Recent progress in QCD sum rule calculations of heavy meson properties

Dmitri Melikhov

SINP, Moscow

I discuss the details of calculating hadron properties from the OPE for correlators of quark currents in QCD, which constitutes the basis of the method of QCD sum rules. The main emphasis is laid on gaining control over the systematic uncertainties of the hadron parameters obtained within this method. We start with examples from quantum mechanics, where bound-state properties may be calculated independently in two ways: exactly, by solving the Schrödinger equation, and approximately, by the method of sum rules. Knowing the exact solution allows us to control each step of the sum-rule extraction procedure. On the basis of this analysis, we formulate several improvements of the method of sum rules. We then apply these modifications to the analysis of the decay constants of heavy charm and beauty mesons.

A QCD sum-rule calculation of hadron parameters involves two steps:

I. Calculating the operator product expansion (OPE) series for a relevant correlator

For heavy-light currents, one observes a very strong dependence of the OPE for the correlator (and, consequently, of the extracted decay constant) on the heavy-quark mass used, i.e., on-shell (pole), or running $\overline{\mathrm{MS}}$ *mass.*

OPE reorganized in terms of $\overline{\mathrm{MS}}$ *mass exhibits a reasonable convergence of the perturbative expansion for hadron observables.*

II. Extracting the parameters of the ground state by a numerical procedure

NEW :

(a) Make use of the new more accurate duality relation based on Borel-parameter-dependent threshold.

Allows a more accurate extraction of the decay constants and provides realistic estimates of the intrinsic (systematic) errors — those related to the limited accuracy of sum-rule extraction procedures.

(b) Study the sensitivity of the extracted value of f_P *to the OPE parameters (quark masses, condensates,...). The corresponding error is referred to as OPE uncertainty, or statistical error.*

1. Basic object in QCD:

$\Pi(p^2) = i\int dx e^{ipx}\langle 0|T\left(j_5(x)j_5^\dagger(0)\right)|0\rangle, \qquad j_5(x) = (m_Q + m)\bar{q}i\gamma_5 Q(x)$

and its Borel transform $\Pi(\tau)$, $p^2 \to \tau$.

Analogue in quantum mechanics:

Polarization operator $\Pi(E)$ is defined through the full Green function $G(E)$:

$$\Pi(E) = \langle \vec{r}_f = 0|\frac{1}{H - E}|\vec{r}_i = 0\rangle.$$

and its Borel transform $E \to T$,

$$\frac{1}{H - E} \to \exp(-HT)$$

which leads to the evolution operator in imaginary time T ($T = 1/M_{\rm Borel}$):

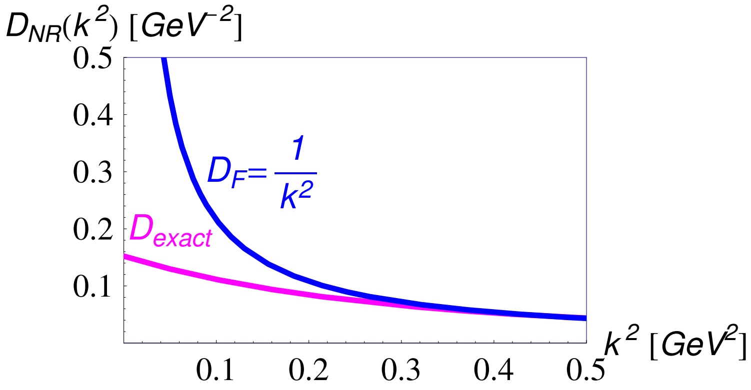
$$\Pi(T) = \langle \vec{r}_f = 0|\exp(-HT)|\vec{r}_i = 0\rangle.$$

Exact and Feynman propagators of a confined particle

Feynman propagator of a NR particle with mass m

$$D_F(E,\vec{k}^2) = \frac{1}{\vec{k}^2 - 2mE - i0}.$$

The plot compares the values at $E = 0$ of D_F and $D_{\rm exact}$ for HO potential $m\omega^2 r^2/2$.



As soon as "soft" momenta in Feynman diagrams are essential, nonperturbative effects in propagators are essential.

At large k^2, one finds

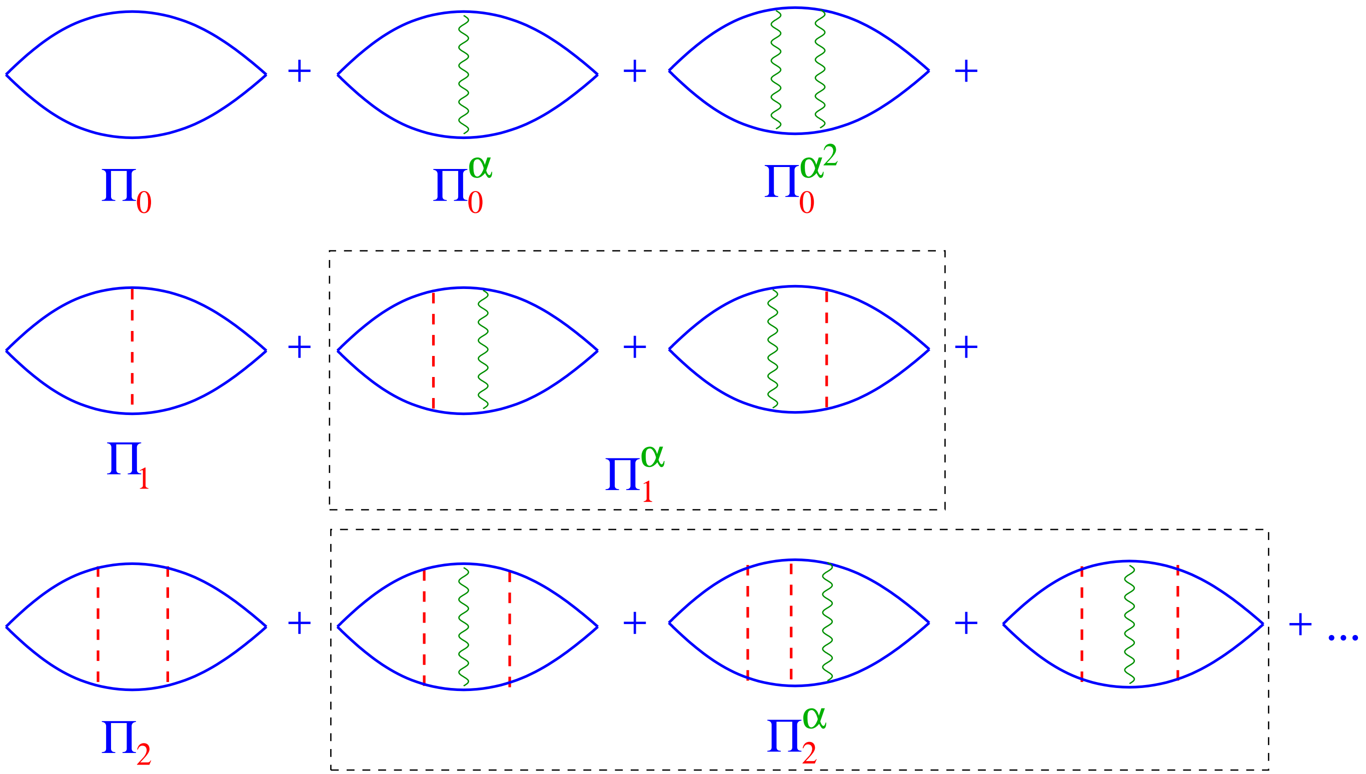
$$D_{\rm exact}(k^2) = D_F(k^2) + \#\frac{\omega^2}{k^4} + \#\frac{\omega^4}{k^6} + \ldots$$

