New definition of regularity for representation of canonical commutation relations algebras

Sergey G. Salinskiy
with M. Mnatsakanova, Yu. Vernov
Moscow State University

September, 2010

Table of content

Canonical commutation relations

$$[\hat{p}, \hat{q}] = -i\,\hat{I}$$

$$L_2(-\infty, +\infty)$$

$$\hat{q} f(q) = q f(q), \quad \hat{p} f(q) = -i \frac{\partial}{\partial q} f(q)$$

A definition of regularity

Rellich-Dixmier's theorem

Operators $\hat{q}$ and $\hat{p}$ form regular representation of CCR algebras if:

1. there exist dense domain $D \in D_q \bigcap D_p$ invariant under the action of $\hat{q}$ and $\hat{p}$ such that CCR hold on D;

2. Operator $(\hat{q}^2 + \hat{p}^2)$ is essentially self-adjoint on D.

Definition

Any representations of CCR algebras which are unitary equivalent to Schrodinger one are regular.

Weyl form

$$e^{it\hat{p}}e^{is\hat{q}} = e^{ist}e^{is\hat{q}}e^{it\hat{p}}$$

$$U(t) = e^{it\hat{p}}$$

$$V(s) = e^{is\hat{q}}$$

Analytical vectors

Definition

Let $\hat{A}$ be a linear operator on a Hilbert space H. A vector $\xi \in H$ is called entire analytic for $\hat{A}$, if ξ is in the domain of $\hat{A}^k$ for every $k \in \mathbb{N}$ and

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|\hat{A}^k \xi\| < +\infty$$

for every $t > 0$.

Main theorem

Let's prove that a representation of CCR algebras is regular if there is a dense domain D in witch any vector $\xi \in D$ obeys conditions

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|\hat{q}^k \xi\| < \infty, \quad \forall\, t > 0; \quad \sum_{k=0}^{\infty} \frac{s^k}{k!} \|\hat{p}^k \xi\| < \infty, \quad \forall\, s > 0$$

and also any regular representation obeys this conditions.

Main theorem: part 1

If ξ is an entire analytic vector for $\hat{A}$ then $exp(t\hat{A})\xi$ make sense for every $t \in \mathbb{C}$ and it is an entire analytic function of t.

$$\hat{p}\,\hat{q}^n - \hat{q}^n\,\hat{p} = -i(\hat{q}^n)', \quad (' = d/dq)$$

$$\hat{p}\,e^{is\hat{q}} - e^{it\hat{p}}\,\hat{p} = -i(e^{it\hat{q}})'$$

$$e^{-it\hat{q}}\,\hat{p}^n\,e^{it\hat{q}} = (\hat{p} + t\hat{I})^n$$

Main theorem: Stone's and von Neumann's theorems

Stone's theorem

Let $\hat{A}$ be a shelf-adjoint operator on a Hilbert space H. Then $U(t) = e^{it\hat{A}}, t \in \mathbb{R}$ is a strongly continuous one-parameter family of unitary operators.

von Neumann's theorem

If on a Hillbert space H self-adjoint operators $\hat{q}$ and $\hat{p}$ are generators of the unitary groups $U(t) = e^{it\hat{p}}$ and $V(s) = e^{is\hat{q}}$, satisfying the Weyl relations, then $\hat{q}$ and $\hat{p}$ satisfy a regular representation of CCR.

Main theorem: part 2

$$\hat{q} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^*), \quad \hat{p} = \frac{1}{i\sqrt{2}}(\hat{a} - \hat{a}^*), \quad \hat{N} = \hat{a}^*\hat{a}$$

$$\psi = \sum_{m}^{m+n} C_k\, \psi_k, \quad \psi_n = (\hat{a}^*)^n\, \psi_0$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \|\hat{q}^k \psi\| \leq$$

$$\leq \sum_{k=0}^{\infty} \frac{t^k}{k!} C^k \left(2\,(m+n+1)!\right)^{k/2} < +\infty$$

Conclusion

- Krein spaces
- Weyl representation