The problem of quantization of lightcone QCD

Alexey V. Popov*

Novgorod State University. (Velikiy Novgorod, Russia)

QFTHEP'2011

Abstract

This talk is devoted to the recent progress in the method of the lightcone coordinates in QCD. We show that boundary gauge fields are crucial for the consistent and complete definition of the theory. The result is important for the theory of high energy QCD evolution, since scattering amplitudes are directly related to the lightcone Hamiltonian, whose complete structure is still unclear on quantum level. Namely, there exists the problem to construct a quantum algebra of observables in lightcone QCD beyond the perturbative regime. Careful analysis shows that we have the problems with: canonical commutation relations, spatial invariance, and the boundary degrees of freedom in the Hamiltonian.

*email: avp@novgorod.net

1 Introduction

There exists three main theoretical methods in high energy QCD:

- Feynman diagrams (*Lipatov, Braun, Kovchegov, ...*).
- Path integral (*McLerran, Hatta, Balitski, ...*).
- Quantum Hamiltonian (*Kovner, Lublinsky, Mueller, ...*).

  There are two typical coordinates here:

  - cartesian coordinates.
  - lightcone coordinates.

While in the cartesian coordinates the complexity of the theory is pushed entirely into the quantum level (wave function), in the lightcone coordinates it remains mostly in the classical level via the complicated structure of the phase space and the complicated Hamiltonian.

Why the complete lightcone QCD Hamiltonian is so important?

- The evolution of amplitudes of high energy scattering in QCD [1].
- JIMWLK/BK equations and NLO corrections. Very simple derivation.
- Powerful generalization of BFKL in the framework of evolution equations. Unification of pomeron loops and pomeron poles.
- Multiple gluon production and other exclusive processes.
- A possible way to the nonperturbative area of QCD?

Advantages of the lightcone method:

- Only operator algebra. No diagrams.
- Life in the physical Hilbert space directly related to observables.
- Trivial vacuum.
- Correctly removed gauge freedom. No ghosts, no zero modes.

2 How to quantize lightcone QCD?

Our road map is:

1. Fix the gauge freedom.
2. Construct the classical phase space.
3. Construct the classical Hamiltonian.
4. Quantize the theory.
   (a) To construct a Poisson algebra with a Hamiltonian. This algebra will be used as an algebra of classical observables.
   (b) To quantize the Poisson algebra. To construct a Hilbert space and canonical commutation relations.
   (c) To resolve possible ordering ambiguities in the quantum-classical operator correspondence.

3 Classical lightcone QCD

To quantize the theory we must identify the degrees of freedom. In lightcone theories, it is the well known fact [2] that boundary conditions at $x^- = \infty$ are crucial for physical consequences. Namely, choosing a different boundary condition, we obtain a different physic. Hence, in addition to the bulk fields, we must consider the boundary degrees of freedom.

- There is no natural choice of a boundary condition in QCD!
- Hence, gauge field at $x^- = \pm\infty$ is not zero.
- Moreover, if the gauge group is non-Abelian and there are four or more space-time dimensions, then the boundary condition at $x^- = \pm\infty$ is neither antisymmetric nor symmetric.
- Using residual gauge freedom, we can set the gauge field at $x^- = -\infty$ to zero.

The main gauge fixing is

$$A_- = 0 \tag{1}$$

The gauge fixing (1) is not complete. There exist gauge transformations that does not change A_-. In fact, any $U(\vec{x})$, which does not depend on x^-, is a symmetry of the dynamical system. So, after imposing the constraint (1), we again have the opportunity to impose an external constraint that does not miss physical information. This is a residual gauge freedom. We use it to impose the following realizable additional gauge constraint:

$$A_i(-\infty, \vec{x}) = 0 \tag{2}$$

Let $\gamma_i(\vec{x})$ be value of a field $A_i(x^-, \vec{x})$ at the boundary $x^- = +\infty$

$$\gamma_i(\vec{x}) = A_i(+\infty, \vec{x}) \tag{3}$$

The field γ_i play the important role in the story. It is not zero and it arises in the Hamiltonian and affect scattering amplitudes in NLO!

4 Failure of the canonical method

- Conventional canonical methods cannot treat fields at boundaries.
- The situation becomes even more difficult if a theory has constraints. To handle second class constraints, one should to construct the Dirac bracket which can be properly defined only in a finite-dimensional case.
- In an infinite-dimensional case, due to the existence of surface integrals, there exists the problem to define the naive variational derivative.

