Extension of Haag's Theorem on Spaces with Arbitrary Dimensions

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1 Introduction

In this report we consider one of the most important result of axiomatic approach in quantum field theory (QFT). In the usual Hamiltonian formalism it is assumed that asymptotic fields are related with interacting fields by unitary transformation.
The Haag's theorem shows that in accordance with Lorentz invariance of the theory the interacting fields are also trivial which means that corresponding S-matrix is equal to unity. So the usual interaction representation can not exist in the Lorentz invariant theory. In four dimensional case in accordance with the Haag's theorem four first Wightman functions coincide in two related by the unitary transformation theories.

Let us recall that n-point Wightman functions $W\,(x_1, \ldots, x_n)$ are $\langle\, \Psi_0, \varphi(x_1) \ldots \varphi(x_n)\, \Psi_0 \,\rangle$, where Ψ_0 is a vacuum vector. Actually in accordance with translation invariance Wightman functions are functions of variables $\xi_i = x_i - x_{i+1}$, $i = 1, \ldots, n-1$. At first Haag's theorem is proved in $S\,O\,(1,3)$ invariant theory in four dimensional case.

Now the theories in spaces of arbitrary dimensions are widely considered. In last years noncommutative generalization of QFT - NC QFT - attracts interest of physicists as in some cases NC QFT is the low energy limit of string theory.

NC QFT is defined by the Heisenberg-like commutation relations between coordinates

$$[x^\mu, x^\nu] = i\,\theta^{\mu\nu}, \qquad (1)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.

It is very important that NC QFT can be formulated in commutative space if the usual product between operators (precisely between corresponding test functions) is substituted by the star (Moyal-type) product.

Thus it is interesting to consider Haag's theorem in the general case of $k+1$ commutative variables (time and k spatial coordinates) and arbitrary number m of noncommutative coordinates. This extension of the Haag's theorem is a goal of our work.

2 Haag's theorem in four dimensional case

In axiomatic QFT fields $\varphi\left(f\right)$ smeared on all four coordinates are unbounded operators in the state vectors space. In the derivation of Haag's theorem it is necessary to assume that fields smeared on the spatial coordinates are well defined operators.

Let us recall Haag's theorem in four dimensional case.

Let $\varphi_1(f,t)$ and $\varphi_2(f,t)$ belong to two sets of irreducible operators in Hilbert spaces H_1 and H_2 correspondingly. We assume that both theories are Poincare invariant, that is

$$U_j(a,\Lambda)\,\varphi_j(x)U_j^{-1}(a,\Lambda)=\varphi_j(\Lambda x+a), \tag{2}$$

$$U_j(a,\Lambda)\Psi_{0j}=\Psi_{0j}, \qquad j=1,2. \tag{3}$$

Let us suppose also that there exists unitary transformation V related fields in question and vacuum states as well in two theories at any t:

$$\varphi_2(f,t)=V\varphi_1(f,t)V^{-1}, \tag{4}$$

$$c\Psi_{02}=V\Psi_{01}, \tag{5}$$

where c is a complex number with module one.

Let as suppose that vacuum states in two theories are invariant under the same unitary transformation.

If in two theories there are not states with negative energies then four first Wightman functions coincide in two theories.

Let us give the idea of the proof.

The invariance of the vacuum states implies that Wightman functions coincide at equal times.

$$(\Psi_{01}, \varphi_1(t, x_1), \ldots, \varphi_1(t, x_n)\Psi_{01}) =$$
$$(\Psi_{02}, \varphi_2(t, x_1), \ldots, \varphi_2(t, x_n)\Psi_{02}) \tag{6}$$

First let us notice that at equal times all points x_i belong to the set of Iost points. Let us recall that Iost points are real points, which belong to the corresponding domain of analyticity of Wightman functions. It is known that the interval between two arbitrary Iost points is space-like:

$$(r_k - r_l)^2 < 0 \tag{7}$$

Thus any Iost point belongs to the set of Iost points with its vicinity. So Iost points fully determine Wightman functions, i.e. two Wightman functions, which coincide at Iost points, precisely, in the open subset of these points, coincide identically.

