No-go results for non-topological solitons in some types of gauge field theories

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The Derrick theorem

R.H. Hobart, *Proc. Phys. Soc.* 82 (1963) 201.

G.H. Derrick, *J. Math. Phys.* 5 (1964) 1252.

$$S = \int dt\, d^3x \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\partial_i\phi\partial_i\phi - V(\phi) \right)$$

$$S_{eff} = -E = \int d^3x \left(-\frac{1}{2}\partial_i\phi\partial_i\phi - V(\phi) \right) = -\Pi - I,$$

$$\Pi = \frac{1}{2}\int d^3x\, (\partial_i\phi\partial_i\phi) \geq 0 \qquad I = \int d^3x V(\phi)$$

$$\phi(\vec{x}) \to \phi(\lambda \vec{x})$$

$$S_{eff}[\phi(\lambda \vec{x})] = -\lambda^{-1}\Pi - \lambda^{-3} I$$

$$\left. \frac{dS_{eff}[\phi(\lambda \vec{x})]}{d\lambda} \right|_{\lambda=1} = 0 = \Pi + 3I$$

$$\boxed{V(\phi) \geq 0 \qquad \to \qquad \phi = \phi_{vac}}$$

The same result can be obtained by multiplying the corresponding equation of motion by

$$x^i \partial_i \phi$$

and integrating over the 3-volume

How to overcome this restriction? For example, one can consider

$$\phi(t, \vec{x}) \sim e^{i\omega t}\varphi(\vec{x})$$

Gauge theories

$$S = \int d^4x \left[\eta^{\mu\nu} (D_\mu \phi)^\dagger D_\nu \phi - V(\phi^\dagger \phi) - \frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} \right]$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g C^{abc} A^b_\mu A^c_\nu$$
$$D_\mu \phi = \partial_\mu \phi - i g T^a A^a_\mu \phi$$

$$V(\phi^\dagger \phi)|_{\phi^\dagger \phi = 0} = 0, \quad \left. \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \right|_{\phi^\dagger \phi = 0} = C, \quad |C| < \infty.$$

Extra conditions

- there are no sources which are external to the system described by the presented action
- solutions to equations of motion are periodic in time with a period T up to a coordinate shift and a spatial rotation, i.e. for all fields on the solution the relation

  $$\Psi(t+T,\vec{x}) \equiv \Lambda(\Omega)\Psi(t,\Omega^{-1}\vec{x}-\vec{l})$$

  must hold for any t

One can always pass to a suitable coordinate system in which

$$\Omega\vec{l} = \vec{l}.$$

$$S = \int_{-\infty}^{\infty} dt \int d^3x L[\Psi(t,\vec{x})] = \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} dt \int d^3x L[\Psi(t,\vec{x})] =$$

$$= \sum_{n=-\infty}^{\infty} \int_0^T dt \int d^3x L[\Psi(t+nT,\vec{x})] = \sum_{n=-\infty}^{\infty} \int_0^T dt \int d^3x L[\Lambda^n(\Omega)\Psi(t,\Omega^{-n}\vec{x} - n\vec{l})] =$$

$$= \sum_{n=-\infty}^{\infty} \int_0^T dt \int d^3x L[\Psi(t,\vec{x})],$$

One can use the effective action

$$\boxed{S_{eff} = \int_0^T dt \int d^3x L[\Psi(t,\vec{x})]}$$

$$\lim_{x^i \to \pm\infty} \phi(t, \vec{x}) = 0, \tag{1}$$

$$\lim_{x^i \to \pm\infty} A_\mu(t, \vec{x}) = 0. \tag{2}$$

$$\int_0^T dt \int d^3x (D_0\phi)^\dagger D_0\phi = \Pi_0 \geq 0, \tag{3}$$

$$\int_0^T dt \int d^3x (D_i\phi)^\dagger D_i\phi = \Pi_1 \geq 0, \tag{4}$$

$$\int_0^T dt \int d^3x \frac{1}{2} F^a_{0i} F^a_{0i} = \Pi_{A0} \geq 0, \tag{5}$$

$$\int_0^T dt \int d^3x \frac{1}{4} F^a_{ij} F^a_{ij} = \Pi_{A1} \geq 0. \tag{6}$$

Non-topological solitons of form (1), (2), periodic in time up to a spatial rotation and a coordinate shift, with integrals (3)-(6) and integrals

$$\int_0^T dt \int d^3x \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi, \int_0^T dt \int d^3x V(\phi^\dagger \phi)$$

finite, are absent in the theory if there exists $\gamma : \ \frac{1}{2} < \gamma \leq \frac{3}{2}$

for which the inequality

$$2\gamma \frac{dV(\phi^\dagger \phi)}{d(\phi^\dagger \phi)} \phi^\dagger \phi - 3V(\phi^\dagger \phi) \geq 0$$

is fulfilled for any ϕ.