Correlator in a realistic potential model : confinement + Coulomb

Polarization operator $\Pi(E) = \langle \vec{r}_f = 0 | \frac{1}{H-E} | \vec{r}_i = 0 \rangle$.

$$H = \frac{k^2}{2m} + V_{\rm conf}(r) - \frac{\alpha}{r}.$$

Expansion of $\Pi(E)$ **in powers of the interaction:**



• **Analogue of the OPE for the Borel image $\Pi(T)$:**

For the case $V_{\rm conf}(r) = \frac{m\omega^2 r^2}{2}$ an explicit double expansion in powers of α and powers of ωT

$$\begin{aligned}
\Pi_{\rm OPE}(T) &= \Pi_{\rm pert}(T) + \Pi_{\rm power}(T),\\
\Pi_{\rm pert}(T) &= \left(\frac{m}{2\pi T}\right)^{3/2}\left[1 + \sqrt{2\pi m T}\alpha + \frac{1}{3}m\pi^2 T\alpha^2\right],\\
\Pi_{\rm power}(T) &= \left(\frac{m}{2\pi T}\right)^{3/2}\left[-\frac{1}{4}\omega^2 T^2\left(1 + \frac{11}{12}\sqrt{2\pi m T}\alpha\right) + \frac{19}{480}\omega^4 T^4\left(1 + \frac{1541}{1824}\sqrt{2\pi m T}\alpha\right)\right]
\end{aligned}$$

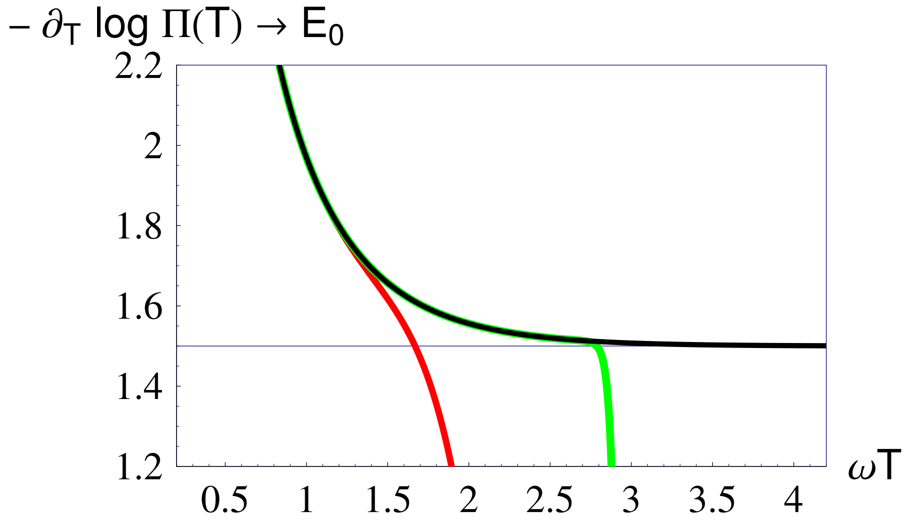
$$\Pi_{\rm pert}(T) = \left(\frac{m}{2\pi}\right)^{3/2}\int\limits_0^\infty dz\,\exp(-zT)\left[2\sqrt{\frac{z}{\pi}} + \sqrt{2\pi m}\alpha + \frac{\pi^{3/2}m\alpha^2}{3\sqrt{z}}\right]$$

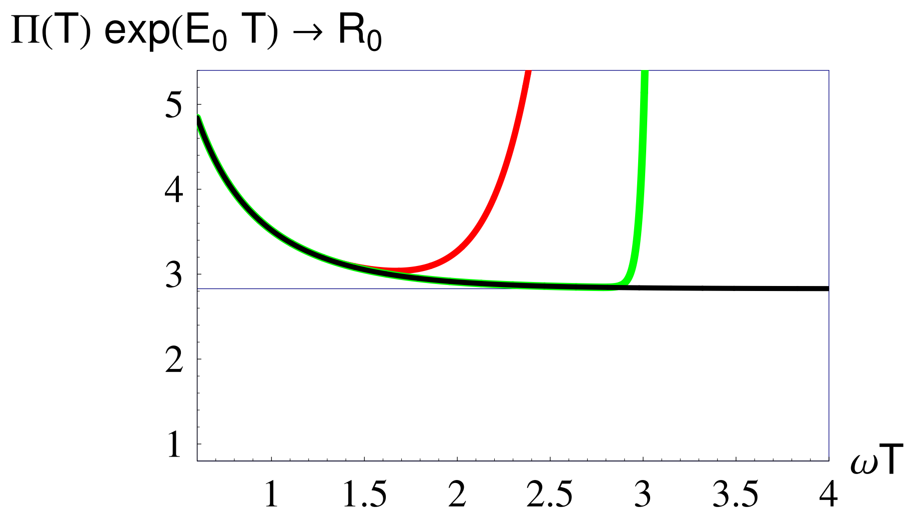
• **The "phenomenological" representation for $\Pi(T)$ – in the basis of hadron eigenstates:**

$$\Pi(T) = \langle\vec r_f = 0|\exp(-HT)|\vec r_i = 0\rangle = \sum_{n=0}^{\infty} R_n \exp(-E_n T),$$

E_n **- energy of the** n**-th bound state,** $R_n = |\Psi_n(\vec r = 0)|^2$.

How to calculate $E_{n=0}$ and $R_{n=0}$ of the ground state from $\Pi(T)$ known numerically?





Black - exact $\Pi(T)$; Red - OPE with 4 power corrections, Green - OPE with 100 power corrections.

With a few power corrections the plateau cannot be reached.

Some other concept: "Quark-hadron duality" assumption will be used.

Sum rule:

$$\Pi_{\rm OPE}(T) = \Pi_{\rm phys}(T).$$

OPE side:

$$\Pi_{\rm OPE}(T) = \int_0^\infty dz e^{-zT}\rho_{\rm pert}(z) + \Pi_{\rm power}(T).$$

Physical side with the "Standard Ansatz" for the excited states:

$$\Pi(T) = \int_0^\infty dz e^{-zT}\rho_{\rm phys}(z), \qquad \rho_{\rm phys}(z) = R_0\delta(z - E_0) + \theta(z - z_{\rm eff})\rho_{\rm pert}(z).$$

This gives

$$R_0 e^{-E_0 T} = \Pi_{\rm dual}(T, z_{\rm eff}) \equiv \int_0^{z_{\rm eff}} dz e^{-zT}\rho_{\rm pert}(z) + \Pi_{\rm power}(T).$$

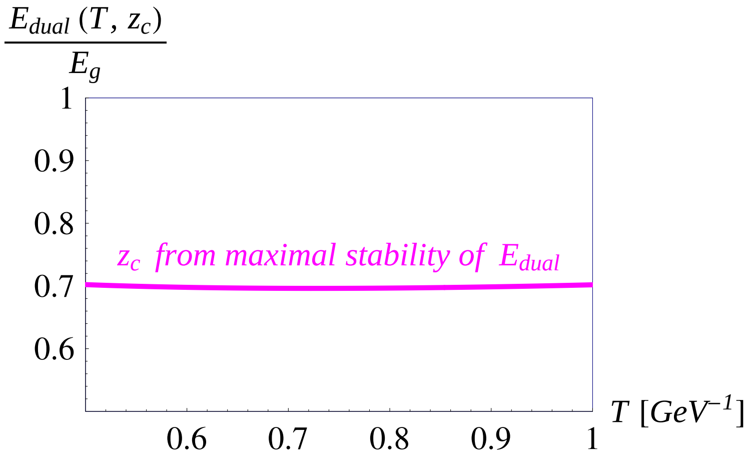
From this sum rule one obtains estimates for R_0 and E_0, $R_{\rm dual}(T, z_{\rm eff})$, $E_{\rm dual}(T, z_{\rm eff}) = -d_T \log \Pi(T, z_{\rm eff})$.