In this work we employ the symplectic Faddeev-Jackiw method [3] which allows to handle fields at the infinity:

- The symplectic method does not use variational derivative.
- A symplectic structure have no problems with the appearance of surface integrals.
- The residual gauge freedom can be properly fixed in a rigorous way.

5 Main features of the symplectic method

The Lagrangian of QCD directly gives the following triple:

- Configuration field manifold V
- Symplectic form ω
- Hamiltonian function H.

If a theory has a gauge invariance, then the symplectic form ω is degenerate. To construct the physical space we have to perform the following recursive procedure:

1. Take the bare phase space: symplectic triple (V, ω, H).
2. Factorization of $\operatorname{Ker}\omega$. $\operatorname{Ker}\omega$ gives a foliation of the symplectic manifold.
3. Reduction of the Hamiltonian on $V/\operatorname{Ker}\omega$.
4. Obtain new symplectic triple (V_1, ω_1, H_1) and go to the step 1.

The procedure ends when new symplectic triple becomes equal to previous one

$$(V_1, \omega_1, H_1) = (V, \omega, H)$$

In QCD, the symplectic method with the gauge fixing gives the following phase structure:

- The symplectic form is

$$\omega = \int \partial_- dA_i^a \wedge dA_i^a \tag{4}$$

- The bare configuration space $\{A_+, A_-, \pi^-, A_i\}$ is reduced to the purely transverse gauge potentials $\{A_i\}$.
- The complete set of Gauss constraints, including boundary fields.
- The Hamiltonian involving boundary fields.
- γ_i is not independent variable and is the functional $\gamma[\tilde{A}]$

During this calculations, we don't impose any boundary condition.

6 The classical Hamiltonian

The phase space of the theory is the space of fields $\tilde{A}_i(x^-, \vec{x})$ that obey $\tilde{A}_i(+\infty, \vec{x}) = -\tilde{A}_i(-\infty, \vec{x})$. The symplectic form is $\omega = \int \partial_- d\tilde{A}_i^a \wedge d\tilde{A}_i^a$. The boundary field $\gamma_i^a(\vec{x})$ is determined by the two following equations:

$$\partial_i \gamma_j^a - \partial_j \gamma_i^a + g f_{abc} \gamma_i^b \gamma_j^c = 0 \tag{5}$$

$$\partial_i \gamma_i^a(\vec{x}) = \frac{g}{2} f_{abc} \gamma_i^b \gamma_i^c + \int\limits_{-\infty}^{+\infty} \left(g f_{abc} \partial_- \tilde{A}_i^b \tilde{A}_i^c - g J_a^+ \right) dx^- \tag{6}$$

where J_a^+ is an external current. The original gauge fields A_i is related to $\tilde{A}_i$ as

$$\tilde{A}_i(x^-, \vec{x}) = A_i(x^-, \vec{x}) - \frac{1}{2} \gamma_i(\vec{x}) \tag{7}$$

The momentum $\pi_a^-(x^-, \vec{x})$ is determined by condition $\pi_a^-(\pm\infty, \vec{x}) = 0$ and the constraint

$$\partial_- \pi_a^- + \partial_- \partial_i A_i^a - g f_{abc} \partial_- A_i^b A_i^c + g J_a^+ = 0 \tag{8}$$

The Hamiltonian is given by

$$H[\tilde{A}_i] = \frac{1}{2}(\pi_a^-)^2 + \frac{1}{4}F_{ij}^a[A_i]F_a^{ij}[A_i] - gA_i^a J_a^i \tag{9}$$

The properties of the obtained Hamiltonian are:

- It involves the boundary field γ_i both explicitly and implicitly.
- For 4D it has infinite number of terms .
- For 4D it has infinite power over the coupling constant g.
- The perturbative expansion can be performed in a controlled way.
- First-order of the perturbative expansion gives $\gamma_i = 0$ and

$$H = H_0 + g\partial_i \tilde{A}_i^a \frac{1}{\partial_-} j_a^+ + g\tilde{A}_i^a j_i^a \tag{10}$$

- In the next order the g^4-correction to JIMWLK equation can be obtained, which involves γ_i.

The complete derivation of the lightcone Hamiltonian is given in Ref. [4].

7 The problem of quantization

There exists the problem to construct a quantum algebra of observables in lightcone QCD beyond the perturbative regime [5]. To quantize the theory we have to construct a Poisson algebra and quantize it.

Careful analysis will show that the Poisson formulation has the problem with either:

- canonical commutation relations
- spatial invariance
- boundary degrees of freedom in the Hamiltonian

We have analyzed three currently known variables: A, $\tilde{A}$, and c.

