Cluster properties in Noncommutative Quantum Field Theory

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1 Introduction

Cluster properties of Wightman functions are very important properties of quantum field theory (QFT).
Let us recall that cluster properties of Wightman functions imply that

$$W\left(x_1, \ldots x_k, x_{k+1} + \lambda a, \ldots x_n + \lambda a\right) \to$$
$$W\left(x_1, \ldots x_k\right) W\left(x_{k+1}, \ldots x_n\right), \tag{1}$$

if $a^2 = -1$, $\lambda \to \infty$, $W\left(x_1, \ldots x_n\right) \equiv \langle\, \Psi_0, \varphi(x_1) \ldots \varphi(x_n)\, \Psi_0 \,\rangle$, where Ψ_0 is a vacuum vector.

Cluster properties are very natural consequence of Lorentz invariance. In fact, quantum field theory is defined by its Wightman functions. Owing to Lorentz invariance quantum field operators are independent in points $x_1, \ldots x_k$ and points $x_{k+1} + \lambda a, \ldots x_n + \lambda a$.

Thus
$$W\left(x_1, \ldots x_k, x_{k+1} + \lambda a, \ldots x_n + \lambda a\right) \rightarrow$$
$$W\left(x_1, \ldots x_k\right) W\left(x_{k+1} + \lambda a, \ldots x_n + \lambda a\right). \qquad (2)$$

In order to come to eq. (1) we have to take into account that Wightman functions are translation invariant.

In recent years noncommutative quantum field theory (NC QFT) attracts a great attention as very natural extension of the commutative theory to high energies. Moreover, in some cases NC QFT is a low energy limit of string theory. Thus it is very interesting to consider cluster properties in NC QFT. In this report we show that cluster properties are valid, if NC QFT is determined by the Heisenberg-like commutative relations:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\,\theta_{\mu\nu}, \qquad (3)$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix.

It is known that NC QFT can be formulated in commutative space if the usual product of quantum field operators is substituted by the $\star$-product. Precisely, it implies that the usual product of the corresponding test functions is substituted by the $\star$-product. NC QFT is a physically important example of such a theory.

In fact, the class of theories, in which usual product is substituted by the $\star$-product, is wider than NC QFT as Heisenberg-like commutation relations imply the existence of $SO(1,1) \otimes SO(2)$ symmetry of the theory. Evidently, theories with the $\star$-product have not to possess such a symmetry in a general case.

Here we show that cluster properties are valid in NC QFT as well as in commutative QFT. Moreover, they are valid in any theory with the $\star$-product of test functions.

2 Space-space Noncommutativity

First let us consider the case when time commutes with spatial variables. It is known that in this case there exists also one spatial variable which commutes with all other variables. Thus in this case we have two commutative variables, say x^0 and x^3 and two noncommutative (x^1 and x^2) ones.

Let us stress that, actually, the number of noncommutative variables can be arbitrary. Let us point out that in the case of space-space noncommutativity there exists one spatial variable, which commutes with all others irrespective of a number of noncommutative variables.

We admit in correspondence with the standard QFT that Wightman functions are tempered distributions in respect with commutative variables. In fact, after integration over noncommutative variables we, actually, obtain two-dimensional commutative QFT. The cluster properties in this case exist on the same ground as in the four-dimensional theory. Thus we can use the standard methods and obtain that, actually, in this case in the theories with mass gap the difference between the left-hand member of the eq.(1) and its right-hand member falls more rapidly than λ^{-n}, where n is an arbitrary positive number. In the theory with massless particals $n = 1$, a is two-dimensional vector: $a = \{a^0, a^3\}$.

3 General Case

The general case when all $\theta^{ij} \neq 0$ is much more nontrivial. Nevertheless we prove that eq.(1) is valid also in this case.