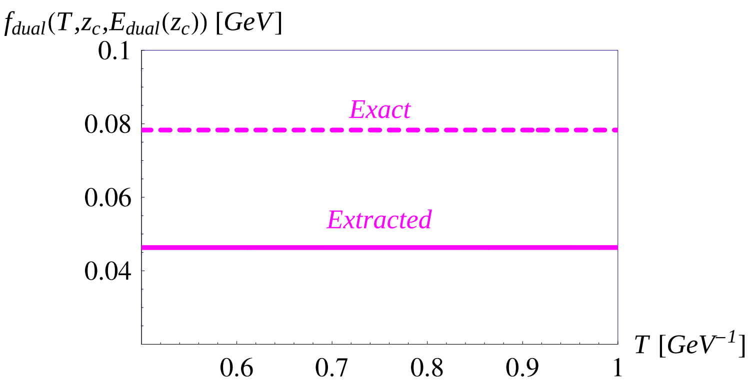
These depend on unphysical parameters T and $z_{\rm eff}$.

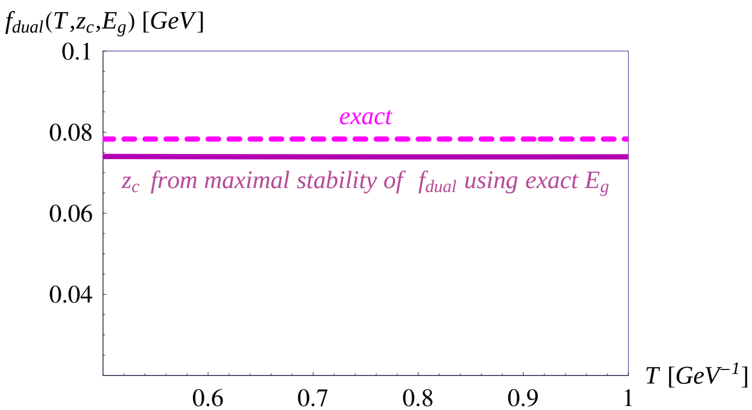
How to fix these parameters?

The standard and almost obvious criterion is "maximal stability":
choose $z_{\rm eff}$ such that the dependence of $E_{\rm dual}$ and $R_{\rm dual}$ on T in the T-"window" is minimal.

How this works in quantum mechanics for $H = \frac{k^2}{2m} + \frac{m\omega^2 r^2}{2} - \frac{\alpha}{r}$.







What is bad?

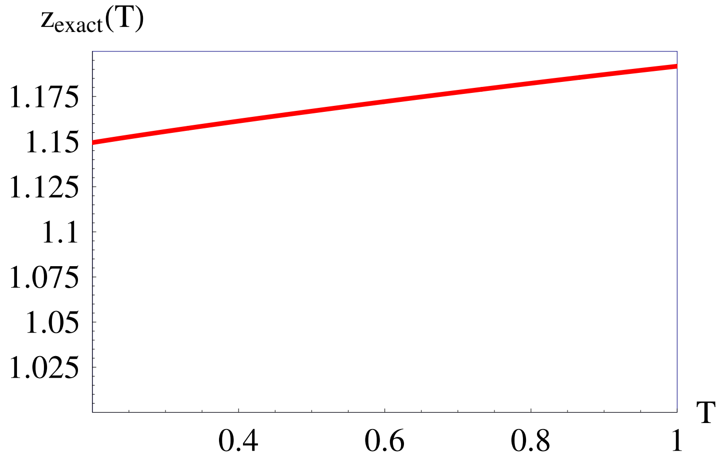
- **A model for the excited states is oversimplified.**
- **Within the assumption $z_{\rm eff} = const$, maximal stability does not automatically lead to a success.**

Exact effective threshold:

Use the exact E_0 and R_0 obtained from Schroedinger equation and solve the relation

$$R_0 e^{-E_0 T} = \int_0^{z_{\rm eff}} dz e^{-zT} \rho_{\rm pert}(z) + \Pi_{\rm power}(T)$$

with respect to $z_{\rm eff}$. The obtained "exact threshold" is a slightly rising function of T, $z_{\rm eff}(T)$.



We have neglected this dependence when calculating $E_{\rm dual} = -d_T \log \Pi_{\rm dual}(T, z_{\rm eff}(T))$.

Correlator in QCD

$$\Pi(p^2) = i\int dx e^{ipx}\langle\Omega|T\left(j_5(x)j_5^\dagger(0)\right)|\Omega\rangle, \qquad j_5(x) = (m_Q+m)\bar{q}i\gamma_5 Q(x)$$

Physical QCD vacuum $|\Omega\rangle$ is complicated and differs from perturbative QCD vacuum $|0\rangle$.

Wilsonian OPE:

$$T\left(j_5(x)j_5^\dagger(0)\right) = C_0(x^2,\mu)\hat{1} + \sum_n C_n(x^2,\mu) : \hat{O}(0,\mu) :$$

Condensates – nonzero expectation values of gauge-invariant operators over physical vacuum:

$$\langle\Omega| : \hat{O}(0,\mu) : |\Omega\rangle \neq 0.$$

Borel transform ($p^2 \to \tau$): *Green functions in Minkowski space → evolution operator in Euclidean space*

$$\Pi(\tau) = \int_{(m_Q+m_u)^2}^{\infty} e^{-s\tau}\rho_{\rm pert}(s,\alpha,m_Q,\mu)\,ds + \Pi_{\rm power}(\tau,m_Q,\mu),$$

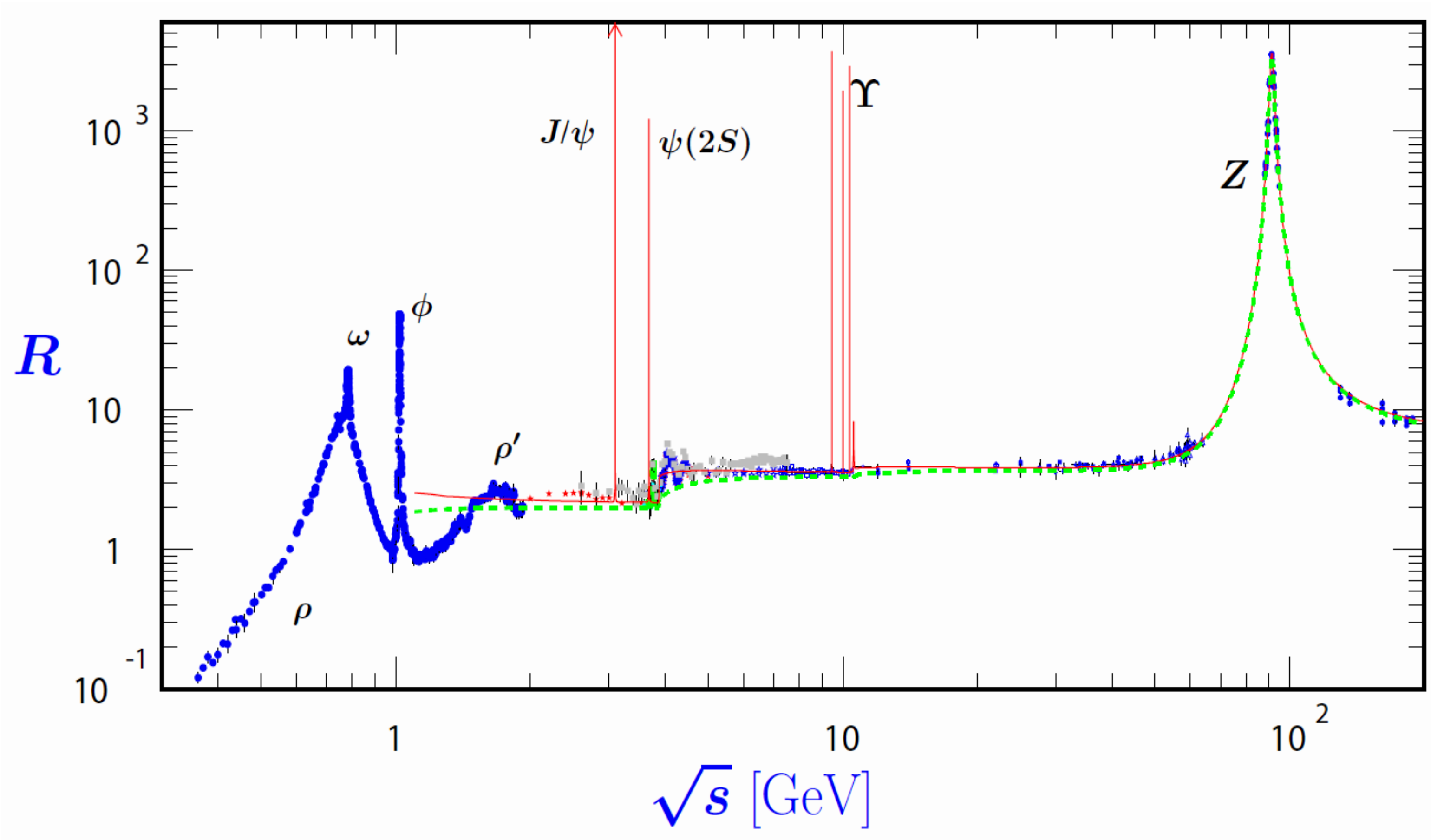
- $\rho_{\rm pert}(s,\mu) = \rho^{(0)}(s) + \frac{\alpha_s(\mu)}{\pi}\rho^{(1)}(s) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\rho^{(2)}(s) + \cdots$
- $\Pi_{\rm power}(\tau,\mu)$ **– power expansion in τ in terms of the condensates:**