$$S = \Pi_0 - \Pi_1 - \int_0^T dt \int d^3x V\left(\phi^\dagger \phi\right) + \Pi_{A0} - \Pi_{A1}$$

$$\boxed{\begin{aligned} \phi(t,\vec{x}) &\to \lambda^\gamma \phi(t,\lambda\vec{x}), \\ A_0^a(t,\vec{x}) &\to A_0^a(t,\lambda\vec{x}), \\ A_i^a(t,\vec{x}) &\to \lambda A_i^a(t,\lambda\vec{x}) \end{aligned}}$$

$$S = \lambda^{2\gamma-3}\Pi_0 - \lambda^{2\gamma-1}\Pi_1 -$$

$$-\lambda^{-3}\int_0^T dt \int d^3x V\left(\lambda^{2\gamma}\phi^\dagger(t,\vec{x})\phi(t,\vec{x})\right) + \lambda^{-1}\Pi_{A0} - \lambda\Pi_{A1}.$$

$$\frac{dS}{d\lambda}|_{\lambda=1} = (2\gamma-3)\Pi_0 - (2\gamma-1)\Pi_1 -$$

$$-\int_0^T dt \int d^3x \left(2\gamma \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - 3V(\phi^\dagger\phi)\right) - \Pi_{A0} - \Pi_{A1} = 0.$$

1. $\frac{1}{2} < \gamma < \frac{3}{2}$. If

$$2\gamma \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - 3V(\phi^\dagger\phi) \geq 0$$

for any ϕ, then $\Pi_0 = \Pi_1 = \Pi_{A0} = \Pi_{A1} \equiv 0$ ($2\gamma \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)} - 3V(\phi^\dagger\phi) = 0$ also), in this case $F^a_{\mu\nu} \equiv 0$ (this equality means that A_μ is a pure gauge and we can set $A_\mu \equiv 0$). From $\Pi_0 = \Pi_1 \equiv 0$ with $A_\mu \equiv 0$ and we get $\phi \equiv 0$.

2. $\gamma = \frac{3}{2}$. If

$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - V(\phi^\dagger\phi) \geq 0$$

for any ϕ, then $\Pi_1 = \Pi_{A0} = \Pi_{A1} \equiv 0$, in this case $A_\mu \equiv 0$, $\phi = \phi(t) \equiv 0$

3. $\gamma = \frac{1}{2}$. If

$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - 3V(\phi^\dagger\phi) \geq 0$$

for any ϕ, then $\Pi_0 = \Pi_{A0} = \Pi_{A1} \equiv 0$, in this case $A_\mu \equiv 0$, $\phi = \phi(\vec{x})$.

$$\phi(t,\vec{x}) = \phi(\vec{x}) \to \lambda^\gamma \phi(\vec{x}),$$
$$A_0^a(t,\vec{x}) = A_0^a(\vec{x}) \to \lambda^\beta A_0^a(\vec{x}),$$
$$A_i^a(t,\vec{x}) = A_i^a(\vec{x}) \to A_i^a(\vec{x})$$

with $\gamma > 0$, $\beta < -\gamma$. Then we get

$$\frac{dS^\phi}{d\lambda}|_{\lambda=1} = 2(\gamma+\beta)\Pi_0 - 2\gamma\left[\Pi_1 + \int_0^T dt \int d^3x \frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi\right] + 2\beta\Pi_{A0} = 0.$$

Thus if

$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi \geq 0,$$

then $\Pi_0 = \Pi_1 = \Pi_{A0} \equiv 0$. Relation $\Pi_{A0} \equiv 0$ implies $A_0 \equiv 0$, from $\Pi_1 \equiv 0$ it follows that $D_i\phi \equiv 0$ and thus $\partial_i(\phi^\dagger\phi) \equiv 0$, which implies $\phi \equiv 0$. Then it is very easy to show that $A_i \equiv 0$

Corollary

1. For $V(\phi^\dagger\phi) \geq 0$, non-topological solitons are absent if

$$\frac{dV(\phi^\dagger\phi)}{d(\phi^\dagger\phi)}\phi^\dagger\phi - V(\phi^\dagger\phi) \geq 0.$$

R.T. Glassey, W.A. Strauss, *Commun. Math. Phys.* 67 (1979) 51

2. The restrictions presented above are valid for the models with the scalar field only, i.e. if we drop the gauge field from the theory.