The crucial point in the derivation of eq.(1) is the fact that any test function $f_i \equiv f_i(x_i)$ belongs to the Gelfand-Shilov space S^β with $\beta < 1/2$. Precisely it was shown in our paper that if $f_i \in S^\beta$ with $\beta < 1/2$, then

$$f_i \star f_{i+1} \equiv \exp\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x_i^\mu}\,\frac{\partial}{\partial x_{i+1}^\nu}\right) f_i(x_i)\,f_{i+1}(x_{i+1}) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\,\theta^{\mu\nu}\,\frac{\partial}{\partial x_i^\mu}\,\frac{\partial}{\partial x_{i+1}^\nu}\right)^n f_i(x_i)\,f_{i+1}(x_{i+1}) \qquad (4)$$

converges. The similar result was obtained also in the paper of Soloviev.

If $f_i \star f_{i+1}$ converges $\forall\, i$, then $f_1 \star \ldots \star f_n$ converges as well if $f_i \in S^\beta$ with $\beta < 1/2$. Moreover this function belongs to S^β with the same β.

Let us remind that in the simplest case of one variable $f(x)$ belongs to the space S^β if

$$\left| x^k \frac{d^q f(x)}{d x^q} \right| \leq C_k\, B^q\, q^{q\beta}, \quad -\infty < x < \infty, \quad k, q \in \mathbb{N}, \tag{5}$$

where the constants C_k and B depend on the function $f(x)$. In fact, for convergence of the series (4) it is sufficient to use the inequality (5) only at $k = 0$. The extension of the last inequality on the case of multivariable function is straightforward.

Convergence of the series (4) implies that

$$f_1(x_1) \star \cdots \star f_n(x_n) = f_1^N(x_1) \star \cdots \star f_n^N(x_n) + \varepsilon(N), \ \varepsilon(N) \to 0, \tag{6}$$

if $N \to \infty$.

$$f_i^N(x_i) \star f_{i+1}^N(x_{i+1}) =$$

$$\sum_{n=0}^{N} \frac{1}{n!} \left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x_i^\mu} \frac{\partial}{\partial x_{i+1}^\nu} \right)^n f_i(x_i) f_{i+1}(x_{i+1}).$$

Let us stress that as B in inequality (5) depends on function $f(x)$, but does not depend on x, thus estimation (4) is the same for every x_i.

Now let us point out that as $f_i^N(x_i) \star f_{i+1}^N$ contains only a finite numbers of derivatives, then the corresponding generalized functions are tempered distributions.

Thus at any N we have cluster properties. This fact gives us the possibility to prove cluster properties in the general case. Let us point out that $W(x_1, \ldots x_n)$ is a functional in the space of functions $f_1(x_1) \star \cdots \star f_n(x_n)$. Actually, on the one hand, for arbitrary λ

$$W(x_1, \ldots x_k, x_{k+1} + \lambda a, \ldots x_n + \lambda a) \rightarrow$$

$$W^N(x_1, \ldots x_k, x_{k+1} + \lambda a, \ldots x_n + \lambda a) + \varepsilon(N), \qquad (7)$$

where $\varepsilon(N) \rightarrow 0$, if $N \rightarrow \infty$. Here a is a four-dimensional vector.

On the other hand, at any N

$$W^N(x_1, \ldots x_k, x_{k+1} + \lambda a, \ldots x_n + \lambda a) =$$

$$W^N(x_1, \ldots x_k) W^N(x_{k+1}, \ldots x_n) + \varepsilon(\lambda), \qquad (8)$$

$\varepsilon(\lambda) \rightarrow 0$, if $\lambda \rightarrow \infty$. Combining the eqs. (7) and (8), we come

to eq. (1):

$$W\left(x_1, \ldots x_k, x_{k+1} + \lambda a, \ldots x_n + \lambda a\right) \rightarrow$$
$$W\left(x_1, \ldots x_k\right) W\left(x_{k+1}, \ldots x_n\right).$$

Thus the cluster properties of Wightman functions are proved in any theory, in which the usual product of test functions is substituted by the $\star$-product.

Physically important example of such a theory is NC QFT. Now let us remark that if time and so one spatial variable commutes with all variables, then it is very natural to assume that Wightman functions are tempered distributions in respect with commuting coordinates. The most important consequence of cluster properties is the uniqueness of vacuum state.

4 Conclusion

In conclusion let us stress that as assertion that test functions belong to the Gelfand-Shilov space S^{β}, $\beta < 1/2$ does not depend on the dimension of the underlying space, all the results are valid in the space of arbitrary dimensions.