$$\Pi_{\rm power}(\tau,\mu=m_Q) = (m_Q+m)^2 e^{-m_Q^2\tau}\left\{-m_Q\langle\bar{q}q\rangle\left[1+\frac{2C_F\alpha_s}{\pi}\left(1-\frac{m_Q^2\tau}{2}\right)+\frac{m_0^2\tau}{2}\left(1-\frac{m_Q^2\tau}{2}\right)\right]+\frac{1}{12}\left\langle\frac{\alpha_s}{\pi}GG\right\rangle\right\}.$$

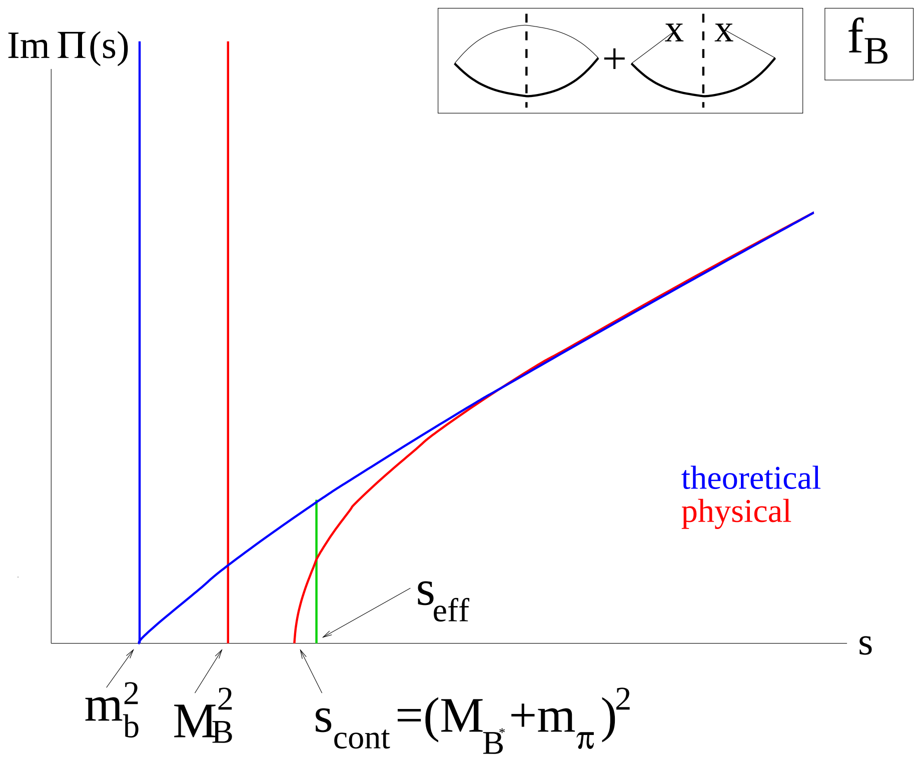
$$\boxed{\textbf{Sum rule: } \Pi_{\rm OPE}(\tau) = \Pi_{\rm hadron}(\tau)}$$

Sum rule: $\Pi_{\mathrm{OPE}}(\tau) = \Pi_{\mathrm{hadron}}(\tau)$

Duality concept: where pQCD calculations may be applied in hadron physics?



Spectral densities of the polarization operator (OPE vs hadron language):



Quark – hadron duality assumption :

$$\int_{s_{\rm eff}}^{\infty} ds\, \exp(-s\tau)\rho_{\rm pert}(s) = \int_{s_{\rm phys.cont.}}^{\infty} ds\, \exp(-s\tau)\rho_{\rm hadr}(s).$$

With the help of the duality assumption, the contribution of the excited states cancels against the high-energy region of the perturbative contribution, and from

$$\Pi_{\rm OPE}(\tau) = \Pi_{\rm hadron}(\tau)$$

we come to

$$\boxed{f_Q^2 M_Q^4 e^{-M_Q^2\tau} = \int\limits_{(m_Q+m_u)^2}^{s_{\rm eff}} e^{-s\tau}\rho_{\rm pert}(s,\alpha,m_Q,\mu)\, ds + \Pi_{\rm power}(\tau,m_Q,\mu) \equiv \Pi_{\rm dual}(\tau,\mu,s_{\rm eff})}$$

Note: nonperturbative contributions are all referred to the ground state.

Extraction of bound-state parameters is possible only if we fix $s_{\rm eff}$ by some "external" criterion.

<u>**For heavy-meson observables one faces two problems:**</u>

1. How to reliably calculate the truncated OPE for the correlator?

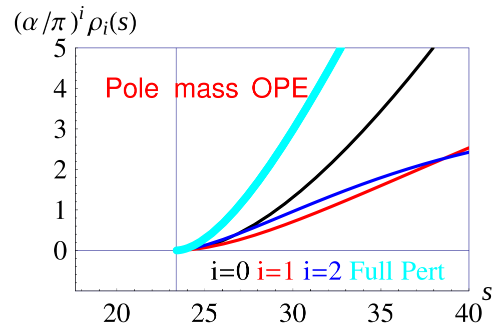
2. How to fix $s_{\rm eff}$ and estimate the errors in the extracted value of f_Q?

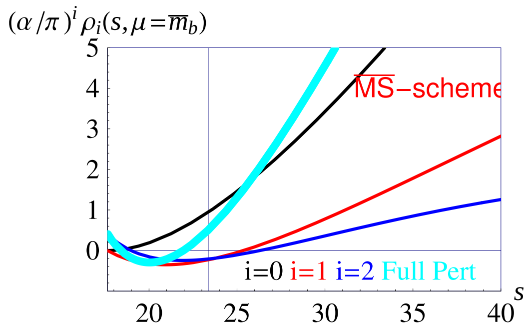
OPE : heavy – quark pole mass or running mass?

