# Novel mathematical aspects of Feynman integrals

O. V. Tarasov

Sochi, Russia , September 25 , 2011

# Outline of the talk:

- Introduction
- Mini review of useful features of
  - Generalized recurrence relations
  - Dimensional recurrences
  - Gröbner bases technique
  - Functional equations
- Finding new relationships for hypergeometric functions by evaluating Feynman integrals
- Conclusions

Radiative corrections to different physical quantities needed for the comparison of theoretical predictions with experimental data to be collected with the CERN Large Hadron Collider (LHC) and, in future, with an International Linear Collider (ILC) and other colliders are expressed in terms of complicated Feynman integrals. In many cases, radiative corrections must be evaluated analytically to achieve reliable accuracies in the calculations.

Researches working for the LHC collider presented famous ”wishlist“ – a list of physical processes where next to leading order radiative corrections are needed. Practically all these corrections require evaluation of radiative corrections with 5-, 6- and more external legs.

Characteristic features of these corrections:

- masses of many particles must be taken into account
- diagrams with many external lines, i.e.many kinematic variables must be calculated

Therefore one should know how to calculate analytically and (or) numerically with very high precision functions of many variables.

To perform such calculations new mathematical approaches are needed!

Rather novel concepts for such calculations were proposed during last several years:

- generalized recurrence relations
- Gröbner bases technique
- the method of dimensional recurrences
- functional equations

These methods and techniques are either recursive or strongly connected with recurrence relations. They do not exploit traditional integral representations or differential equations.

To extend applicability of these approaches their mathematical background should be further developed and certainly that will be useful in other fields of research like it was with computer algebra systems at the beginning of seventies.

In general Feynman diagrams are sums of tensor integrals. These integrals can be expressed as combinations of scalar integrals multiplied by products of tensors made of Mankowski tensor and external momenta.

There are essentially two different methods for reducing tensor integrals to scalar ones. One method (Passarino-Veltman) based on Ansatz for such integrals in terms of all possible combinations of Mankowski tensor and external momenta multiplied by unknown coefficients. For example,

$$\int \frac{d^dk\; k_\mu k_\nu}{k^2(k-p_1)^2(k-p_2)^2} = x_1 g_{\mu\nu} + x_2 p_{1\mu}p_{1\nu} + x_3 p_{1\mu}p_{2\nu} + x_4 p_{2\mu}p_{1\nu} + x_5 p_{2\mu}p_{2\nu}$$

Contracting this Ansatz with $g_{\mu\nu}$, $p_{1\mu}p_{1\nu}$, ... one get system of equations for $x_1, ..., x_5$. Solution for $x_1, ..., x_5$ will be given in terms of scalar integrals. In a similar way one can get representation for more complicated integrals. For higher rank tensor integrals such method leads to very big complicated systems of equations.

Another method for reducing tensor integrals is attributed to introducing auxiliary vectors $v_j$, representing products of integration vectors as derivatives w.r.t. these vectors,

$$k_{1\mu_1}\ldots k_{L,\mu_r} = \frac{1}{i^r}\frac{\partial}{\partial v_{\mu_1}}\ldots\frac{\partial}{\partial v_{\mu_r}}\exp[iv_jk_j]|_{v_s=0}\,,$$

transforming the resulting momentum integrals into integrals over Feynman parameters. From this parametric representation for an arbitrary tensor integral one can obtain the following formula:

$$\int d^dk_1\ldots\int d^dk_L\frac{k_{1\mu}\ldots k_{N\nu}}{(\overline{k}_1^2-m_1^2)^{\nu_1}\ldots(\overline{k}_N^2-m_N^2)^{\nu_N}}$$
$$= T_{\mu,\ldots,\nu}(q,\partial,\mathbf{d}^+)\int d^dk_1\ldots\int d^dk_L\frac{1}{(\overline{k}_1^2-m_1^2)^{\nu_1}\ldots(\overline{k}_N^2-m_N^2)^{\nu_N}}$$

where

$$\mathbf{d}^+G^{(d)} = G^{(d+2)} \quad \text{and} \quad \partial_j = \frac{\partial}{\partial m_j^2}$$

A general formula for the polynomial tensor operator $T_{\mu,\ldots,\nu}(q,\partial,\mathbf{d}^+)$ was given by O.T., Phys.Rev. D54 .