It can be shown that the equality of Wightman functions at equal times and their analyticity lead to equality of four first Wightman functions in two theories related by unitary transformation at equal times.

Let us stress that noncommutative coordinates belong to the boundary of analyticity of Wightman functions. As in the derivation of Haag's theorem only transformations of coordinates which belong the domain of analyticity are essential, we omit the dependence of vectors under consideration on noncommutative variables.

3 Extension of Haag's theorem

Let us obtain the extension of Haag's theorem on the $SO(1,k)$ invariant theory.

As at $n > k$ vectors $\xi_i = (0, \vec{\xi_i})$ are linear dependent, then vectors related by them with Lorentz transformation are linear dependent too and thus can not be on the open set of Iost points. Thus they can not determine Wightman functions.

Let us show that if $n \leq k$, then corresponding Iost points fully determine Wightman functions.

We give the sketch of the proof.
As the vectors $\vec{\xi_i}$ are arbitrary we can choose the set of vectors $\xi_i = (0, \vec{\xi_i})$ in such a way that they all be orthogonal one to another. As $\xi_i \perp \xi_j$ if $i \neq j$, then also $\alpha\,\xi_i \perp \beta\,\xi_j$, $\alpha, \beta \in \mathbb{R}$ are arbitrary. If $\tilde{\xi_i} = L\,\xi_i$, where L is a real Lorentz transformation,

then $\tilde{\xi}_i \perp \tilde{\xi}_j,\ i \neq j$ and also $\alpha\,\tilde{\xi}_i \perp \beta\,\tilde{\xi}_j$. So these points form the open subset of Iost points and thus fully determine Wightman functions.

As in two theories related by an unitary transformation at equal times first $k+1$ Whigtman functions coincide on the open set of Iost points, then these functions coincide in all points.

4 Consequens of the Haag's Theorem

4.1 $SO(1,1)$ invariant theories

First let us consider Haag's theorem in the $SO(1,1)$ invariant field theory. In accordance with previous result equality of only two-point Wightman functions takes place. Let us prove that if one of considered theories is trivial, that is the corresponding S-matrix is equal to unity, then another is trivial too.

Let us point out that in the $SO(1,1)$ invariant theory it is sufficient that the spectral condition, which implies non existence of tachions, is valid only in respect with commutative coordinates. Also it is sufficient that translation invariance is valid only in respect with the commutative coordinates. In this case the equality of two-point Wightman functions in the two theories leads to the

following conclusion:
if local commutativity condition in respect with commutative coordinates is fulfilled and the current in one of the theories is equal to zero, then another current is zero as well.
Indeed as $W_1(x_1, x_2) = W_2(x_1, x_2)$, then also

$$< \Psi_0^1, j_{\bar{f}}^1 j_f^1 \ \Psi_0^1 >=< \Psi_0^2, j_{\bar{f}}^2 j_f^2 \Psi_0^2 >= 0. \qquad (8)$$

If, for example, $j_f^1 = 0$, then $j_f^2 \Psi_0^2 = 0$, where

$$j_f^i = (\Box + m^2)\, \varphi_f^i.$$

Hence,

$$j_f^2 \Psi_0^2 = 0. \qquad (9)$$

From the latter formula and local commutativity condition it follows that

$$j_f^2 \equiv 0.$$

Our statement is proved.

4.2 $SO(1,3)$ invariant theories

Let us proceed now to the $SO(1,3)$ invariant theory, precisely, to the theory with four commutative coordinates and arbitrary number of noncommutative ones. In this case we show that from the equality of the four-point Wightman functions for the fields $\varphi^1_f(t)$ and $\varphi^2_f(t)$, related by the conditions (4) and (5), which takes place in the commutative theory, an essential physical consequence follows. Namely, for such fields the elastic scattering amplitudes in the corresponding theories coincide, hence, due to the optical theorem, the total cross-sections coincide as well.
In particular, if one of these fields, for example, φ^1_f is a trivial field, i.e. the corresponding S matrix is equal to unity, also the field φ^2_f is trivial. In the derivation of this result the local commutativity condition is not used.