Spectral densities

$$\rho(m_b,\alpha_s,s) \to \Pi(m_b,\alpha_s,\tau) \to \Pi(m_b(\overline{m}_b,\alpha_s),\alpha_s,\tau) \to \Pi(\overline{m}_b,\alpha_s,\tau) \to \rho(\overline{m}_b,\alpha_s,s)$$

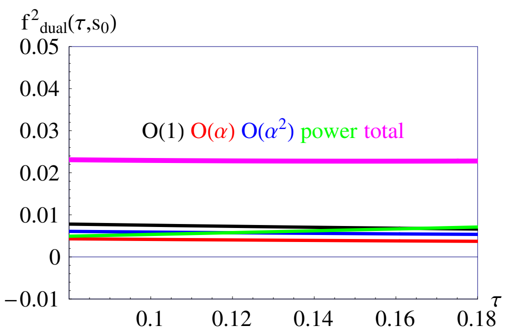
To α_s^2-accuracy, $m_{b,pole} = 4.83$ **GeV** $\leftrightarrow$ $\overline{m}_b(\overline{m}_b) = 4.20$ **GeV**

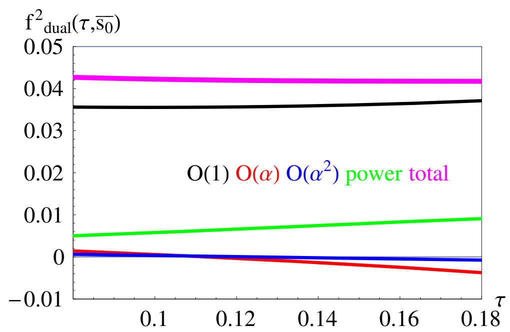




- **In pole mass scheme poor convergence of perturbative expansion**
- **In $\overline{\rm MS}$ scheme the perturbative spectral density has negative region**

Extracted decay constant





- **Decay constant in pole mass shows NO hierarchy of perturbative contributions**
- **Decay constant in $\overline{\rm MS}$-scheme shows such hierarchy. Numerically, f_P using pole mass $\ll$ f_P using $\overline{\rm MS}$ mass.**

Quark – hadron duality assumption :

$$\boxed{f_Q^2 M_Q^4 e^{-M_Q^2\tau} = \int_{(m_Q+m_u)^2}^{s_{\rm eff}} e^{-s\tau}\rho_{\rm pert}(s,\alpha,m_Q,\mu)\,ds + \Pi_{\rm power}(\tau,m_Q,\mu) \equiv \Pi_{\rm dual}(\tau,\mu,s_{\rm eff})}$$

In order the l.h.s. and the r.h.s. have the same τ-behavior

$s_{\rm eff}$ is a function of τ (and μ) : $s_{\rm eff}\,(\tau,\,\mu)$

The "dual" mass: $\boxed{M^2_{\rm dual}(\tau) = -\frac{d}{d\tau}\log\Pi_{\rm dual}(\tau, s_{\rm eff}(\tau)).}$

If quark-hadron duality is implemented "perfectly", then $M_{\rm dual}$ should be equal to M_Q;
The deviation of $M_{\rm dual}$ from the actual meson mass M_Q measures the contamination of the dual correlator by excited states. Better reproduction of M_Q → more accurate extraction of f_Q.

Taking into account τ-dependence of $s_{\rm eff}$
improves the accuracy of the duality approximation.

Obviously, in order to predict f_Q, we need to fix $s_{\rm eff}$. How to fix $s_{\rm eff}$?

• *For a given trial function $s_{\rm eff}(\tau)$ there exists a variational solution which minimizes the deviation of the dual mass from the actual meson mass in the τ-"window".*

Our new algorithm for extracting ground – state parameters when M_Q is known

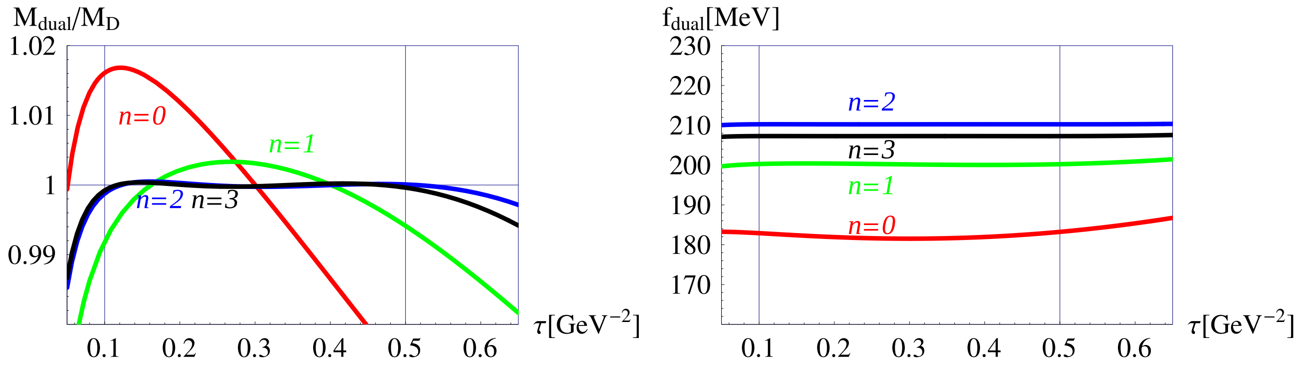
(i) Consider a set of Polynomial τ-dependent Ansaetze for $s_{\rm eff}$: $s_{\rm eff}^{(n)}(\tau) = \sum_{j=0}^{n} s_j^{(n)}(\tau)^j$.

(ii) Minimize the squared difference between the "dual" mass $M_{\rm dual}^2$ and the known value M_Q^2 in the τ-window. This gives us the parameters of the effective continuum threshold.

(iii) Making use of the obtained thresholds, calculate the decay constant.

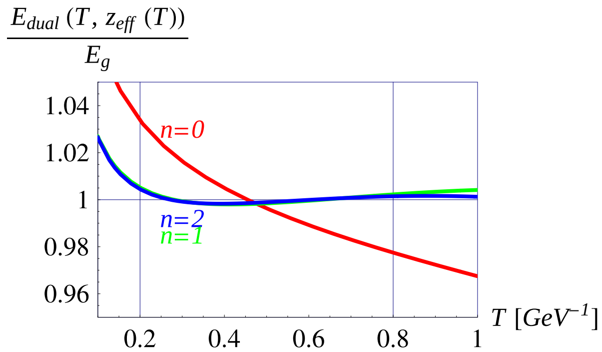
(iv) Take the band of values provided by the results corresponding to <u>linear</u>, <u>quadratic</u>, and <u>cubic</u> effective thresholds as the characteristic of the intrinsic uncertainty of the extraction procedure.