This method is very efficient and it is easily implementable on computers. There is no need to solve huge systems of linear equations. However integrals with different shifts of the space-time dimension do appear. To reduce all such scalar integrals to basic set of integrals the method of generalized recurrence relations was developed O.V. T., Phys.Rev. D.54(1996) 6479. To obtain recurrence relations one can use 't Hooft and Veltman idea (Nucl.Phys. B44 (1972) 189) that

$$\int d^dk_1 \ldots \int d^dk_L \frac{\partial}{\partial k_{j\mu}} \frac{l_{j\mu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}} = 0,$$

where $\overline{k}_j$ are linear combinations of external and integration momenta, $l_\mu$- is either integration or external momentum.

Scalar products emerging after differentiation w.r.t. $k$, can be represented as combinations of factors in denominators, masses and external momenta (IBP method):

$$k_1 q_1 = \frac{1}{2}\{[(k_1+q_1)^2 - m_1^2] - [k_1^2 - m_1^2] - q_1^2\}, \ldots.$$

When it is not possible one should introduce into consideration artificial factors in denominators. As a result one gets recurrence relations connecting integrals with different powers of propagators $\nu_j$.

One can represent emerging scalar products in terms of integrals with shifted space-time dimension by exploiting the formula:

$$\int d^dk_1 \ldots \int d^dk_L \frac{k_{1\mu} \ldots k_{N\nu}}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}}$$
$$= T_{\mu,\ldots,\nu}(q, \partial, \mathbf{d}^+) \int d^dk_1 \ldots \int d^dk_L \frac{1}{(\overline{k}_1^2 - m_1^2)^{\nu_1} \ldots (\overline{k}_N^2 - m_N^2)^{\nu_N}}$$

As a result one gets recurrence relations connecting integrals with different powers of propagators $\nu_j$ and also integrals with different dimensionality $d$.

Important: These recurrence relations additionally to $\nu_j$ have new recurrence parameter - $d$. For this reason we call them generalized recurrence relations

With this new parameter one can construct very efficient algorithms for reducing scalar integrals to a set of bases integrals.

The system of generalized recurrence relations is strongly overdetermined. To find minimal set of recurrence relations allowing to reduce scalar integrals to minimal set of integrals it was proposed to use

Theory of Gröbner bases and Buchberger algorithm.

To use theory of Gröbner bases for recurrence relations for Feynman integrals for the first time was proposed by

1. O.V. T
Reduction of Feynman graph amplitudes to a minimal set of basic integrals ,
Acta Physica Polonica, v B29 (1998) 2655

2. O. V. T,
Computation of Gröbner bases for two-loop propagator type integrals,
Talk at ACAT-2003
Nucl. Instrum. Meth. A 534 (2004) 293 [arXiv:hep-ph/0403253]

## Gröbner bases for IBP relations for the two-loop self energy integrals

$$J_3^{(d)}(\nu_1,\nu_2,\nu_3) = \frac{1}{(i\pi^{d/2})^2}\int\int \frac{d^dk_1 d^dk_2}{(k_1^2-m_1^2)^{\nu_1}((k_1-k_2)^2-m_2^2)^{\nu_2}(k_2^2-m_3^2)^{\nu_3}}.$$