G. Rosen, *J. Math. Phys.* 9 (1968) 999

3. Non-topological solitons satisfying the conditions presented above are absent in the pure Yang-Mills theory.

S. Deser, *Phys. Lett.* B 64 (1976) 463; H. Pagels, *Phys. Lett.* B 68 (1977) 466; S.R. Coleman, *Commun. Math. Phys.* 55 (1977) 113; S.R. Coleman, L. Smarr, *Commun. Math. Phys.* 56 (1977) 1; R. Weder, *Commun. Math. Phys.* 57 (1977) 161; M. Magg, *J. Math. Phys.* 19 (1978) 991; R.T. Glassey, W.A. Strauss, *Commun. Math. Phys.* 65 (1979) 1

Charged massive vector field

$$S = \int d^4x \left[-\frac{1}{2} \eta^{\mu\rho} \eta^{\nu\sigma} W^{-}_{\mu\nu} W^{+}_{\rho\sigma} + m^2 \eta^{\mu\nu} W^{-}_{\mu} W^{+}_{\nu} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right]$$

with $m \neq 0$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

$$D_\mu W^{\pm}_{\nu} = \partial_\mu W^{\pm}_{\nu} \mp ie A_\mu W^{\pm}_{\nu} ,$$

$$W^{\pm}_{\mu\nu} = D_\mu W^{\pm}_{\nu} - D_\nu W^{\pm}_{\mu} .$$

Again we suppose that:

1. there are no sources which are external to the system
2. all fields are smooth and vanish at spatial infinity
3. solutions to equations of motion are periodic in time with a period T up to a spatial rotation and a coordinate shift

$$\int_0^T dt \int d^3x W_{0i}^- W_{0i}^+ = \Pi_{W0} \geq 0,$$

$$\int_0^T dt \int d^3x \frac{1}{2} W_{ij}^- W_{ij}^+ = \Pi_{W1} \geq 0,$$

$$m^2 \int_0^T dt \int d^3x W_0^- W_0^+ = V_0 \geq 0,$$

$$m^2 \int_0^T dt \int d^3x W_i^- W_i^+ = V_1 \geq 0,$$

$$\int_0^T dt \int d^3x \frac{1}{2} F_{0i} F_{0i} = \Pi_{A0} \geq 0,$$

$$\int_0^T dt \int d^3x \frac{1}{4} F_{ij} F_{ij} = \Pi_{A1} \geq 0.$$

$$\boxed{\begin{aligned} W_0^{\pm}(t,\vec{x}) &\to \lambda^{\beta-1} W_0^{\pm}(t,\lambda\vec{x}), \\ W_i^{\pm}(t,\vec{x}) &\to \lambda^{\beta} W_i^{\pm}(t,\lambda\vec{x}), \\ A_0^a(t,\vec{x}) &\to A_0^a(t,\lambda\vec{x}), \\ A_i^a(t,\vec{x}) &\to \lambda A_i^a(t,\lambda\vec{x}) \end{aligned}}$$

$$S = \lambda^{2\beta-3}\Pi_{W0} - \lambda^{2\beta-1}\Pi_{W1} + \lambda^{2\beta-5}V_0 - \\ -\lambda^{2\beta-3}V_1 + \lambda^{-1}\Pi_{A0} - \lambda\Pi_{A1}.$$

$$\beta = \frac{3}{2}.$$

$$\frac{dS}{d\lambda}|_{\lambda=1} = -2\Pi_{W1} - 2V_0 - \Pi_{A0} - \Pi_{A1} = 0.$$

$F_{\mu\nu} \equiv 0$ and we can set $A_\mu \equiv 0$, $W_0^{\pm} \equiv 0$ and $W_{ij}^{\pm} \equiv 0$.

With $A_\mu \equiv 0$ we can rewrite $W_{ij}^{\pm} \equiv 0$ as

$$\partial_i W_j^{\pm} - \partial_j W_i^{\pm} \equiv 0.$$

from equations of motion for the field $W_\mu^{\pm}$ with $A_\mu \equiv 0$ we get

$$\partial^\mu W_\mu^{\pm} = 0.$$

Using the fact that $W_0^{\pm} \equiv 0$

$$\partial^i W_i^{\pm} = 0.$$

$$\operatorname{div} \vec{W}^{\pm} = 0, \qquad \operatorname{rot} \vec{W}^{\pm} = 0,$$

where $\vec{W}^{\pm} = (W_1^{\pm}, W_2^{\pm}, W_3^{\pm})$.

$$\vec{W}^{\pm} = \operatorname{grad} \varphi^{\pm}$$

$$\Delta \varphi^{\pm} = 0$$

where $\varphi^{\pm} = \varphi^{\pm}(t, \vec{x})$, $(\varphi^{+})^{*} = \varphi^{-}$

The condition $\int d^3x W_i^{-} W_i^{+} = \int d^3x \partial_i \varphi^{-} \partial_i \varphi^{+} < \infty$ clearly

leads to $\varphi^{\pm} = \varphi^{\pm}(t)$

Thus $\quad \vec{W}^{\pm} \equiv 0.$

$$\boxed{W_{\mu}^{\pm} \equiv 0.}$$