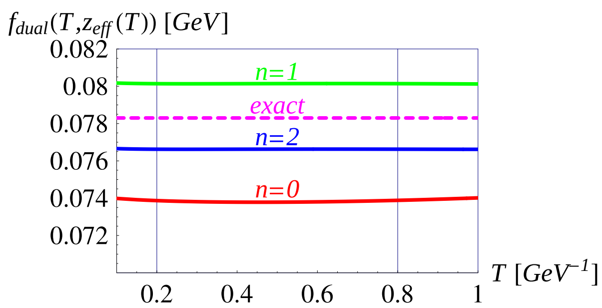
Illustration: D-meson



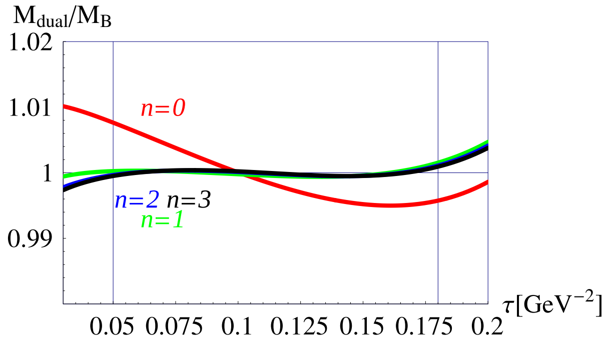
Extraction of f_P: QCD vs potential model

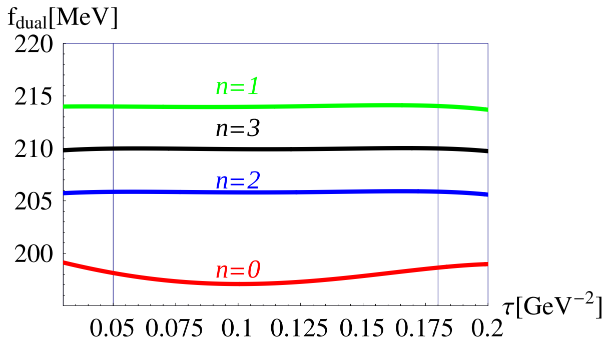
Potential Model (HO + Coulomb)





QCD (f_B for $\bar{m}_b(\bar{m}_b) = 4.20$ GeV)



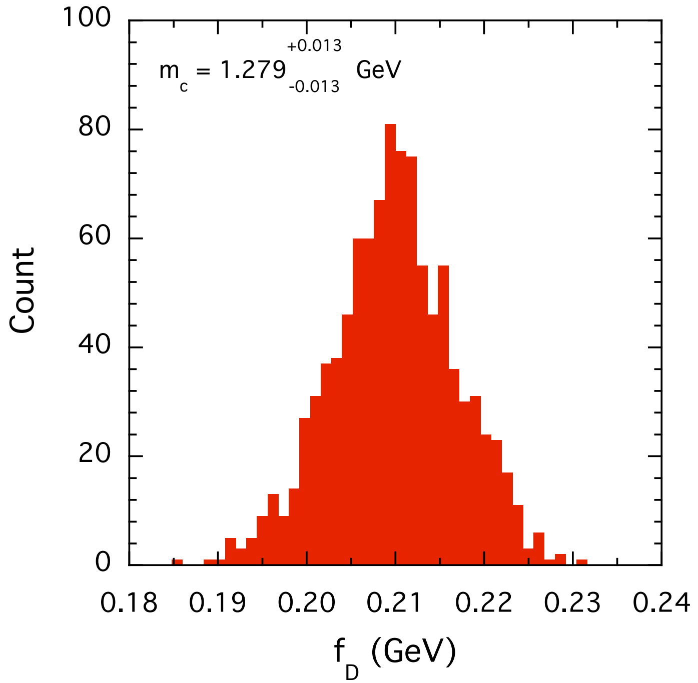


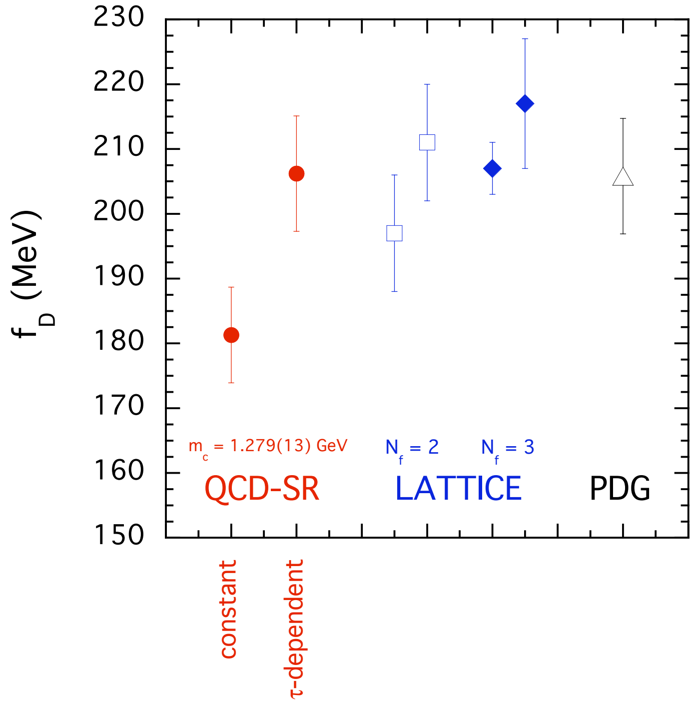
Surprising? No:

As soon as quark-hadron duality is implemented as a cut on the perturbative correlator, the extraction of the ground-state parameters in QCD and in potential model are very similar.

Extraction of f_D

$m_c(m_c) = 1.279 \pm 0.013\,\mathrm{GeV}, \mu = 1 - 3\,\mathrm{GeV}.$





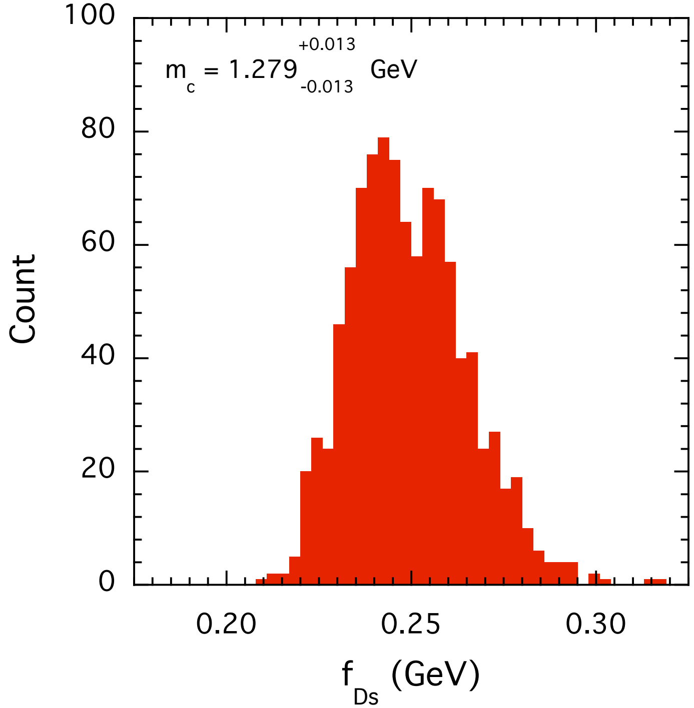
$f_D = 206.2 \pm 7.3_{OPE} \pm 5.1_{syst}$ MeV

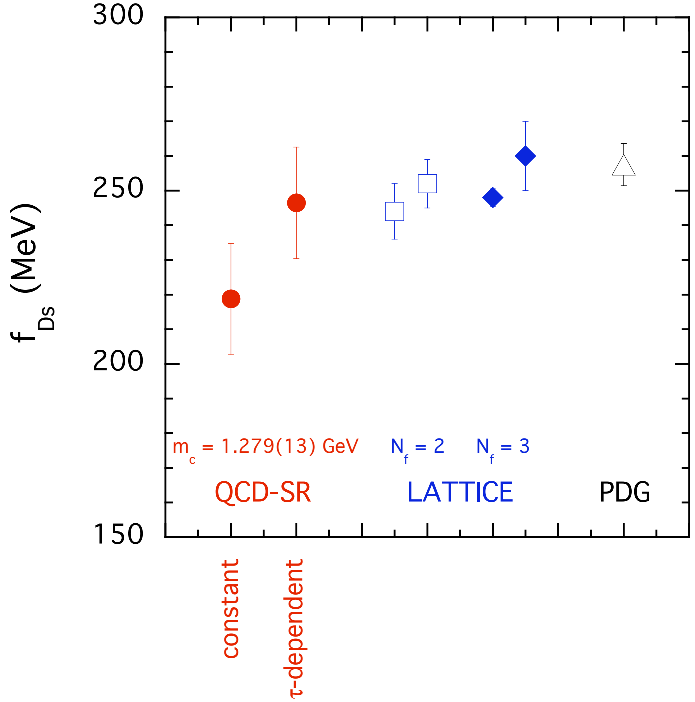
f_D (const) = $181.3 \pm 7.4_{OPE}$ MeV

The effect of τ-dependent threshold is visible!