Gröbner bases for IBP recurrence relations:

$$\Delta_{123}\nu_1 \mathbf{1}^+ J_3^{(d)}(\nu_1\nu_2\nu_3) = \big\{u_{123}(d-\nu_1-2\nu_2)+2m_2^2(\nu_1-\nu_2)$$
$$+u_{312}\nu_1\mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^-) + 2m_2^2\nu_2\mathbf{2}^+(\mathbf{1}^- - \mathbf{3}^-)\big\}\, J_3^{(d)}(\nu_1\nu_2\nu_3).$$

$$\Delta_{123}\nu_2 \mathbf{2}^+ J_3^{(d)}(\nu_1\nu_2\nu_3) = \big\{u_{213}(d-\nu_2-2\nu_1)+2m_1^2(\nu_2-\nu_1)$$
$$+u_{321}\nu_2\mathbf{2}^+(\mathbf{1}^- - \mathbf{3}^-) + 2m_1^2\nu_1\mathbf{1}^+(\mathbf{2}^- - \mathbf{3}^-)\big\}\, J_3^{(d)}(\nu_1\nu_2\nu_3).$$

$$\Delta_{123}\nu_3 \mathbf{3}^+ J_3^{(d)}(\nu_1\nu_2\nu_3) = \big\{u_{312}(d-\nu_3-2\nu_1)+2m_1^2(\nu_3-\nu_1)$$
$$+u_{231}\nu_3\mathbf{3}^+(\mathbf{1}^- - \mathbf{2}^-) + 2m_1^2\nu_1\mathbf{1}^+(\mathbf{3}^- - \mathbf{2}^-)\big\}\, J_3^{(d)}(\nu_1\nu_2\nu_3).$$

where $\mathbf{1}^\pm J_3^{(d)}(\nu_1,\nu_2\nu_3) = J_3^{(d)}(\nu_1\pm 1,\nu_2,\nu_3)$,..., $u_{ijk}=m_i-m_j-m_k$ and

$$\Delta_{ijk} = m_i^4+m_j^4+m_k^4-2m_i^2m_j^2-2m_i^2m_k^2-2m_j^2m_k^2.$$

Gröbner bases for generalized recurrence relations:

$$(d-2)\nu_1\mathbf{1}^+ J_3^{(d)}(\nu_1\nu_2\nu_3) = \left\{-u_{123} - \mathbf{1}^- + \mathbf{2}^- + \mathbf{3}^-\right\} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3),$$
$$(d-2)\nu_2\mathbf{2}^+ J_3^{(d)}(\nu_1\nu_2\nu_3) = \left\{-u_{213} - \mathbf{2}^- + \mathbf{1}^- + \mathbf{3}^-\right\} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3),$$
$$(d-2)\nu_3\mathbf{3}^+ J_3^{(d)}(\nu_1\nu_2\nu_3) = \left\{-u_{321} - \mathbf{3}^- + \mathbf{2}^- + \mathbf{1}^-\right\} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3),$$

$$(d-2)(d-\nu_1-\nu_2-\nu_3)J_3^{(d)}(\nu_1,\nu_2,\nu_3) = \\ -\left\{\Delta_{123} + u_{123}\mathbf{1}^- + u_{213}\mathbf{2}^- + u_{312}\mathbf{3}^-\right\} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3).$$

By exploiting Gröbner bases either for IBP relations or for generalized recurrence relations and explicit formula for tadpole integral:

one can reduce any integral $J_3^{(d)}(\nu_1,\nu_2,\nu_3)$ with integer $\nu_,\nu_2,\nu_3$ to the set of basic integrals $J_3^{(d)}(1,1,1)$, $J_3^{(d)}(0,1,1)$ , $J_3^{(d)}(1,0,1)$ , $J_3^{(d)}(1,1,0)$.

It turns out that Gröbner bases for generalized recurrence relations is much more efficient than for IBP relations!!

Example: reduction of the integral $J_3^{(d)}(3,5,4)$
IBP relations: 72 sec
Generalized recurrence relations: 9 sec

For higher powers of propagators the difference in time is more than 20 times.