The statement follows directly from the Lehmann-Symanzik-Zimmermann reduction formulas.
Here and below dealing with the commutative case in order not to complicate formulas we consider operators $\varphi_1(x)$ and $\varphi_2(x)$ as they are given in a point.

Let $< p_3, p_4 | p_1, p_2 >_i$, $i = 1, 2$ be elastic scattering amplitudes for the fields $\varphi_1(x)$ and $\varphi_2(x)$ respectively. Owing to the reduction formulas,

$$< p_3, p_4 | p_1, p_2 >_i \sim \int d\,x_1 \cdots d\,x_4\, e^{i\,(-p_1\,x_1 - p_2\,x_2 + p_3\,x_3 + p_4\,x_4)}\,.$$

$$\prod_{j=1}^{4} (\Box_j + m^2)\ < 0 | T\,\varphi_i\,(x_1)\ \cdots\ \varphi_i\,(x_4) | 0 >, \qquad (10)$$

where $T\,\varphi_i\,(x_1)\ \cdots\ \varphi_i\,(x_4)$ is the chronological product of operators.

From the equality

$$W_1\left(x_1, \ldots, x_4\right) = W_2\left(x_1, \ldots, x_4\right)$$

it follows that

$$< p_3, p_4 | p_1, p_2 >_1 = < p_3, p_4 | p_1, p_2 >_2 \tag{11}$$

for any p_i. Having applied this equality for the forward elastic scattering amplitudes, we obtain that, according to the optical theorem, the total cross-sections for the fields $\varphi_1(x)$ and $\varphi_2(x)$ coincide. If now the S-matrix for the field $\varphi_1(x)$ is unity, then it is also unity for field $\varphi_2(x)$. We stress that the equality of the four-point Wightman in four dimension space in two theories related by the unitary transformation are valid only in the commutative field theory, but not in the noncommutative case.

4.3 General Case

Now let us proceed to the general case, i.e. to $SO(1,k)$ invariant theory. We prove that in addition to equality of elastic and total cross sections the equality of amplitudes of some inelastic processes takes place. In accordance with reduction formula

$$\left\langle p'_1 \ldots p'^{\,out}_l \mid p_1 \ldots p^{in}_m \right\rangle =$$

$$i^{l+m} \int dy_1 \ldots dy_l \, dx_1 \ldots dx_m \, f^*_{p'_1}(y_1) \times \ldots$$
$$\times f^{\star}_{p'_l}(y_l) \vec{K}_{y_1} \ldots \vec{K}_{y_l} \Big(\Psi_0, T\big(\varphi(y_1) \ldots \varphi(x_m)\big) \Psi_0 \Big) \times$$
$$\times \vec{K}_{x_1} \ldots \vec{K}_{x_m} f_{p_1}(x_1) \ldots f_{p_m}(x_m), \quad (12)$$

where $K_x = \left(\partial_\mu \partial^\mu\right)_x + m^2$ is Klein-Gordon operator and $f_p(x) = \frac{e^{-ipx}}{(2\pi)^{3/2}}$ is a corresponding wave function.

Let us consider $2 \Rightarrow k-1$ processes. We see that amplitudes of these processes coincide in two theories.

5 Conclusions

- The Haag's theorem has been proved in $SO(1,k)$ invariant theory, which can include arbitrary number of noncommutative coordinates in addition.
- We have proved that if one of $SO(1,1)$ invariant theories is trivial, then other such a theory, connected with the first one by unitary transformation, is trivial as well.
- In $SO(1,3)$ invariant theory we have proved equality of elastic and consequentely total cross sections in two theories under consideration.
- In $SO(1,k)$ invariant theory in addition we have proved the equality of amplitudes of some inelastic processes.