Extraction of f_{Ds}

$m_c(m_c) = 1.279 \pm 0.013\,\text{GeV}, \mu = 1 - 3\,\text{GeV}.$

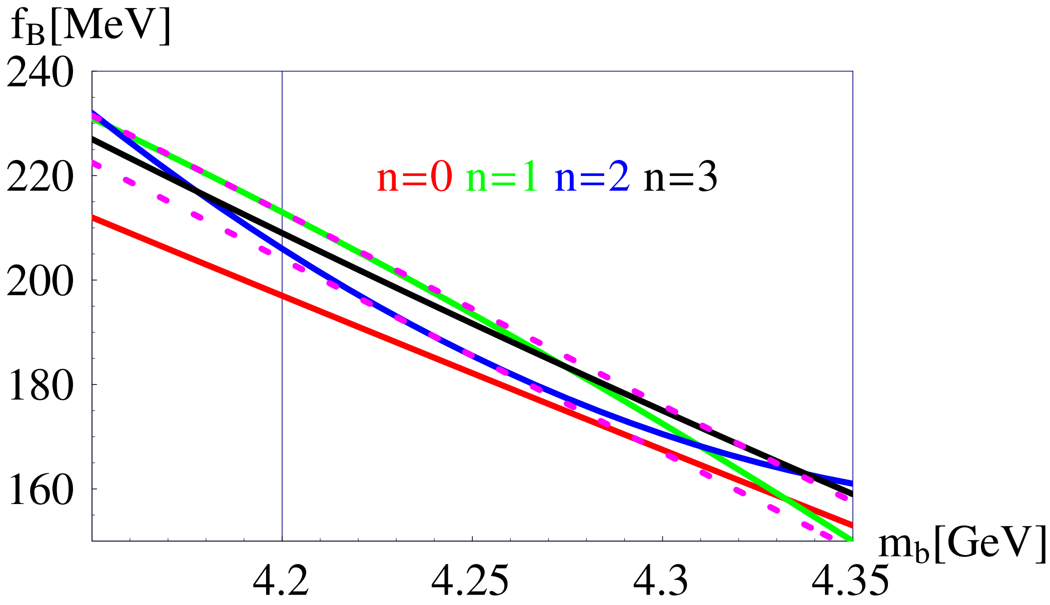




$$f_{D_s} = 246.5 \pm 15.7_{OPE} \pm 5_{syst}\ \text{MeV}$$

f_{Ds} (const) = 218.8 $\pm$ 16.1$_{OPE}$ MeV

Extraction of f_B. Problem 1: a very strong sensitivity to $m_b(m_b)$



τ-dependent effective threshold:

$$f_B^{\mathrm{dual}}(m_b, \langle\bar{q}q\rangle, \mu = m_b) = \left[206.5 \pm 4 - 37\left(\frac{m_b - \mathbf{4.245\ GeV}}{\mathbf{0.1\ GeV}}\right) + 4\left(\frac{\langle\bar{q}q\rangle^{1/3} - \mathbf{0.267\ GeV}}{\mathbf{0.01\ GeV}}\right)\right] MeV,$$

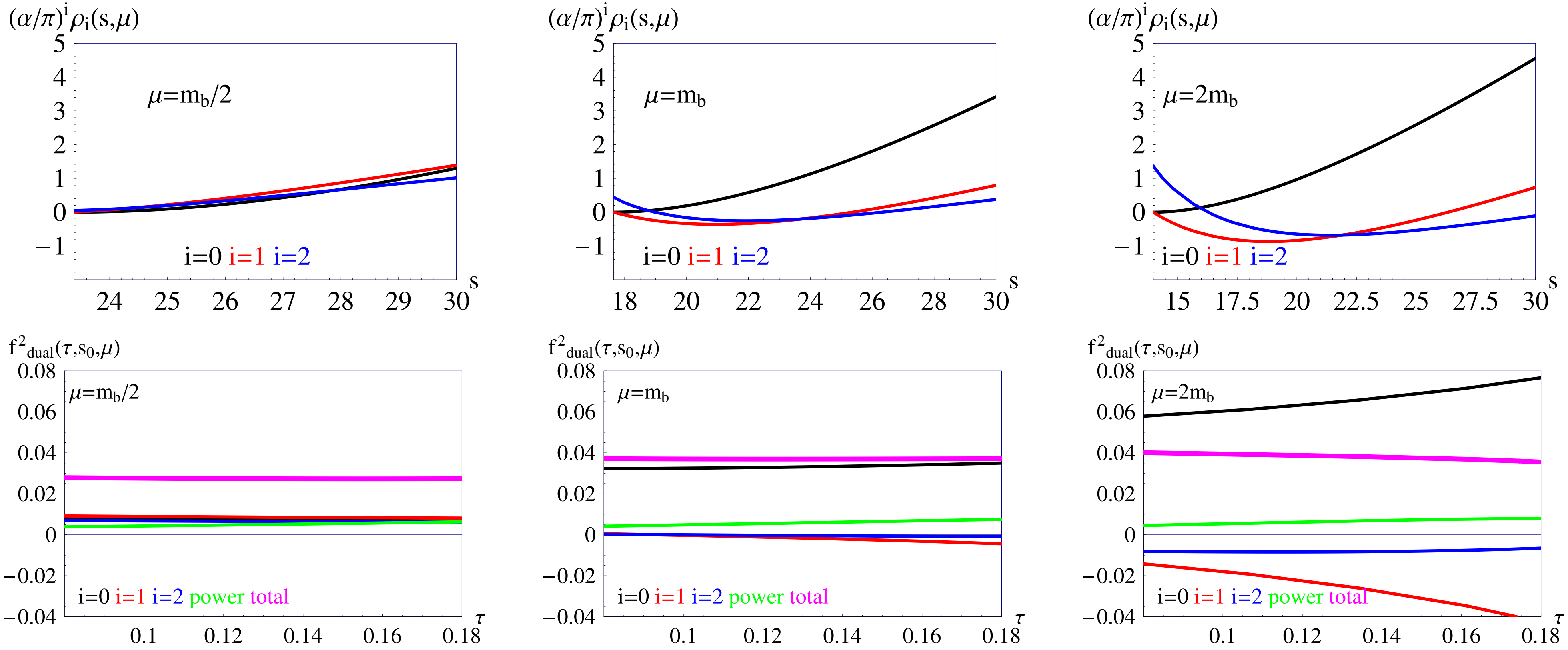
$\pm$ **10 MeV on** $m_b \to \mp$ **37 MeV on** f_B**!**

Problem 2. The dependence on the renormalization scale μ.

Even with NNLO corrections to the correlator, the sensitivity to the choice of μ is rather large.