There are even more efficient relations. For the considered example they are:

$$(d-2)\nu_1\nu_2\mathbf{1}^+\mathbf{2}^+J_3^{(d)}(\nu_1,\nu_2,\nu_3) = \left\{-2m_3^2\nu_3\mathbf{3}^+ + (d-2-2\nu_3)\right\} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3),$$
$$(d-2)\nu_1\nu_3\mathbf{1}^+\mathbf{3}^+J_3^{(d)}(\nu_1,\nu_2,\nu_3) = \left\{-2m_2^2\nu_1\mathbf{2}^+ + (d-2-2\nu_2)\right\} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3),$$
$$(d-2)\nu_2\nu_3\mathbf{2}^+\mathbf{3}^+J_3^{(d)}(\nu_1,\nu_2,\nu_3) = \left\{-2m_1^2\nu_1\mathbf{1}^+ + (d-2-2\nu_1)\right\} J_3^{(d-2)}(\nu_1,\nu_2,\nu_3),$$
$$\left\{\nu_1 1^+ + \nu_2 2^+ + \nu_3 3^+ - (d-\nu_1-\nu_2-\nu_3)\right\} J_3^{(d)}(\nu_1,\nu_2,\nu_3) = 0.$$

To find these relations we used Gröbner bases. For the considered integral $J_3^{(d)}(3,5,4)$ exploiting above relations only 3 seconds were needed to reduce it to basic integrals.

The reason is that this optimal set of relations has no explicit dependence on kinematical Gram determinants! This was one of the criteria for finding those relations. Reduction of $d \to d-2$ was very essential! Gram determinants disappear only in relations connecting integrals with different dimensions of the space-time!

Similar recurrence relations were discovered for the one-loop multi-leg integrals. The calculations of the one-loop five gluon amplitude are now in progress. Depending on diagram evaluations are from 10 to 100 times faster than with Gröbner bases for generalized recurrence relations.

## Dimensional recurrences

Dimensional recurrences are particular case of generalized recurrence relations. They include integral with fixed powers of propagators but with different shifts of the space - time dimension and simpler integrals considered as inhomogeneous part of the equation.

General solution of dimensional recurrences can be written in the form:

$$M_k(d, \{m_j\}, \{p_ip_k\}) = \sum_s \Phi_s(d, \{m_j\}, \{p_ip_k\})\ w_s(d, \{m_j\}, \{p_ip_k\})$$

where $\Phi_s$ are functions from the fundamental set of solutions for and $w_s$ are the so-called ‘periodics‘ satisfying the following condition:

$$w_s(d+2, \{m_j\}, \{p_ip_k\}) = w_s(d, \{m_j\}, \{p_ip_k\})$$

They can be found, for example, from the comparison of the above solution with the asymptotic value of the integral at $d \to \infty$. In some cases one can obtain simple differential equation with respect to kinematic variables for $w_s(d, \{m_j\}, \{p_ip_k\})$.

With the help of these algorithms new analytic results were obtained:

- hypergeometric representation for the one - loop integrals corresponding to diagrams with three- and four external legs
- analytic formula for the one-loop massless pentagon type integral
- hypergeometric representation for the two-loop 'sunrise' propagator type integral

Recently Li and Smirnov applied this method in calculating four loop massless propagator type integrals satisfying first order dimensional recurrence relation.

## Functional equations for Feynman integrals

Feynman integrals satisfy recurrence relations which we write in the form

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where $Q_j, R_k$ are polynomials in masses, scalar products of external momenta, $d$, and powers of propagators. $I_{k,r}$ - are integrals with $r$ external lines. In recurrence relations some integrals are more complicated than the others: they have more arguments than the others.

