# One Rigorous Negative Result in Noncommutative Quantum Field Theory

M. N. Mnatsakanova
Institute of Nuclear Physics, Moscow State University,
Moscow, Russia

based on the joint work with
Yu.S. Vernov (Institute for Nuclear Research)

# 1 Introduction

Negative results play important role in axiomatic QFT. They show that relation between asymptotic and interacting fields is very nontrivial. The most important example of such results is Haag's theorem.

Here we consider another well-known result and show that it is possible to obtain it at weaker conditions. I mean the following

Theorem

*If any local field $\varphi(x)$ is irreducible and*

$$[\varphi(x), \varphi(y)] = B(x-y), \tag{1}$$

*then operator $B(x)$ is multiple of unit operator, that is $\varphi(x)$ is an asymptotic field.*

We show that this results is true also in the theories, in which Lorentz symmetry is broken up to $SO(1,1) \otimes SO(2)$.

As an example of such class of theories we can name noncommutative quantum field theory (NC QFT), which is important in the view of physics.

In this report we show that, actually, the commutator in question can not be an operator depending on the difference between one spatial coordinate in points $x$ and $y$. Our result is most interesting in the case of noncommutative theory, precisely, in the case of space-space noncommutativity, in which time commutes with spatial variables and, as a consequence, one spatial variable, say $x_3$, commutes with others. In what follows we consider just that very case.

In our proof we use the following general principles of axiomatic field theory:

*i)* Local commutativity condition (LCC);

*ii)* Irreducibility of the set of field operators.

For simplicity we consider the case of neutral scalar fields.

Local commutativity means that

$$[\varphi(x), \varphi(y)] = 0, \quad \text{if} \quad x \sim y. \tag{2}$$

The condition $x \sim y$ in usual (commutative) theory means that

$$(x-y)^2 < 0.$$

In noncommutative quantum field theory (NC QFT) LCC can be fulfilled with respect to commutative variables only.

The reason is that test functions, corresponding to noncommutative variables, belong to the one of Gelfand-Shilov spaces $S^{\beta}$ with $\beta < 1/2$, which does not contain functions with finite support and so corresponding field operators can not satisfy LCC.

Thus in NC QFT we have the following LCC:

$$[\varphi(x), \varphi(y)] = 0, \quad \text{if} \quad (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \tag{3}$$

Let us stress that our result is valid in any theory, where this condition is fulfilled.

Now let us recall the condition of irreducibility.
The set of field operators $\varphi(x)$ is irreducible if the bounded operator, which commutes with all field operators, has to be $C\,\mathbb{I}$, where $\mathbb{I}$ is identical operator and $C$ is some function.

Our proof is the modification of the classical proof given in the book of N. N. Bogoliubov, A. A. Logunov and I. T. Todorov.

## 2 Proof

Let us prove that

*if*

$$[\varphi(x), \varphi(y)] = A(x_3 - y_3, X, Y), \tag{4}$$

*where we denote all other variables as* $X, Y$,

*then*

$$A(x_3 - y_3, X, Y) = C\,\mathbb{I}$$

Let me remind the Jacobi identity:

$$[\varphi(x), [\varphi(y), \varphi(z)]] + [\varphi(y), [\varphi(z), \varphi(x)]] + [\varphi(z), [\varphi(x), \varphi(y)]] = 0 \tag{5}$$

If

$$(z_0 - y_0)^2 - (z_3 - y_3)^2 < 0,$$
$$(z_0 - x_0)^2 - (z_3 - x_3)^2 < 0,$$

then in accordance with LCC from Jacobi identity it follows that

$$[\varphi(z), [\varphi(x), \varphi(y)]] = 0. \tag{6}$$

The necessary conditions: $z - x$ as well as $z - y$ are space-like vectors in respect with commutative coordinates, are fulfilled if

$$x_3 = \lambda + x_3', \quad y_3 = \lambda + y_3'$$

$$x_3', \; y_3' \quad \text{are arbitrary}, \quad \lambda = (0, 0, \lambda, 0) \quad \lambda^2 \to -\infty;$$

then $\qquad x_3 - y_3 = \lambda + x_3' - \lambda - y_3' = x_3' - y_3'.$

So, $A(x_3 - y_3, X, Y)$, which we have in (4), is:

$$A(x_3 - y_3, X, Y) = B(x_3' - y_3', X, Y).$$

Let me remind eq. (6):

$$[\varphi(z),[\varphi(x),\varphi(y)]]=[\varphi(z),A(x_3-y_3,X,Y)]=$$
$$[\varphi(z),B(x'_3-y'_3,X,Y)]=0, \tag{7}$$

where $z, x'$ and $y'$ are arbitrary. So we see that $B(x'_3-y'_3,X,Y)]$ commutes with $\varphi(z)$ at arbitrary $z$.

Owing to irreducibility of the set of quantum field operators, $[\varphi(x'),\varphi(y')]=C\,\mathbb{I}$, where $C$ is some function.

Thus we have proved that commutator $[\varphi(x'),\varphi(y')]$ has to be a function. It is known that in this case any Wightman function

$$\langle\psi_0,\varphi(x_1),\ldots\varphi(x_n)\,\psi_0\rangle$$

has to be some superposition of two-point Wightman functions and so in this case the set of Wightman functions can not define any nontrivial theory.

Let us stress that our result is valid in a space of arbitrary dimensions.