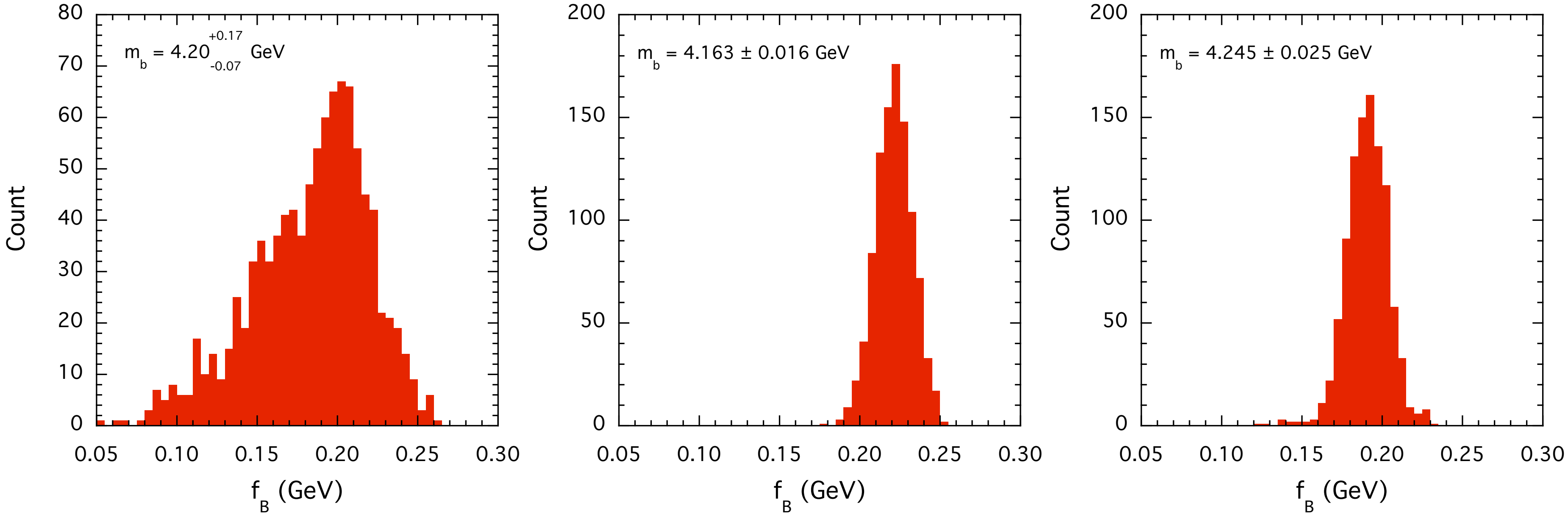
This signals that NNNLO (4 loops) are non-negligible.

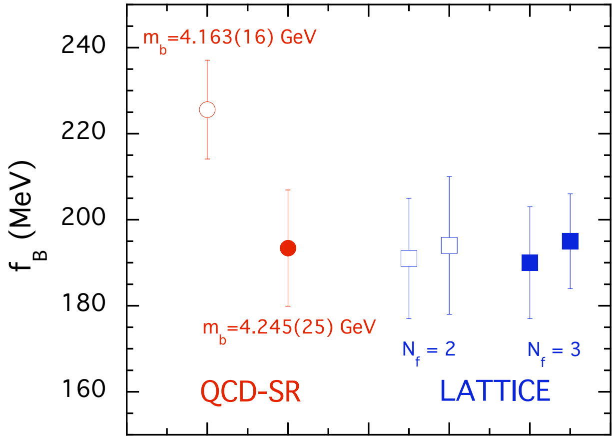
Often, the contribution of the omitted higher orders is probed by the variation of the scale μ. "Standard" in B-physics: $m_b/2 < \mu < 2m_b$. But why?



What is the relevant range of the μ-variation to probe higher-order contributions?

The prediction for f_B is not feasible without a very precise knowledge of m_b:



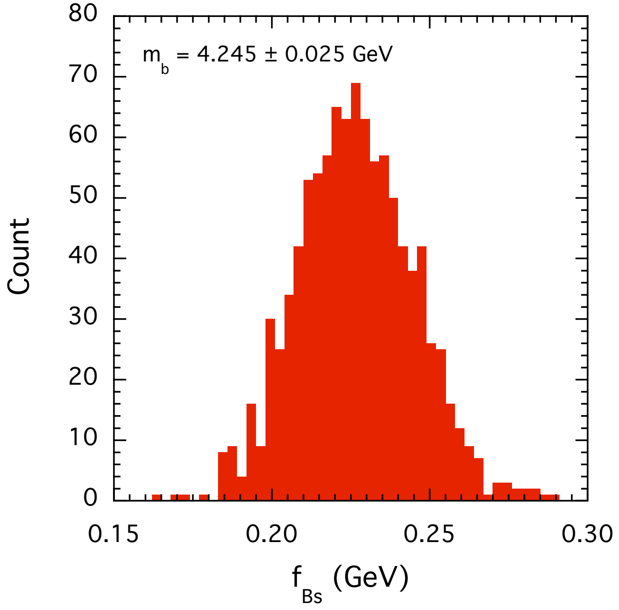


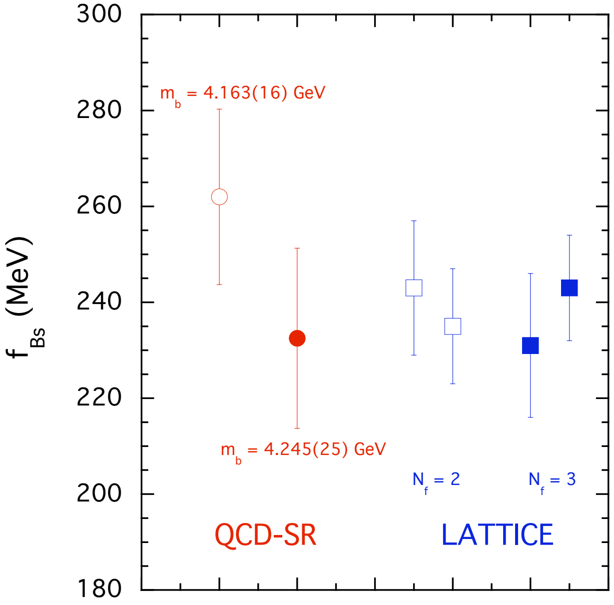
Our estimate : $\overline{m}_b(\overline{m}_b) = 4.245 \pm 0.025$ GeV

$f_B = 193.4 \pm 12.3_{OPE} \pm 4.3_{syst}$ MeV

f_B (const) = $184 \pm 13_{OPE}$ MeV

Extraction of f_{Bs}





$$f_{Bs} = 232.5 \pm 18.6_{OPE} \pm 2.4_{syst}\ \text{MeV}$$

$$f_{Bs}\ (\text{const}) = 218 \pm 18_{OPE}\ \text{MeV}$$

Conclusions

The effective continuum threshold $s_{\rm eff}$ is an important ingredient of the method of dispersive sum rules which determines to a (very) large extent the numerical values of the extracted hadron parameter. Finding a criterion for fixing $s_{\rm eff}$ poses a problem in the method of sum rules.

• $s_{\rm eff}$ depends on the external kinematical variables (e.g., momentum transfer in sum rules for 3-point correlators and light-cone sum rules) and "unphysical" parameters (renormalization scale μ, Borel parameter τ). Borel-parameter τ-dependence of $s_{\rm eff}$ emerges naturally when trying to make quark-hadron duality more accurate.

• We proposed a new algorithm for fixing τ-dependent $s_{\rm eff}$, for those problems where the ground-state mass M_Q is known. We have tested that our algorithm leads to more accurate values of ground-state parameters than the "standard" algorithms used in the context of dispersive sum rules before. Moreover, our algorithm allows one to probe "systematic" (≡ "intrinsic") uncertainties related to the limited accuracy of the extraction procedure in the method of QCD sum rules.

• We reported the decay constants of D, D_s, B, B_s mesons which along with the "statistical" errors related to the uncertainties in the QCD parameters, for the first time include realistic "systematic" errors.

Many other results are to come.