### General method for deriving functional equations:

By choosing kinematic variables, masses, indices of propagators remove most complicated integrals, i.e. impose conditions :

$$Q_j = 0$$

keeping at least some other coefficients different from zero

$$R_k \neq 0$$

Example: one-loop $n$-point integrals

Integrals $I_n^{(d)}$ satisfy generalized recurrence relations O.T. in Phys.Rev.D54 (1996) p.6479

$$G_{n-1}\nu_j \mathbf{j}^+ I_n^{(d+2)} - (\partial_j \Delta_n) I_n^{(d)} = \sum_{k=1}^{n} (\partial_j \partial_k \Delta_n) \mathbf{k}^- I_n^{(d)},$$

where $\mathbf{j}^{\pm}$ shifts indices $\nu_j \to \nu_j \pm 1$,

$$\partial_j \equiv \frac{\partial}{\partial m_j^2},$$

$$G_{n-1} = -2^n \begin{vmatrix} p_1p_1 & p_1p_2 & \dots & p_1p_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_1p_{n-1} & p_2p_{n-1} & \dots & p_{n-1}p_{n-1} \end{vmatrix},$$

$$\Delta_n = \begin{vmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{1n} & Y_{2n} & \dots & Y_{nn} \end{vmatrix}, \qquad Y_{ij} = m_i^2 + m_j^2 - p_{ij}, \qquad p_{ij} = (p_i - p_j)^2,$$

At $n = 3,\ j = 1$ we get equation:

$$\begin{aligned}
&G_2 \mathbf{1}^+ I_3^{(d+2)}(m_1^2, m_2^2, m_3^2, p_{23}, p_{13}, p_{12}) \\
&\quad -(\partial_1 \Delta_3) I_3^{(d)}(m_1^2, m_2^2, m_3^2, p_{23}, p_{13}, p_{12}) \\
&= 2(p_{12} + p_{23} - p_{13}) I_2^{(d)}(m_1^2, m_2^2, p_{12}) \\
&\quad +2(p_{13} + p_{23} - p_{12}) I_2^{(d)}(m_1^2, m_3^2, p_{13}) - 4p_{23} I_2^{(d)}(m_2^2, m_3^2, p_{23}).
\end{aligned}$$

where

$$\begin{aligned}
G_2 &= 2p_{12}^2 + 2p_{13}^2 + 2p_{23}^2 - 4p_{13}p_{23} - 4p_{12}p_{13} - 4p_{23}p_{12}, \\
\Delta_3 &= 2(m_2^2 - m_3^2)[(m_1^2 - m_2^2)p_{13} - (m_1^2 - m_3^2)p_{12}] - 2m_1^2 p_{23}^2 - 2m_3^2 p_{12}^2 \\
&\quad -2m_2^2 p_{13}^2 - 2(m_1^2 - m_3^2)(m_1^2 - m_2^2)p_{23} + 2(m_2^2 + m_3^2)p_{12}p_{13} \\
&\quad +2(m_3^2 + m_1^2)p_{23}p_{12} + 2(m_2^2 + m_1^2)p_{13}p_{23} - 2p_{12}p_{13}p_{23}
\end{aligned}$$

Coefficients in front of $I_3$ depend on 6 variables $p_{12}, p_{13}, p_{23}$, $m_1^2, m_2^2, m_3^2$. To remove $I_3$ from the equation we must solve system of equations

$$
\begin{aligned}
G_2 &= 2p_{12}^2 + 2p_{13}^2 + 2p_{23}^2 - 4p_{13}p_{23} - 4p_{12}p_{13} - 4p_{23}p_{12} = 0, \\
\partial_1 \Delta_3 &= -2p_{23}^2 - 4m_1^2 p_{23} + 2m_2^2 p_{23} + 2m_3^2 p_{23} + 2p_{12} m_3^2 \\
&\quad + 2m_2^2 p_{13} - 2m_3^2 p_{13} + 2p_{13}p_{23} - 2m_2^2 p_{12} + 2p_{23}p_{12} = 0
\end{aligned}
$$