Natural variables A_i^a. To construct a Poisson algebra from our symplectic dynamical system we have to invert the symplectic form ω and to construct a Poisson brackets. Let us try to invert the symplectic form in the linear space of fields obeying the boundary condition $A_i^a(-\infty) = 0$.

We have to find a linear operator $P(x-y)$ such that

$$\omega\left(A, \int P(x-y)B(y)dy\right) = \int ABdx \tag{11}$$

It is easy to check that for the variables A_i^a there exists only one such operator

$$P(x) = \frac{1}{2}\theta(-x) \tag{12}$$

where $\theta(x)$ is the standard step function. The result (12) has a fatal problem: the kernel $P(x)$ is not antisymmetric. Hence, a naive Poisson brackets does not exist and the variables A_i^a cannot generate a Poisson algebra of observables.

Antisymmetric variables $\tilde{A}_i^a$. From the wide practice of lightcone field theories we know that the antisymmetric conditions induces a well-defined Poisson algebra. The symplectic form ω is invertible and the corresponding inverse kernel is

$$P(x) = -\frac{1}{4}\varepsilon(x) \tag{13}$$

However, the fatal problem here is that the lightcone QCD Hamiltonian has an implicit dependence over the boundary field γ_i. To extract it, we calculate an infinitesimal variation $\delta\tilde{H}$ over a variation of the canonical variables $\delta\tilde{A}$. The boundary contribution to the variation $\delta\tilde{H}$ is

$$\delta\tilde{H}\Big|_{\text{boundary}} = -\int\limits_{x^-,\vec{x}} \frac{g}{4} f_{abc}\pi^-_a\delta\gamma^b_i\gamma^c_i \sim O(g^3) \tag{14}$$

The boundary variation can not be converted to a bulk one, since there exists an obstruction that the color-space 1-form $f_{abc}d\gamma^b\gamma^c$ is not exact and not closed, except the case of a one-dimensional color space. So, the Hamiltonian is not an element of the bulk Poisson algebra and the standard method of quantization can not be applied.

Boundaryless variables c^a_i. The another attempt to separate the boundary contribution is proposed in Ref. [1]. The idea is to express a boundary contribution to a bulk one. Let us define new fundamental variables c^a_i with zero boundary conditions as

$$A^a_i(x^-,\vec{x}) = c^a_i(x^-,\vec{x}) + \gamma^a_i(\vec{x})\varphi(x^-) \tag{15}$$

where $\varphi(x^-)$ is an arbitrary fixed global function such that

$$\begin{aligned} \varphi(-\infty) &= 0 \\ \varphi(+\infty) &= 1 \end{aligned} \tag{16}$$

It is helpful to imagine $\varphi(x^-)$ as a typical smooth monotonic kink-like function. So, we have $c_i^a(\pm\infty) = 0$. This gives the new Hamiltonian $H[c]$, which variation $\delta H[c]$ is <u>well defined</u>.

The first problem of this method is that the fundamental brackets $\{c(x_1), c(x_2)\}$ becomes very nonlinear. Although we have no methods how to quantize this nonlinear brackets, in principle, we can hope that a solution exist. The usage of variables c_i^a has the one more problem. Since the function $\varphi(x^-)$ explicitly depends on x^-, it breaks the longitudinal spatial invariance of the Hamiltonian. This problem is most serious because there are no preferred points in the initial formulation of the theory.

8 Conclusions

The theory of lightcone QCD has the following features:

- The boundary fields at $x^- = \infty$ play the important role.
- The symplectic method is most productive in this case.
- The boundary fields γ_i affect on physical scattering amplitudes beyond the leading order.
- The structure of the classical phase space is linear symplectic space.
- For 4D space-time and non-Abelian group the Hamiltonian has:
  - infinite number of terms.
  - infinite power over the coupling constant g.
- The quantum structure of lightcone QCD is still not clear. Beyond leader order the theory can not be directly quantized.
- Careful analysis shows that a Poisson formulation has the problems with: canonical commutation relations, spatial invariance, and the boundary degrees of freedom in the Hamiltonian.

References

[1] A. Kovner, M. Lublinsky, U. Wiedemann, JHEP 0706, 075 (2007), arXiv:0705.1713.

[2] P. Steinhardt, Ann. Phys. 128, 425 (1980).

[3] L. Faddeev, R. Jackiw, Phys. Rev. Lett. 60, 1692 (1988).

[4] A. V. Popov, Nucl. Phys. A 836 (2010) 136158, arXiv:0908.0139.

[5] A. V. Popov, arXiv:1107.4528.

Acknowledgments

This work and the participation in QFTHEP'2011 was supported by RFFI 11-02-01395-a grant.

Thank you for attention!