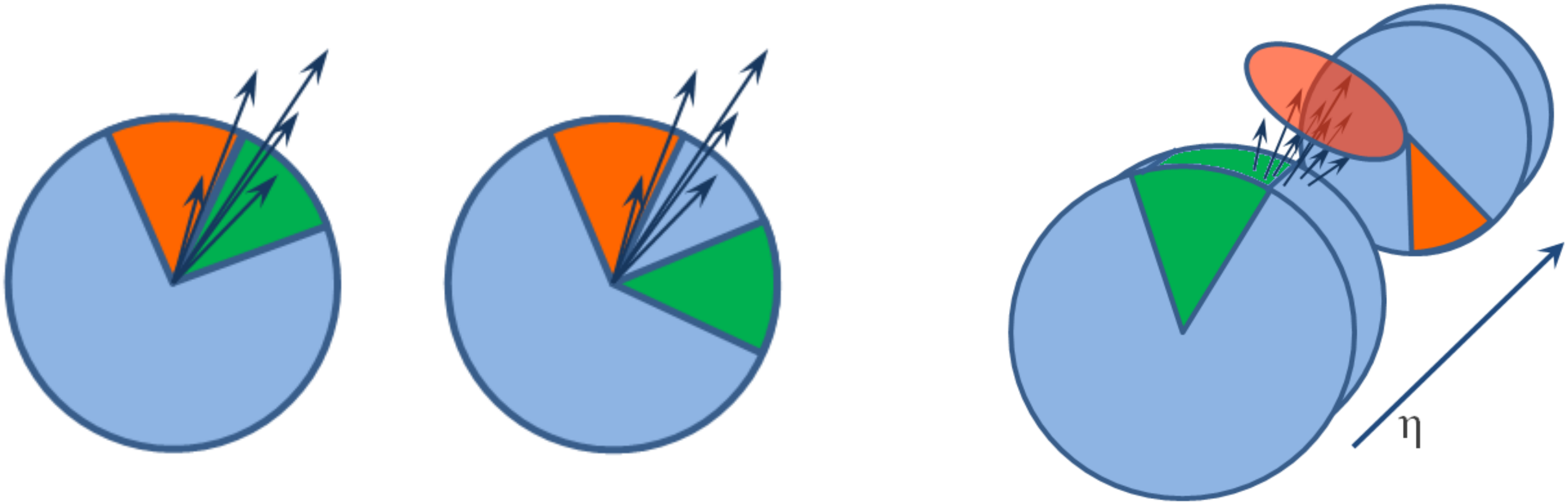
M.A. Braun, R.S. Kolevatov, C. Pajares, V.V. Vechernin, Eur. Phys. J. C32, 535 (2004).
V.V. Vechernin, arXiv:1012.0214, 2010.



Additional mechanism (II)

(to be updated)

Simultaneous contribution of one minijet in two our windows (η_{gap} and φ_{gap} dependent)



Alternative definitions correlation b_{corr}

$$\langle n_B \rangle_{n_F} = a + b\, n_F \ .$$

$$b = \frac{\langle n_F n_B \rangle - \langle n_F \rangle \langle n_B \rangle}{D_{n_F}} \ ,$$

$$b_{rel} = \frac{\langle \nu_F \nu_B \rangle - 1}{\langle \nu_F^2 \rangle - 1} = \frac{\langle n_F \rangle}{\langle n_B \rangle}\, b \ ,$$

$$b_{sym} = \frac{\langle n_F n_B \rangle - \langle n_F \rangle \langle n_B \rangle}{\sqrt{D_{n_F} D_{n_B}}} \ ,$$

Note that in the case of symmetric windows, when $\langle n_F \rangle = \langle n_B \rangle$ and $D_{n_F} = D_{n_B}$, all these definitions lead to the same result

$$b_{rel} = b_{sym} = b \ .$$

Another observables:

$$b_{mod} \equiv \frac{\langle n_F n_B \rangle - \langle n_F \rangle \langle n_B \rangle}{\langle n_F \rangle \langle n_B \rangle} = \left\langle \frac{n_F}{\langle n_F \rangle} \frac{n_B}{\langle n_B \rangle} \right\rangle - 1 \ .$$

$$b_{rob} \equiv \frac{\langle n_F n_B \rangle - \langle n_F \rangle \langle n_B \rangle}{\sqrt{D_{n_F} - \langle n_F \rangle} \sqrt{D_{n_B} - \langle n_B \rangle}}$$

(*) see V.V. Vechernin, arXiv: 1305.0857, 2013