This system can be resolved w.r.t. $p_{13}, p_{23}$. There is a nontrivial solution

$$
p_{13} = s_{13}(m_1^2, m_2^2, m_3^2, p_{12}) = \frac{\Delta_{12} + 2p_{12}(m_1^2 + m_3^2) - (p_{12} + m_1^2 - m_2^2)\lambda}{2p_{12}},
$$

$$
p_{23} = s_{23}(m_1^2, m_2^2, m_3^2, p_{12}) = \frac{\Delta_{12} + 2p_{12}(m_2^2 + m_3^2) + (p_{12} - m_1^2 + m_2^2)\lambda}{2p_{12}}.
$$

where

$$
\lambda = \pm\sqrt{\Delta_{12} + 4p_{12}m_3^2}.
$$

$$
\Delta_{ij} = p_{ij}^2 + m_i^4 + m_j^4 - 2p_{ij}m_i^2 - 2p_{ij}m_j^2 - 2m_i^2 m_j^2.
$$

This solution leads to the following functional equation

$$I_2^{(d)}(m_1^2, m_2^2, p_{12}) = \frac{p_{12} + m_1^2 - m_2^2 - \lambda}{2p_{12}} \; I_2^{(d)}(m_1^2, m_3^2, s_{13}(m_1^2, m_2^2, m_3^2, p_{12}))$$
$$+ \frac{p_{12} - m_1^2 + m_2^2 + \lambda}{2p_{12}} \; I_2^{(d)}(m_2^2, m_3^2, s_{23}(m_1^2, m_2^2, m_3^2, p_{12})).$$

Substituting $m_3^2 = 0$ into functional equation we have :

$$
\begin{aligned}
I_2^{(d)}(m_1^2, m_2^2, p_{12}) &= \frac{p_{12} + m_1^2 - m_2^2 - \alpha_{12}}{2p_{12}}\, I_2^{(d)}(m_1^2, 0, s_{13}) \\
&+ \frac{p_{12} - m_1^2 + m_2^2 + \alpha_{12}}{2p_{12}}\, I_2^{(d)}(0, m_2^2, s_{23})
\end{aligned}
$$

where

$$
\begin{aligned}
s_{13} &= \frac{\Delta_{12} + 2p_{12}m_1^2 - (p_{12} + m_1^2 - m_2^2)\alpha_{12}}{2p_{12}}, \\
s_{23} &= \frac{\Delta_{12} + 2p_{12}m_2^2 + (p_{12} - m_1^2 + m_2^2)\alpha_{12}}{2p_{12}}, \\
\alpha_{12} &= \pm\sqrt{\Delta_{12}}.
\end{aligned}
$$

Integral with arbitrary mases and momentum can be expressed in terms of integrals with one propagator massless !!!

Analytic result for $I_2^{(d)}(0, m^2, p^2)$ is known
Bollini and Giambiagi (1972b), Boos and Davydychev (1990rg):

$$I_2^{(d)}(0, m^2, p^2) = I_2^{(d)}(0, m^2, 0)\, {}_2F_1\left[\begin{array}{cc} 1, 2-\frac{d}{2}; & \frac{q^2}{m^2} \\ \frac{d}{2}; & \end{array}\right].$$

where

$$I_2^{(d)}(0, m^2, 0) = -\Gamma\left(1 - \frac{d}{2}\right) m^{d-4}.$$

Substituting this expression for $I_2^{(d)}(0, m^2, p^2)$ into functional equation we get complete agreement with the known result for $I_2^{(d)}(m_1^2, m_2^2, p_{12})$

Setting $m_2 = 0$ into the previous functional equation we have :

$$I_2^{(d)}(m_1^2, 0, p_{12}) = \frac{m_1^2}{p_{12}} I_2^{(d)}\left(m_1^2, 0, \frac{m_1^4}{p_{12}}\right) + \frac{(p_{12}-m_1^2)}{p_{12}} I_2^{(d)}\left(0, 0, \frac{(p_{12}-m_1^2)^2}{p_{12}}\right).$$

where

$$I_2^{(d)}(0,0,p^2) = \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma(d-2)}(-p^2)^{\frac{d}{2}-2}.$$

Integral $I_2^{(d)}$ on the right hand side has inverse argument . In fact this equation corresponds to the well known formula for analytic continuation:

$$ {}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2}\,; \\ \frac{d}{2}\,; \end{array} z\right] = \frac{1}{z}\ {}_2F_1\left[\begin{array}{c} 1, 2-\frac{d}{2}\,; \\ \frac{d}{2}\,; \end{array} \frac{1}{z}\right] + \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(d-2)}(-z)^{\frac{d}{2}-2}\left(1-\frac{1}{z}\right)^{d-3}.$$

Very similar functional equations do exist for the three-, four-, e.t.c integrals. Functional equations here play the same role as 24 Kummer relations for Gauss' hypergeometric function! Therefore functional equations can be used for analytic continuation of functions with several variables.

It is not so easy to obtain formulae for analytic continuation for hypergeometric functions with several variables. For the rather simple Appell function $F_1$ explicit representation in terms of Gauss hypergeometric function was used.

For analytic continuation of Feynman integrals with the help of functional equations explicit representation is not needed!

It will be interesting to obtain functional equations for some Green functions to all orders of perturbation theory. Probably one can use use Dyson-Schwinger equations and exploit functional equations for the kernels of this equations.

As was realized many years ago in papers by Regge (1967), Feynman integrals are generalized hypergeometric functions. This conjecture was confirmed through the evaluations of specific Feynman integrals. From numerous results we know that Feynman integrals can be expressed in terms of generalized hypergeometric functions, Appell functions $F_1$, $F_2$,$F_3$, $F_4$ Laurichella, Laurichella-Saran e.t.c functions. . These results were obtained using rather different methods, e.g.

- by directly evaluating the integrals from their Feynman parameter representations,
- by applying Mellin-Barnes integral representations,
- by solving recurrence relations,
- by making use of the negative-dimension approach,
- or by using spectral representations.

As a method for finding relations between hypergeometric functions, Srivastava and Karlsson in their book advocated the evaluation of integrals reducible to hypergeometric functions by several different methods and the comparison of the results thus obtained. In this respect, the evaluation of Feynman integrals may be considered as a rich source for finding relations between hypergeometric functions.

As an example we consider the evaluation of the one-loop propagator type integral with arbitrary masses and arbitrary powers of propagators:

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12}) = \int \frac{d^dq}{i\pi^{d/2}} \frac{1}{[(q-p_1)^2 - m_1^2]^{\nu_1}[(q-p_2)^2 - m_2^2]^{\nu_2}}.$$

This integral can be written as an integral over Feynman parameters:

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12}) = (-1)^{\nu_1+\nu_2} \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 \frac{dx\ x^{\nu_1-1}(1-x)^{\nu_2-1}}{[s_{12}x^2 + x(m_1^2 - m_2^2 - s_{12}) + m_2^2]^{\nu_1+\nu_2-\frac{d}{2}}}.$$

Representing the quadratic polynomial in the denominator as

$$s_{12}x^2 + x(m_1^2 - m_2^2 - s_{12}) + m_2^2 = m_2^2(1 - x_+ x)(1 - x_- x),$$

where

$$x_\pm = \frac{1 + x - y \pm \sqrt{x^2 + y^2 + 1 - 2xy - 2x - 2y}}{2}, \qquad x = \frac{s_{12}}{m_2^2}, \qquad y = \frac{m_1^2}{m_2^2},$$

and then comparing our integral with the integral representation for the Appell function $F_1$

the following result follows:

$$I^{(d)}_{\nu_1\nu_2}(m_1^2, m_2^2;\ s_{12}) = \frac{(-1)^{\nu_1+\nu_2}\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)(m_2^2)^{\nu_1+\nu_2-d/2}}$$
$$\times F_1\left(\nu_1, \nu_1+\nu_2-\frac{d}{2}, \nu_1+\nu_2-\frac{d}{2}; \nu_1+\nu_2; x_-, x_+\right),$$

An analytic result for this integral in terms of two Appell functions $F_4$ was derived by E. Boos and A. Davydychev. Comparing both results we can derive relation between $F_1$ and $F_4$ functions. Just for simplicity we consider the case $m_1 = m_2$. By using Mellin-Barnes representation E. Boos and A. Davydychev obtained the following result

$$I^{(d)}_{\nu_1\nu_2}(m^2, m^2;\ s_{12}) = (-1)^{\nu_1+\nu_2}(m^2)^{d/2-\nu_1-\nu_2}$$
$$\times \frac{\Gamma\left(\nu_1+\nu_2-\frac{d}{2}\right)}{\Gamma(\nu_1+\nu_2)}\, {}_3F_2\left[\begin{array}{c} \nu_1, \nu_2, \nu_1+\nu_2-\frac{d}{2};\ \frac{x_1}{x_2} \\ \frac{\nu_1+\nu_2}{2}, \frac{\nu_1+\nu_2+1}{2};\ \end{array}\right].$$

Comparing this formula with our result taken at $m_1^2 = m_2^2 = m^2$, we obtain:

$$F_1\left(\alpha, \beta, \beta; \gamma;\ x-\sqrt{x^2-2x},\ x+\sqrt{x^2-2x}\ \right) = {}_3F_2\left[\begin{array}{c} \alpha, \gamma-\alpha, \beta;\ \frac{x}{2} \\ \frac{\gamma}{2}, \frac{\gamma+1}{2};\ \end{array}\right],$$

which may be rewritten as:

$$F_1\left(\alpha,\beta,\beta;\gamma;\ x,\ \frac{x}{x-1}\right) = {}_3F_2\left[\begin{matrix}\alpha,\gamma-\alpha,\beta\,; \\ \frac{\gamma}{2},\frac{\gamma+1}{2}\,;\end{matrix}\ \frac{x^2}{4(x-1)}\right].$$

To the best of our knowledge, there is no such a relation in the mathematical literature.

Another interesting relation can be obtained from the comparison of imaginary part of the two-loop "sunrise integral"calculated by two different methods:

$${}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\,; \\ 1\,;\end{matrix}\ \frac{(x-3)(x+1)^3}{(x+3)(x-1)^3}\right] = \frac{\sqrt{3(x+3)(x-1)^3}}{(x^2+3)}\,{}_2F_1\left[\begin{matrix}\frac{1}{3},\frac{2}{3}\,; \\ 1\,;\end{matrix}\ \frac{x^2(x^2-9)^2}{(x^2+3)^3}\right].$$

The hypergeometric function ${}_2F_1$ on the left-hand side of this equation is proportional to the complete elliptic integral of the first kind. Relations between hypergeometric functions with parameters $1/2, 1/2, 1$ and $1/3, 2/3, 1$ but with arguments different from that in the above equation were first derived by Ramanujan.

Several other relations one can find in the paper:

B. A. Kniehl and O.V. T., (arXiv:1108.6019 [math-ph])

## Summary

- generalized recurrence relations provide us a tool for efficient evaluation of Feynman integrals but further investigation concerning optimal sets (Gröbner bases) of recurrence relations is needed
- the method of dimensional recurrences can be used in calculation of multiscale integrals as well as multiloop integrals. Dimensional recurrences are simpler than differential equations because singularity structure of differential equations w.r.t kinematic variables is more complicated than w.r.t. $d$
- functional equations represent a powerful instrument for analytic continuation of Feynman integrals. For a detailed classification of these equations group theoretical approach should be formulated.
- computational machinery for Feynman integrals can be used to obtain relationships for hypergeometric functions that will be useful in other applications. From the already known results we can essentially extend lists of formulae given in the well known books by Baitmen-Erdely and Brychkov,Marichev,Prudnikov.